The constrained TV-flow

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Based on joint works with

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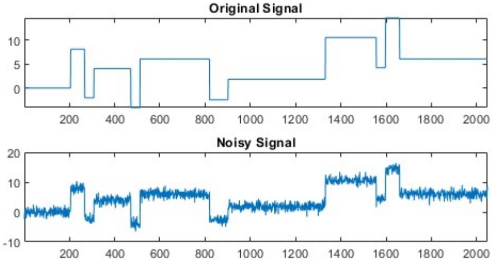
Denoising problem (simplified).

Let $u_0 : \Omega \subset \mathbb{R}^N \to \mathbb{R}^+$ be a given noisy (observed) image. Suppose that the observed image differs from the real image $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^+$ by a gaussian noise n with known variance; i.e.

$$u_0 = u + n\,, \quad \int_\Omega n dx = 0 \quad \int_\Omega n^2 dx = \sigma^2 \quad \Rightarrow u?$$



The use of total variation in image processing.

Let $\mathbf{u} : \Omega \subset \mathbb{R}^m \to \mathbb{R}^N$,

$$E[\mathbf{u}] = \int_\Omega d|D\mathbf{u}| := \sup \left\{ \int_\Omega u^\ell \operatorname{div} \varphi^\ell \mathrm{d}x : \ \varphi \in [C_0^1(\Omega; \mathbb{R}^N)]^m, \ \|\varphi\|_\infty \leq 1 \right\}.$$

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Some features:

- It allows discontinuities (minimizers can be discontinuous).
- It is a convex functional.
- Applied to indicator functions, it gives the perimeter.

L. Rudin, S. Osher, E. Fatemi (1990)

$$\min \left\{ \int_\Omega d|Du| : u \in BV(\Omega)\,, \quad \int_\Omega u = \int_\Omega u_0\,, \quad \int_\Omega (u-u_0)^2 = \sigma^2 \right\}$$

Need for TV into manifolds

Color images: B.Tang, G. Sappiro, V. Caselles '00.

- Given $I : \Omega \subset \mathbb{R}^N \to \mathbb{R}^3$ a noisy image, separate it into its brightness $M : \Omega \to \mathbb{R}^+$

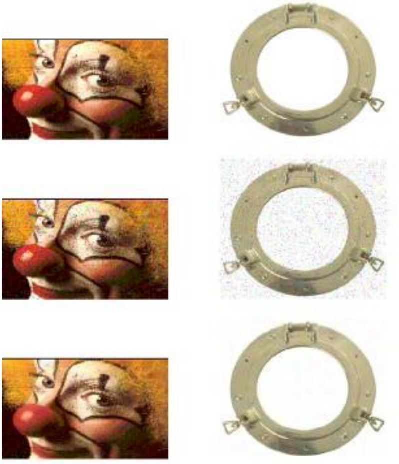
  $$M(x) = \sqrt{\sum_{i=1}^{3} I_i(x)^2}$$

  and chromaticity $\mathbf{u} : \Omega \to (S^2)^+$,

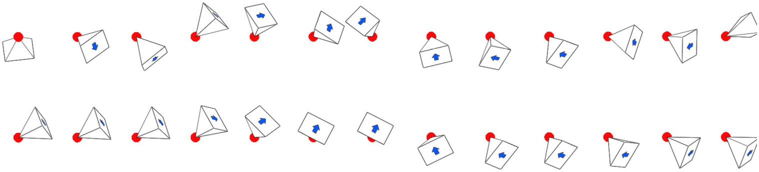
  $$\mathbf{u}(x) := \frac{I(x)}{M(x)}.$$

- Denoise the brightness by scalar TV-flow.
- Denoise the chromaticity by the constrained TV-flow ($(S^2)^+$-valued).
- Recover the denoised image.

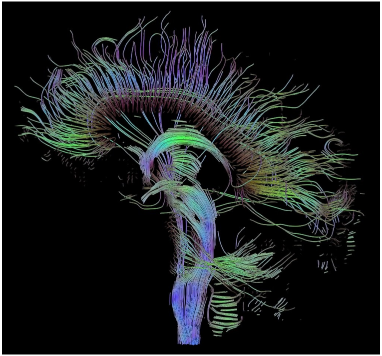
Example of denoising.

