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An Existence Result for a Discretized Constrained Gradient System of Total Variation Flow in Color Image Processing

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We consider a constrained gradient system of total variation flow. Our system is often used in color image processing to remove a noise from picture. In this paper, using abstract convergence theory of convex functions, we show the global existence of solutions to our problem with piecewise constant initial data.

KEYWORDS: total variation, 1-harmonic map flow, subdifferential

1. Introduction

This is a preliminary report on 1-harmonic map flow. We consider a constrained gradient system of total variation flow for $u : [0,T) \times \Omega \to S^{n-1} \subset \mathbb{R}^n$ with Neumann boundary condition as follows:

$$\begin{cases} u' = -\pi_u\left(-\operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right)\right) & \text{in} \quad (0,T) \times \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $u'(t) := \frac{d}{dt}u(t)$, Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and ν is the outer normal unit vector. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ $(n \geq 1)$, *i.e.*

$$S^{n-1} := \{w \in \mathbb{R}^n; |w| = 1\}.$$

For each element $u \in S^{n-1}$, let $\pi_u : \mathbb{R}^n \to T_uS^{n-1}$ be an orthogonal projection from $\mathbb{R}^n = T_u\mathbb{R}^n$ to tangent space T_uS^{n-1} of S^{n-1} at u. The given initial data u_0 is a map from Ω to S^{n-1}.

The problem (1.1) is proposed by B. Tang, G. Sapiro and V. Caselles [TSC] in order to remove a noise from the chromaticity of the initial image $u_0(x,y)$ preseving the brightness of $u(t,x,y)$ for all $t \in (0,T)$ and $(x,y) \in \mathbb{R}^2$.

In 2003, Giga and Kobayashi [GK] considered the problem (1.1) in the one-dimensional case. Then, they showed that for each piecewise constant initial data u_0 on Ω, there is an unique global solution u on $[0,\infty)$ such that $u(t)$ is a piecewise constant on Ω. Moreover, They studied the stationary problem in the case when the manifold is the unit circle S^1 in $\mathbb{R}^2$.

In 2004, Giga, Kashima and Yamazaki [GKY] studied the general n-dimensional case. In [GKY], they assumed that the initial data u_0 is (sufficiently) small in some sense, and they showed the local solution to (1.1) in the torus domain $\Omega := \mathbb{T}^n = \prod_{i=1}^n(\mathbb{R}/\omega_i\mathbb{Z})$ for given $\omega_i > 0$ $(i = 1,2,\ldots,n)$, by applying the theory of p-harmonic map flow equation with $p > 1$

$$u' = -\pi_u(-\operatorname{div}(|\nabla u|^{p-2}\nabla u)) \quad \text{in } (0,T) \times \mathbb{T}^n. \tag{1.2}$$

In this paper we consider a solvability of a discretized problem to (1.1) by using the same argument in [GK]. Namely, the problem (1.1) is to find the piecewise constant solution on Ω. Then, the problem is reduced to a system of ordinary

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differential equations unless two different values merges. Of course, merging may occur, so, it is very difficult to study the detailed dynamics in 2-dimensional case. Different from one dimensional problem, discretized version may not corresponed to a solution of an original problem with a piecewise constant initial data. Such a difficulty is also observed in the unconstrained problem of crystalline flow [BNP] and [GGK], for instance.

In image processing the Gaussian filter is often used for a grey-level function. In other words for a given initial grey-level function u_0, we solve the heat equation

$$\begin{cases} u' - \Delta u = 0 & \text{in} \quad (0, T) \times \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0 & \text{in} \quad \Omega \end{cases}$$

to get a denoised grey-level function $u(t, \cdot)$ at scale t. However, this way has a drawback since all characteristic function is mollified and a sharp contrast become ambiguous. If one use a (unconstrained) gradient system of total variation flow

$$u' - \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0 \quad \text{in} \quad (0, T) \times \Omega,$$

then a Heaviside type function

$$u(t, x, y) = \begin{cases} 1 & x > 0, \\ 0 & x \le 0, \end{cases}$$

is a stationary solution so such a grey-level function is not mollified. When we try to remove the noise from chromaticity, as proposed by [TSC] (2-)harmonic map flow is a typical way and it corresponds to the way using the heat equation for a grey-level function. So this has a similar drawback. A constrained system of total variation flows corresponds to the gradient system of total variation and it preserves a Heaviside like map.

The main object of this paper is to show the global existence of solution to the discretized problem of (1.1). In Section 2, we present the discretized problem and subdifferential formulation of our problem (1.1). Moreover we mention main result (Theorem 2.1) in this paper, which is concerned with the global existence of solution. In Section 3, we consider the approximating problem in order to prove Theorem 2.1. In Section 4, we recall the abstract convergence theorem established in [GKY]. Then we shall give the proof of Theorem 2.1.

2. Discretized Problem

To solve global-in-time solvability for (1.1) we introduce a discretized equation approximating (1.1).

2.1 Subdifferential formulation

In this subsection we reformulate the problem (1.1) to the nonlinear evolution equation in a real Hilbert space. To do so, we use the concept of subdifferential of convex function. Here, we recall the definition of subdifferential of convex function.

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and $\varphi : H \to (-\infty, +\infty]$ be a proper (*i.e.*, not identically equal to infinity), l.s.c. (lower semi-continuous) and convex function on H. Then, the subdifferential of φ is defined by this set

$$\partial\varphi(u) = \{f \in H \mid \varphi(u + h) - \varphi(u) \ge \langle f, h \rangle \text{ for any } h \in H\}.$$

If φ is differentiable, then the subdifferential of φ coincides with its classical derivative. Therefore the subdifferential is an extension of differential. For basic properties of subdifferential, we refer to the monograph by Brézis [B].

To reformulate the problem (1.1), we use the following notations.

For the space $L^2(\Omega; \mathbb{R}^n)$ of $\mathbb{R}^n$-valued square integrable functions, let $L^2(\Omega; S^{n-1})$ be the closed subset of $L^2(\Omega; \mathbb{R}^n)$ of the form

$$L^2(\Omega; S^{n-1}) := \{v \in L^2(\Omega; \mathbb{R}^n) \ ; \ v(x) \in S^{n-1} \text{ a.e. } x \in \Omega\}.$$

For the bounded domain Ω in $\mathbb{R}^2$, let $\mathcal{C}$ be a rectangular decomposition of $\mathbb{R}^2$. In other words $\mathcal{C}$ is a disjoint family of open rectangles $R_j = (a_j, b_j) \times (c_j, d_j)$ which covers $\mathbb{R}^2$ expect a Lebesgue measure zero set, *i.e.* $\mathcal{C} = \{R_j\}_{j \in \Lambda}$. Let Δ be a decomposition of Ω associated with $\mathcal{C}$ defined by $\Delta = \{\Omega_i\}_{i \in I}$ with $\Omega_i = R_i \cap \Omega$, $I = \{i \in \Lambda \ ; \ \Omega_i \neq \emptyset\}$. Note that I is a finite index set, since Ω is a bounded domain. In this paper, we fix the family Δ. Here, we set $c_{ij} = \mathcal{H}^1(\partial\Omega_i \cap \partial\Omega_j)$, where $\mathcal{H}^1$ is the Haussdorff measure, more precisely c_{ij} implies a length of $\partial\Omega_i \cap \partial\Omega_j$.

For the fixed family $\Delta = \{\Omega_i\}_{i \in I}$, let H_Δ be the set of all step functions on $\bigcup_{i \in I} \Omega_i$, *i.e.*

$$H_\Delta = \left\{ \sum_{i \in I} a_i \chi_{\Omega_i} \ ; \ a_i \in \mathbb{R}^n \right\},$$

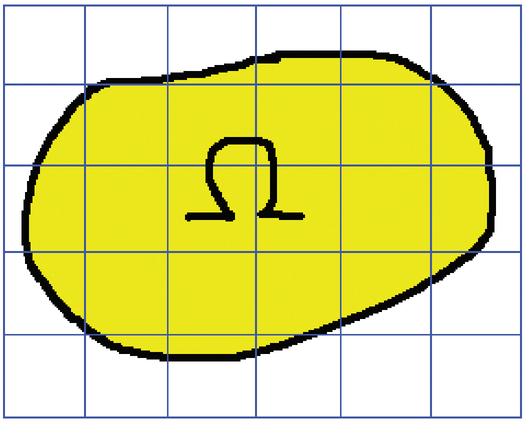

Fig. 1. Rectangular decomposition Δ.

where χ_{Ω_i} is the characteristic function on Ω_i. We easily see that H_Δ is the subset of $L^2(\Omega;\mathbb{R}^n)$. Moreover we observe that if $u \in H_\Delta$, then the total variation of u is like this form

$$\int_\Omega |\nabla u| = \sum_{i<j} c_{ij}|a_i - a_j|.$$

For their precise definitions and basic properties of total variation, see monographs by Evans and Gariepy [EG] or Giusti [G], for instance.