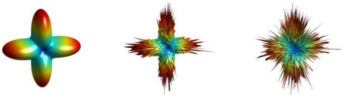


- **Optical flow**: $\mathbf{u} : \Omega \subset \mathbb{R}^2 \to S^2$.
- **LCh–color space**: $\mathbf{u} : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \times S^1$.
- **Camera trajectories**: $\mathbf{u} : [0, T] \to SE(3)$



- **Brain images**: $\mathbf{u} : \Omega \to \mathbb{P}(3)^+$





p-harmonic functions and p-harmonic flows

Let $\Omega \subset \mathbb{R}^m$ and $\mathbf{u} : \Omega \to \mathcal{N}$, where $(\mathcal{N}, g)$ is a smooth manifold. We embed isometrically $\mathcal{N}$ in $\mathbb{R}^N$ and denote again $\mathbf{u} := \iota \circ \mathbf{u}$ and consider

$$|\nabla \mathbf{u}|^2 := {u^i_{x_j}}^2$$

Definition

The $p-$energy of $\mathbf{u}$ is defined as $E_p(\mathbf{u}) := \frac{1}{p} \int_\Omega |\nabla \mathbf{u}|^p \, dx$

Definition

The p*-harmonic functions* are the regular critical points of E_p:

$$0 = \frac{\delta E_p}{\delta \mathbf{u}} = -\pi_{\mathbf{u}} \left(-\mathrm{div}(|\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u}) \right) , \quad \mathbf{u}(\Omega) \subseteq \mathcal{N}$$

The p-harmonic flow is the formal gradient descent flow in L^2 of E_p:

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\delta E_p}{\delta \mathbf{u}} , \quad \mathbf{u}(\Omega) \subseteq \mathcal{N} \quad (t, x) \in (0, T) \times \Omega$$

Some known results.

- Non-positive sectional curvature:
  - $p = 2$ [Eels-Sampson, '64]. Existence of global solutions to the flow and convergence of a subsequence to a harmonic map. The heat flow solved the homotopy problem.
  - $p > 1$ [Fardoun-Regbaoui '02]. Existence and uniqueness, regularity and convergence to a $p-$harmonic map.
- $\mathcal{N} = S^{N-1}$
  - $p = 2$, [Chen-Struwe '89] Existence of a weak solution possibly singular in a finite number of points of $D^2 \times [0, +\infty($. Blow up in finite time [Chan-Ding-Ye '92]
  - $p > 2$: [Chen-Hong-Hungerbuehler '94].
  - $1 < p < 2$: [Liu '97], [Misawa '02].
  - Blow-up in finite time, $p \notin \{1, 2\}$ [Iagar-M, '14]
  - Uniqueness? NOT in general [Bertsch-Dal Passo, van der Hout '02].

The case $p = 1$

- Regular solutions
  - Smooth and small data (periodic) [Giga & coll. '03–]
  - Lipschitz data M. Łasica, L. Giacomelli, S.M. '17.
- BV-solutions in the case of $\mathcal{N} = S^{N-1}$.
  - Piecewise constant evolution [Giga & coll. '05–]
  - Rotational symmetry, blow-up (Dirichlet) [DalPasso-Giacomelli-M., '08-'10]
  - S^1. Existence and uniqueness [Giacomelli, Mazón, S. M. '13]
  - Existence in the case $(S^{N-1})^+$. [Giacomelli, Mazón, M. '14]
- BV-solutions in the case $\mathcal{N}$ is a smooth curve in the plane. [DiCastro-Giacomelli '16]
- BV-solutions for 1-D domain. Giacomelli, Łasica, M. '18

The regular case

$\mathbf{u}_t = \pi_{\mathbf{u}} \mathrm{div}\left(\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}\right)$ (TVF)

The meaning of $\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}$ as a multivalued function:

$$\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|} : (t, \mathbf{x}) \mapsto \begin{cases} \frac{\nabla \mathbf{u}(t,\mathbf{x})}{|\nabla \mathbf{u}(t,\mathbf{x})|} & \text{if } \nabla \mathbf{u}(t,\mathbf{x}) \neq \mathbf{0} \\ B(0,1) \subset \mathbb{R}^m \times T_{\mathbf{u}(t,\mathbf{x})}\mathcal{N} & \text{if } \nabla \mathbf{u}(t,\mathbf{x}) = \mathbf{0} \end{cases}$$