Now, let us define two functions on real Hilbert spaces. For the fixed family $\Delta = \{\Omega_i\}_{i\in I}$, we put

$$\varphi_\Delta(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u| & \text{if } u \in H_\Delta, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

Then we easily see that φ_Δ is the proper, l.s.c. and convex function on $L^2(\Omega;\mathbb{R}^n)$ (See [EG] or [G], for instance). We also define the function Φ_Δ^T by this form

$$\Phi_\Delta^T(u) = \int_0^T \varphi_\Delta(u(t))\,dt \quad \text{for all } u \in L^2(0,T;L^2(\Omega;\mathbb{R}^n)). \tag{2.2}$$

Then, by the slight modification of [GKY, Proposition 2.1] we observe that Φ_Δ^T is also the proper, l.s.c. and convex function on $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$.

For any $g \in L^2(0,T;L^2(\Omega;S^{n-1}))$ we define a map $P_g(\cdot) : L^2(0,T;L^2(\Omega;\mathbb{R}^n)) \longrightarrow L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ by

$$P_g(f)(x,t) = \pi_{g(x,t)}(f(x,t)) \quad \text{for a.e. } (x,t) \in \Omega \times [0,T] \tag{2.3}$$

for any $f \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$.

By these notations as above, we easily see that the problem (1.1) can be reformulated as in the following form:

$$\begin{cases} u' \in -P_u(\partial\Phi_\Delta^T(u)) & \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^n)), \\ u|_{t=0} = u_0 & \text{in } L^2(\Omega;S^{n-1}), \end{cases} \tag{2.4}$$

where $u_0 \in L^2(\Omega, S^{n-1})$ is a given initial data.

The main object of this paper is to show the global existence of a solution to (2.4), since the initial boundary value problem (1.1) can be regarded as a mathematical formulation of (2.4).

Now, let us give the definition of a solution to (1.1).

Definition 2.1. A function $u : \Omega \times [0,T] \to \mathbb{R}^n$ is a solution of (1.1), if u belongs to $L^2(0,T;L^2(\Omega;S^{n-1})) \cap C([0,T], L^2(\Omega;\mathbb{R}^n))$ and satisfies (2.4).

2.2 Main result

In this subsection, let us mention our main result in this paper as follows.

Theorem 2.1. *Suppose the initial data* $u_0 \in H_\Delta$ *with* $u_0 \in L^2(\Omega;S^{n-1})$. *Then for any time* $T > 0$, *there exists at least one solution* u *on* $[0,T]$ *to the problem (1.1).*

It is very difficult to analyze our problem (1.1) directly. We cannot apply the general theory established by Brézis [B] and Kōmura [Ko], because of the projection π_u. So, by using the approximating problem of (1.1), we shall prove Theorem 2.1 in Section 4.

3. Approximating Problem

In this section we consider the approximating problem of (1.1). At first we shall define the approximating energy function to (2.1)

3.1 Approximating energy

For each $\varepsilon > 0$, let us define the function $\varphi_\Delta^\varepsilon$ by this form

$$\varphi_\Delta^\varepsilon(u) = \begin{cases} \sum_{j<k} c_{jk}\sqrt{|a_j - a_k|^2 + \varepsilon^2} & \text{if } u \in H_\Delta, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

Clearly, we see that $\varphi_\Delta^\varepsilon$ is the proper, l.s.c. and convex function on $L^2(\Omega; \mathbb{R}^n)$. Moreover, $\varphi_\Delta^\varepsilon$ is the approximating functional of our energy φ_Δ defined by (2.1).

Note that the function $\varphi_\Delta^\varepsilon$ has no singularity, because of $\varepsilon > 0$. Therefore, by the standard calculation, we can verify that $\partial\varphi_\Delta^\varepsilon(\cdot)$ is single-valued and

$$\partial\varphi_\Delta^\varepsilon(u) = \left\{ \sum_{i,j} \frac{c_{ij}}{|\Omega_i|} \frac{a_i - a_j}{\sqrt{|a_i - a_j|^2 + \varepsilon^2}} \chi_{\Omega_i} \right\} \tag{3.2}$$

for all $u = \sum_{i\in I} a_i \chi_{\Omega_i} \in H_\Delta$, where $|\Omega_i|$ is the volume of Ω_i.

3.2 Solvability of approximating problem

In this subsection, let us discuss the solvability of the approximating problem to (2.4). In Subsection 3.1, we define the approximating functional $\varphi_\Delta^\varepsilon$ on $L^2(\Omega; \mathbb{R}^n)$ of φ_Δ. Hence, we see that the approximating problem to (2.4) is of the form

$$\begin{cases} u_\varepsilon' = -P_{u_\varepsilon}(\partial\varphi_\Delta^\varepsilon(u_\varepsilon)) & \text{in} \quad L^2(\Omega; \mathbb{R}^n), \quad \text{a.e. } t \in (0, T), \\ u_\varepsilon|_{t=0} = u_0 & \text{in} \quad L^2(\Omega; S^{n-1}). \end{cases} \tag{3.3}$$

By taking account of (2.3) and (3.2), we easily get the existence-uniqueness of solution to the approximating problem (3.3) as follows.

Proposition 3.1. *For any time $T > 0$ and the initial data $u_0 \in H_\Delta$ with $u_0 \in L^2(\Omega; S^{n-1})$, there exists at most one solution u_ε on $[0, T]$ to the approximating problem (3.3).*

Proof. By the similar argument in [GK, Subsection 4.3], we can get the conclusion of this Proposition. By $u_0 \in H_\Delta$, (2.3) and (3.2), we easily observe that the approximating problem (3.3) implies the ODE system for $a_i(t)$, which is the problem to find a unique solution $u_\varepsilon(t) = \sum_{i\in I} a_i(t)\chi_{\Omega_i}$. Thus we have only to consider the ODE system for $a_i(t)$.

By the classical method, we can show that there is a unique solution $u_\varepsilon(t) := \sum_{i\in I} a_i(t)\chi_{\Omega_i}$ on $[0, \infty)$ such that $a_i(t)$ is Lipschitz continuous from $[0, \infty)$ to S^{n-1}, and $a_i(t)$ satisfies

$$\frac{da_i(t)}{dt} = -\pi_{a_i(t)}\left(\sum_j \frac{c_{ij}}{|\Omega_i|} \frac{a_i(t) - a_j(t)}{\sqrt{|a_i(t) - a_j(t)|^2 + \varepsilon^2}} \right) \quad \text{on } \Omega_i \tag{3.4}$$

for each $i \in I$. In fact, by taking account of $a_i(t) \in S^{n-1}$ and the projection $\pi_{a_i(t)} : \mathbb{R}^n \to \mathrm{T}_{a_i(t)}S^{n-1}$, we can show that the right hand side of (3.4) is bounded independent of t. Hence, by the theory of ordinary differential equations (*e.g.* [M]), we can get the unique global solution to the ODE system (3.4). Thus, we can obtain the unique global solution u_ε of our approximating problem (3.3). □

4. Proof of Theorem 2.1

In this section, we shall prove our main Theorem 2.1 by applying the abstract convergence theory established in [GKY].

4.1 Abstract convergence theory

In this subsection, let us recall the abstract convergence theory established in [GKY].

Now, let H be a real Hilbert space and G be a non-empty closed subset of H. Let $L^2(0, T; G)$ denote the closed subset of $L^2(0, T; H)$ of the form

$$L^2(0,T;G) := \{u \in L^2(0,T;H)\ ;\ u(t) \in G \text{ a.e. } t \in [0,T]\}.$$

Let B_R denote a closed ball of $L^2(0,T;H)$ defined by

$$B_R := \{u \in L^2(0,T;H)\ ;\ \|u\|_{L^2(0,T;H)} \le R\} \quad \text{for each } R > 0.$$

Here, let us recall the notion of Graph-convergence for multi-valued operators on a real Hilbert space.

Definition 4.1 (*e.g.* [A]). For (multi-valued) operators A_m $(m = 1, 2, \ldots)$ and A on a real Hilbert space H, we say that A_m converges to A in the sense of Graph as $m \to +\infty$, if for any $(u, v) \in Graph(A)$ there exists $(u_m, v_m) \in Graph(A_m)$ such that $u_m \to u$ and $v_m \to v$ strongly in H as $m \to +\infty$.

Next, let us introduce the class $\mathcal{L}(K)$ of the operator $B(\cdot)(\cdot) : L^2(0,T;G) \times L^2(0,T;H) \to L^2(0,T;H)$.