Definition

Let $T \in]0, \infty]$. We say that $\mathbf{u} \in W^{1,2}(]0, T[\times\Omega, \mathcal{N})$ with $\nabla \mathbf{u} \in L^\infty_{loc}([0, T[\times\overline{\Omega}, \mathbb{R}^{m\cdot N})$ is a (regular) solution to (TVF) if there exists $\mathbf{Z} \in L^\infty(]0, T[\times\Omega, \mathbb{R}^m \times \mathbb{R}^N)$ with $\mathrm{div}\,\mathbf{Z} \in L^2_{loc}([0, T[\times\overline{\Omega}, \mathbb{R}^N)$ satisfying $\mathcal{L}^{1+m}$ – a. e. in $]0, T[\times\Omega$.

$$\mathbf{Z} \in \frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|},$$

$$\mathbf{u}_t = \pi_{\mathbf{u}} \mathrm{div}\,\mathbf{Z}$$

Homogeneous Neumann boundary condition:

$$\nu^\Omega \cdot \mathbf{Z} = \mathbf{0}\,, \quad \mathcal{L}^1 \otimes \mathcal{H}^{m-1} - \text{a. e. in }]0, T[\times\partial\Omega.$$

Existence and regularity

$$\mathcal{K}_{\mathcal{N}} = \sup_{\mathbf{p}\in\mathcal{N}} \max_{\mathbf{v},\mathbf{w}\in T_{\mathbf{p}}\mathcal{N}\setminus\{0\}} \frac{\mathbf{v}\cdot\mathcal{R}^{\mathcal{N}}_{\mathbf{p}}(\mathbf{v},\mathbf{w})\mathbf{w}}{|\mathbf{v}|^2|\mathbf{w}|^2-(\mathbf{v}\cdot\mathbf{w})^2}.$$

Theorem

Suppose that Ω is convex, the embedding is closed and $\sup\mathcal{K}_{\mathcal{N}} < \infty$. Given $\mathbf{u}_0 \in W^{1,\infty}(\Omega,\mathcal{N})$, there exists $T = T(\mathcal{N}, \|\nabla\mathbf{u}_0\|_{L^\infty})$ and a regular solution $\mathbf{u} \in L^\infty(]0,T[, W^{1,\infty}(\Omega,\mathcal{N}))$ satisfying

$$\mathrm{ess}sup_{t\in[0,T[} \int_\Omega |\nabla\mathbf{u}(t,\cdot)| + \int_0^T \int_\Omega \mathbf{u}_t^2 \le \int_\Omega |\nabla\mathbf{u}_0|,$$

and

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0$$

The approximate system

$$\begin{cases} \mathbf{u}_t^\varepsilon = \pi_{\mathbf{u}^\varepsilon} \operatorname{div}\left(\frac{\nabla \mathbf{u}^\varepsilon}{\sqrt{\varepsilon^2 + |\nabla \mathbf{u}^\varepsilon|^2}}\right) & \text{in }]0,T[\times\Omega \\ \nu^\Omega \cdot \nabla \mathbf{u}^\varepsilon = 0 & \text{in }]0,T[\times\partial\Omega \\ \mathbf{u}^\varepsilon(0,\cdot) = \mathbf{u}_0 & \end{cases}$$

Theorem

Suppose that Ω is smooth and convex, that $\sup \mathcal{K}_\mathcal{N} < \infty$. Let $\mathbf{u}_0 \in C^{3+\alpha}(\Omega, \mathcal{N})$ satisfy Neumann b. c. . + a compatibility condition. Then, There exist $T_\dagger = T_\dagger(\|\nabla \mathbf{u}_0^\varepsilon\|_{L^\infty}, \mathcal{K}_\mathcal{N}) > 0$, and unique solution

$$\mathbf{u}^\varepsilon \in C_{loc}^{\frac{3+\alpha}{2}, 3+\alpha}(\overline{\Omega}_{[0,T_\dagger[}, \mathcal{N})$$

to the system with initial datum $\mathbf{u}_0$.

Sketch of proof.