Definition 4.2 (*cf.* [GKY, Section 3]). *We denote by $B \in \mathcal{L}(K)$ the set of all operator $B(\cdot)(\cdot) : L^2(0,T;G) \times L^2(0, T;H) \to L^2(0,T;H)$ satisfying the following three conditions:*

(i) For any $u \in L^2(0,T;G)$, $B(u)(\cdot)$ is a bounded linear operator from $L^2(0,T;H)$ to $L^2(0,T;H)$.

(ii) There exists a constant $K > 0$ such that $\sup_{u \in L^2(0,T;G)} \|B(u)(\cdot)\|_{\mathcal{L}} \le K$.

(iii) If a sequence $\{u_k\}_{k=1}^{+\infty} \subset L^2(0,T;G)$ strongly converges to some u in $L^2(0,T;G)$, then there exists a subsequence $\{u_{k(l)}\}_{l=1}^{+\infty} \subset \{u_k\}_{k=1}^{+\infty}$ such that

$$B(u_{k(l)})^*(v) \longrightarrow B(u)^*(v) \text{ strongly in } L^2(0,T;H)$$

for any $v \in L^2(0,T;H)$, where $B(u)^*(\cdot)$ denotes the adjoint operator of $B(u)(\cdot)$.

Now, let us recall the abstract convergence theory established in [GKY].

Proposition 4.1 (Abstract theorem) (*cf.* [GKY, Theorem 3.1]). Let Ψ_m $(m = 1, 2, \ldots)$ and Ψ be proper, convex, l.s.c. functionals on $L^2(0,T;H)$. Let $B \in \mathcal{L}(K)$. Assume that $\partial\Psi_m$ converges to $\partial\Psi$ in the sense of Graph. Assume that $u_m \in L^2(0,T;H)$ $(m = 1, 2, \ldots)$ satisfies following conditions;

$$\begin{cases} u_m' \in -B(u_m)(\partial\Psi_m(u_m) \cap B_R) & \textit{in } L^2(0,T;H), \\ u_m \in L^2(0,T;G), \\ u_m|_{t=0} = u_{0,m} \in G. \end{cases}$$

In addition, assume that

$$u_m \to u \textit{ in } C([0,T],H),$$
$$u_{0,m} \to u_0 \textit{ strongly in } H.$$

Then, u satisfies that

$$\begin{cases} u' \in -B(u)(\partial\Psi(u)) & \textit{in } L^2(0,T;H), \\ u \in L^2(0,T;G), \\ u|_{t=0} = u_0 \in G. \end{cases}$$

4.2 Proof of main theorem

In this subsection, let us prove Theorem 2.1 by applying Proposition 4.1. In order to show Theorem 2.1, we use a notion of convergence for convex functions.

Definition 4.3 (*cf.* [Mo]). Let ψ, ψ_n $(n \in N)$ be proper, l.s.c. and convex functions on H. Then we say that ψ_n converges to ψ on H as $n \to +\infty$ in the sense of Mosco [Mo], if the following two conditions are satisfied:

(i) For any subsequence $\{\psi_{n_k}\} \subset \{\psi_n\}$, if $z_k \to z$ weakly in H as $k \to +\infty$, then

$$\liminf_{k \to +\infty} \psi_{n_k}(z_k) \ge \psi(z).$$

(ii) For any $z \in D(\psi)$, there is a sequence $\{z_n\}$ in H such that

$$z_n \to z \text{ in } H \text{ as } n \to +\infty, \qquad \lim_{n \to +\infty} \psi_n(z_n) = \psi(z).$$

To apply Proposition 4.1, we prepare the following key lemma.

Lemma 4.1. *Let φ_Δ (resp. $\varphi_\Delta^\varepsilon$) be the proper, l.s.c. and convex function defined in (2.1) (resp. (3.1)). Then, we have:*
(i) The function $\varphi_\Delta^\varepsilon$ converges to φ_Δ on $L^2(\Omega;\mathbb{R}^n)$ in the sense of Mosco as $\varepsilon \to 0$.
(ii) The function $\Phi_\Delta^{T,\varepsilon}$ converges to Φ_Δ^T on $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ in the sense of Mosco as $\varepsilon \to 0$, where $\Phi_\Delta^{T,\varepsilon}$ is the proper, l.s.c. and convex function on $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ defined by

$$\Phi_{\Delta}^{T,\varepsilon}(u) = \int_0^T \varphi_{\Delta}^{\varepsilon}(u(t))\,dt \quad \textit{for all } u \in L^2(0,T;L^2(\Omega;\mathbb{R}^n)).$$

(iii) $\partial\Phi_{\Delta}^{T,\varepsilon}$ *converges to* $\partial\Phi_{\Delta}^{T}$ *on* $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ *in the sense of Graph as* $\varepsilon \to 0$.

Proof. Taking account of the lower semicontinuity of the total variation, we can easily show (i). Moreover, by the general theory of convex analysis, the assertions (ii) and (iii) can be verified, so we omit the proof. For the detailed proof, see [A], or [GKY, Appendix]. □

Proof of Theorem 2.1. By applying the abstract convergence theory [Proposition 4.1], we shall prove Theorem 2.1, namely, we can show that the function u_ε of (3.3) converges to the solution of our problem (2.4) as $\varepsilon \to 0$.

To do so, we choose $L^2(\Omega;\mathbb{R}^n)$ as a real Hilbert space H, and take $L^2(\Omega;S^{n-1})$ as a non-empty closed subset.

By the definition of the projection (2.3) and Lemma 4.1, we observe that $P \in \mathcal{L}(K)$, and that $\partial\Phi_{\Delta}^{T,\varepsilon}$ converges to $\partial\Phi_{\Delta}^{T}$ on $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ in the sense of Graph as $\varepsilon \to 0$.

Now, let us show the boundedness of subdifferential $\partial\Phi_{\Delta}^{T,\varepsilon}$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$. Note that the subdifferential $\partial\varphi_{\Delta}^{\varepsilon}(u)$ is bounded in $L^2(\Omega;\mathbb{R}^n)$ uniformly in ε, because $\partial\varphi_{\Delta}^{\varepsilon}(u)$ has the form (3.2). Hence we see that $\partial\Phi_{\Delta}^{T,\varepsilon}(u)$ is also bounded in $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ uniformly in ε for each $T > 0$, namely, there is a closed ball B_R of $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ such that

$$\partial\Phi_{\Delta}^{T,\varepsilon}(u_\varepsilon) \subset B_R \quad \text{for any } \varepsilon > 0,$$

for each $T > 0$.

Moreover, by the same argument in [GKY, Proposition 5.4] we have the following energy equation:

$$\int_0^t \int_\Omega |u_\varepsilon'(\tau,x)|^2 dx d\tau + \varphi_{\Delta}^{\varepsilon}(u_\varepsilon(t)) = \varphi_{\Delta}^{\varepsilon}(u_0) \text{ for any } t \in [0,T]. \tag{4.1}$$

By the above equality (4.1), Lemma 4.1 (i), the compactness theory (*cf.* [G, Theorem 1.19]), it is easy to see that $\{u_\varepsilon(t)\}$ is relatively compact in $L^2(\Omega;\mathbb{R}^n)$ for any $t \in [0,T]$. Thus, it follows from Ascoli-Arzela's theorem that there exists a subsequence $\{u_{\varepsilon_m}\}_{m=1}^{+\infty} \subset \{u_\varepsilon\}$ where ε_m goes to zero as m tends to infinity, and $u \in C([0,T];L^2(\Omega;\mathbb{R}^n))$ such that

$$u_{\varepsilon_m} \longrightarrow u \text{ strongly in } C([0,T];L^2(\Omega;\mathbb{R}^n)) \quad \text{as } m \to \infty.$$

We observe that all assumptions of the abstract convergence theory [Proposition 4.1] are fulfilled. Thus, by applying Proposition 4.1, we can prove Theorem 2.1, namely, we get the solution u on $[0,T]$ of our problem (1.1) for each $T > 0$. □

Remark 4.1. It seems that we can treat more general initial data by tending the size of decomposition Δ in H_Δ to zero. But it is impossible, and is the delicate problem. In the proof of our main Theorem 2.1, we need the uniform boundedness in ε of the subdifferential $\partial\Phi_{\Delta}^{T,\varepsilon}(u_\varepsilon) \subset B_R$, where the radius R is independent of ε. However, the radius R depends on the decomposition Δ. So, if the size of decomposition Δ tends to zero, then R goes to infinity. Hence we can not apply the same convergence argument to the general problem with initial data $u_0 \in L^2(\Omega;S^{n-1})$, for instance.

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