- Step 1. Uniform bounds: Let $\mathbf{u}^\varepsilon \in C^{\frac{3+\alpha}{2},3+\alpha}(\overline{\Omega}_{[0,T[},\mathcal{N})$ be a solution. Then,

  (i) Energy estimate:

$$\sup_{t\in[0,T[}\int_\Omega \sqrt{\varepsilon^2+|\nabla \mathbf{u}^\varepsilon|^2}+\int_0^T\int_\Omega \mathbf{u}_t^2 \leq \int_\Omega \sqrt{\varepsilon^2+|\nabla \mathbf{u}_0^\varepsilon|^2}.$$

  (ii) Parabolic Bochner formula:

$$\frac{1}{2}\frac{d}{dt}|\nabla \mathbf{u}|^2 = \operatorname{div}(\nabla \mathbf{u}\cdot\nabla Z) - (\pi_{\mathbf{u}}\nabla^2\mathbf{u}) : \nabla Z + Z_i \cdot \mathcal{R}_{\mathbf{u}}^{\mathcal{N}}(\mathbf{u}_{x^i},\mathbf{u}_{x^j})\mathbf{u}_{x^j}\,.$$

  (iii) Lipschitz bound:

$$\|\sqrt{\varepsilon^2+|\nabla \mathbf{u}^\varepsilon(t\cdot)|^2}\|_{L^\infty} \leq \frac{\sqrt{\varepsilon^2+|\nabla \mathbf{u}_0|^2}}{1-t\mathcal{K}_N\sqrt{\varepsilon^2+|\nabla \mathbf{u}_0|^2}}$$

- Step 2. Unconstraining the problem:
  (i) Construct a totally geodesic embedding ι of $(\mathcal{N}, g)$ into a Riemannian manifold $(\mathbb{R}^N, h)$. Then,

$$u_t^i = \operatorname{div} \frac{\nabla u^i}{\sqrt{\varepsilon^2 + |\nabla \mathbf{u}|_h^2}} + \frac{1}{\sqrt{\varepsilon^2 + |\nabla \mathbf{u}|_h^2}} \Gamma_{jk}^i(\mathbf{u}) u_{x_l}^j u_{x_l}^k, \quad i = 1, \ldots, N,$$

  (ii) Local existence and uniqueness of solution $\mathbf{u}^\varepsilon \in C^{1+\frac{\alpha}{2}}([0, T_0], L^p(\Omega, \mathbb{R}^N)) \cap C^{\frac{\alpha}{2}}([0, T_0], W^{2,p}(\Omega, \mathbb{R}^N))$ with a $T_0 > 0$ for $\mathbf{u}_0 \in C^{2+\alpha}(\overline{\Omega}, \mathcal{N})$ satisfying Neumann b. c. (Acquistapace and Terreni)
  (iii) Hölder regularity+maximal time of existence by classical results.
- Step 3. The solution stays into $\mathcal{N}$ by uniqueness.

Sketch of passing with $\varepsilon \to 0$

- Case $u_0 \in C^{3+\alpha}(\Omega; \mathcal{N})$+ compatibility conditions. Since the bounds do not depend on ε, there exists a subsequence $\mathbf{u}_k$ from $(\mathbf{u}^\varepsilon)$ such that

$$\begin{cases} \mathbf{u}_k \to \overline{\mathbf{u}} & \text{in } C(]0,T[\times\overline{\Omega}) \\ \nabla\mathbf{u}_k \rightharpoonup \nabla\overline{\mathbf{u}} & \text{weakly in } L^2(]0,T[\times\Omega) \end{cases} .$$

  Then, one can pass to the limit in the weak formulation in extrinsic coordinates:

$$\mathbf{u}_t^\varepsilon = \operatorname{div} \frac{\nabla\mathbf{u}^\varepsilon}{|\nabla\mathbf{u}^\varepsilon|} + A(\mathbf{u}^\varepsilon)(\mathbf{Z}^\varepsilon, \nabla\mathbf{u}^\varepsilon),$$

- General case: By approximation with $C^\infty(\overline{\Omega}; \mathcal{N})$ functions.
- A general convex domain.

Theorem

Suppose that $\mathbf{u}_1, \mathbf{u}_2 \in L^\infty(]0,T[, W^{1,\infty}(\Omega,\mathcal{N}))$ *are two regular solutions to (TVF) such that* $\mathbf{u}_1(0,\cdot) = \mathbf{u}_2(0,\cdot) = \mathbf{u}_0$. *Then* $\mathbf{u}_1 \equiv \mathbf{u}_2$.

Theorem

Given $\mathbf{p}_0 \in \mathcal{N}$ *and* $\mathbf{u}_0 \in W^{1,\infty}(\Omega)$ *such that* $\mathbf{u}_0(\Omega) \subset \overline{B_g(\mathbf{p}_0, R)}$, $R > 0$. *Then, there exist:*

- *A constant* $R_*(\mathbf{p}_0)$ *such that if* $R < R_*$, *then* $\mathbf{u}(t,\Omega) \subset \overline{B_g(\mathbf{p}_0, R)}$, *for all* $t > 0$.
- *constants* $0 < \tilde{R}_*(\mathbf{p}_0) < R_*$, $C(\mathbf{p}_0)$ *and* $u_* \in \mathcal{N}$ *such that, if* $R < \min\{\tilde{R}, \frac{T}{C}\}$, *then* $u(t,\cdot) \equiv u_*$ *for* $t \in]CR, T[$.

Finite extinction time. Sketch of proof.

Consider the barycenter

$$m(t) := \text{argmin}_{w\in\mathcal{N}} \int_\Omega d(u(x), w)^2\, dx.$$

Then, consider geodesic polar coordinates centered at $m(t)$ and estimate

$$\frac{1}{2}\frac{d}{dt}\int_\Omega d(\mathbf{u}(x,t), m(t))^2\, dx = \frac{1}{2}\frac{d}{dt}\int_\Omega (u^r)^2\, dx$$

using the equation in this coordinates, , integration by parts and Poincaré-Sobolev inequality, obtaining

$$\frac{1}{2}\frac{d}{dt}\int_\Omega d(\mathbf{u}(x,t), m(t))^2\, dx \le -CR^{\frac{2}{m}-1}\left(\frac{}{2}\int_\Omega d(\mathbf{u}(x,t), m(t))^2\, dx\right)^{1-\frac{1}{m}}.$$

□

Theorem

Suppose that Ω *is convex and* $\mathcal{K}_{\mathcal{N}} \leq 0$ *(or suppose that* $m = 1$*). Let* $\mathbf{u}_0 \in W^{1,2}(\Omega, \mathcal{N}) \cap L^\infty(\Omega, \mathcal{N})$*. There exists a global regular solution. Furthermore, if* $\mathbf{u}_0 \in W^{1,p}(\Omega, \mathcal{N})$*,* $2 \leq p \leq \infty$*, then* $\mathbf{u} \in L^\infty(]0, \infty[, W^{1,p}(\Omega, \mathcal{N})$ *with*

$$\operatorname*{ess\,sup}_{t>0} \|\nabla \mathbf{u}(t, \cdot)\|_{L^p(\Omega)} \leq \|\nabla \mathbf{u}_0\|_{L^p(\Omega)}.$$

The homotopy problem

Given $(\mathcal{M}, \gamma)$ and $(\mathcal{N}, g)$ two Riemannian manifolds and $\mathbf{u}_0 : \mathcal{M} \to \mathcal{N}$, is there a weak 1-harmonic map $\mathbf{u}_*$ homotopic to $\mathbf{u}_0$?

$$TV_{\mathcal{M}}^{\mathcal{N}}[\mathbf{u}] := \int_{\mathcal{M}} |d\mathbf{u}|_\gamma d\mu_\gamma$$

$$(TVF) : \mathbf{u}_t = \pi_{\mathbf{u}} \left(\mathrm{div}_\gamma \frac{d\mathbf{u}}{|d\mathbf{u}|} \right)$$

$$\frac{\mathrm{d}\mathbf{u}}{|\mathrm{d}\mathbf{u}|_\gamma} : (t, \mathbf{x}) \mapsto \begin{cases} \frac{\mathrm{d}\mathbf{u}(t,\mathbf{x})}{|\mathrm{d}\mathbf{u}(t,\mathbf{x})|_\gamma} & \text{if } \mathrm{d}\mathbf{u}(t,\mathbf{x}) \neq 0 \\ B_\gamma(0,1) \subset T_{\mathbf{x}}^*\mathcal{M} \times T_{\mathbf{u}(t,\mathbf{x})}\mathcal{N} & \text{if } \mathrm{d}\mathbf{u}(t,\mathbf{x}) = 0. \end{cases}$$

Definition

Let $T \in]0, \infty]$. We say that

$$\mathbf{u} \in W^{1,2}_{loc}([0,T[\times \mathcal{M}, \mathcal{N}) \text{ with } \mathrm{d}\mathbf{u} \in L^{\infty}_{loc}([0,T[\times T^*\mathcal{M} \times \mathbb{R}^N)$$

is a solution if there exists $\mathbf{Z} \in L^{\infty}(]0,T[\times T^*\mathcal{M} \times \mathbb{R}^N)$ with $\mathrm{div}_{\gamma}\mathbf{Z} \in L^2_{loc}([0,T[\times \mathcal{M}, \mathbb{R}^N)$ satisfying

$$\mathbf{Z} \in \frac{\mathrm{d}\mathbf{u}}{|\mathrm{d}\mathbf{u}|_{\gamma}},$$

$$\mathbf{u}_t = \pi_{\mathbf{u}}(\mathrm{div}_{\gamma}\mathbf{Z})$$

$\mathcal{L}^{1+m} -$ a. e. in $]0,T[\times \mathcal{M}$.

$$Ric_{\mathcal{M}} = \min_{\mathbf{p}\in\mathcal{M}} \min_{\mathbf{v},\mathbf{w}\in T_p\mathcal{M}\setminus\{\vec{0}\}} \frac{\mathcal{R}ic^{\mathcal{M}}_{\mathbf{p}}(\mathbf{v},\mathbf{w})}{|\mathbf{v}|_\gamma |\mathbf{w}|_\gamma}.$$

Theorem

Let $(\mathcal{M},\gamma)$ be a compact, orientable and let $(\mathcal{N},g)$ be a compact submanifold in $\mathbb{R}^N$. Given $\mathbf{u}_0 \in W^{1,\infty}(\mathcal{M},\mathcal{N})$, there exists $T \in]0,\infty]$ and a unique regular solution in $[0,T[$.

If $K_{\mathcal{N}} \leq 0$, the solution exists in $[0,\infty[$.

If $Ric_{\mathcal{M}} \geq 0$, there exists $(t_k) \subset]0,\infty[$, $t_k \to \infty$, $\mathbf{u}_* \in W^{1,\infty}(\mathcal{M},\mathcal{N})$ and $\mathbf{Z}_* \in L^\infty(T^*\mathcal{M} \times \mathbb{R}^N)$ with $\operatorname{div}_\gamma \mathbf{Z}_* \in L^\infty(\mathcal{M},\mathbb{R}^N)$ such that

$$\pi_{\mathbf{u}_*}(\operatorname{div}_\gamma \mathbf{Z}_*) = \vec{0}, \qquad \mathbf{Z}_* \in \frac{\mathrm{d}\mathbf{u}_*}{|\mathrm{d}\mathbf{u}_*|_\gamma},$$

$$\mathbf{u}(t_k,\cdot) \to \mathbf{u}_* \text{ in } C(\mathcal{M},\mathcal{N}).$$

BV-solutions

BV-solution

Proper interpretation of

$$\mathbf{u}_t = \mathrm{div}\left(\frac{D\mathbf{u}}{|D\mathbf{u}|}\right) + A(\mathbf{u})\left(\frac{D\mathbf{u}}{|D\mathbf{u}|}, D\mathbf{u}\right)$$

for $\mathbf{u} \in BV(\Omega;\mathcal{N})$,?

$$\mathbf{u}_t = \mathrm{div}\,(\mathbf{Z}) + \boldsymbol{\mu}$$

- Special cases:

- $\mathcal{N} = \mathbb{S}^{n-1}_{+}$: $\quad \boldsymbol{\mu} = \mathbf{u}|\tilde{D}\mathbf{u}| + \frac{\mathbf{u}^*}{|\mathbf{u}^*|}|D^j\mathbf{u}|, \quad (\mathbf{Z}, D\mathbf{u}) = |\mathbf{u}^*||D\mathbf{u}|$

- $\mathcal{N} \subset \mathbb{R}^2$: $\quad \boldsymbol{\mu} = -\kappa(\mathbf{u})N(\mathbf{u})|\tilde{D}\mathbf{u}| + (T(\mathbf{u}_-) - T(\mathbf{u}_+))\mathcal{H}^{m-1} \llcorner J_{\mathbf{u}}$

The equation for $\mathcal{N} = S^{N-1}$

$$\mathbf{u}_t = \operatorname{div}\left(\frac{D\mathbf{u}}{|D\mathbf{u}|}\right) + \mathbf{u}|D\mathbf{u}|$$

Approximation. Parabolic regularization.

Lemma (Barrett-Feng-Prohl '08)

Let $\varepsilon > 0$*,* $T > 0$ *and* $\alpha > 0$*. If* $\mathbf{u}_0^\varepsilon \in W^{1,2}(\Omega; \mathbb{S}^{N-1})$*, then there exists* $\mathbf{u}^\varepsilon \in L^\infty(0,T; W^{1,2}(\Omega;\mathbb{R}^N)) \cap W^{1,2}(0,T; L^2(\Omega;\mathbb{R}^N))$ *such that* $\mathbf{u}^\varepsilon(0,\cdot) = \mathbf{u}_0^\varepsilon$*,* $|\mathbf{u}^\varepsilon| = 1$ *a.e. in* Q_T*, and* $\mathbf{u}^\varepsilon$ *is a weak solution to*

$$\begin{cases} \mathbf{u}_t^\varepsilon = \pi_{\mathbf{u}^\varepsilon} \operatorname{div}(\mathbf{Z}^\varepsilon) & \text{in } Q_T \\ [\mathbf{Z}^\varepsilon, \nu] = 0 & \text{in } S_T, \end{cases},$$

where $\mathbf{Z}^\varepsilon := \varepsilon^\alpha \nabla \mathbf{u}^\varepsilon + \frac{\nabla \mathbf{u}^\varepsilon}{\sqrt{|\nabla \mathbf{u}^\varepsilon|^2 + \varepsilon^2}}$*.*

We let

$$\boldsymbol{\mu}^\varepsilon := \varepsilon^\alpha \mathbf{u}^\varepsilon |\nabla \mathbf{u}^\varepsilon|^2 + \mathbf{u}^\varepsilon \frac{|\nabla \mathbf{u}^\varepsilon|^2}{\sqrt{|\nabla \mathbf{u}^\varepsilon|^2 + \varepsilon^2}}$$

With good a-priori estimates and approximation of the initial datum, one can pass to the limit,

$$\mathbf{u}_t - \operatorname{div} \mathbf{Z} = \boldsymbol{\mu} \quad \text{in } [L^2(0,T; C_0(\Omega; \mathbb{R}^N))]',$$

$$\begin{aligned} \mathbf{Z}^T \mathbf{u} &= 0 && \text{a.e. in } Q_T, \\ \mathbf{u}_t \cdot \mathbf{u} &= 0 && \text{a.e. in } Q_T, \\ \mathbf{u}_t(t) \wedge \mathbf{u}(t) &= \operatorname{div}(\mathbf{Z}(t) \wedge \mathbf{u}(t)) && \text{in } L^2(\Omega; \Lambda_2(\mathbb{R}^N)) \text{ for a.e. } t \in [0,T]. \end{aligned}$$

Then,

$$\boldsymbol{\mu}(t) = ((\mathbf{Z}(t) \wedge \mathbf{u}(t)) \wedge D\mathbf{u}(t)) \Longrightarrow |\boldsymbol{\mu}(t)| << |D\mathbf{u}(t)|$$

Therefore,

$$\boldsymbol{\mu}(t) = \frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} \left(|\nabla \mathbf{u}(t)| \mathcal{L}^m + |D^c(\mathbf{u}(t))|\right) + \frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} |\mathbf{u}(t)_+ - \mathbf{u}(t)_-| \mathcal{H}^{m-1} \llcorner J_{\mathbf{u}(t)}$$

If we show that

$$\frac{\boldsymbol{\mu}(t)}{|D\mathbf{u}(t)|} \cdot \frac{\mathbf{u}(t)^*}{|\mathbf{u}(t)^*|} \geq 1 \quad |D\mathbf{u}(t)| - a.e,$$

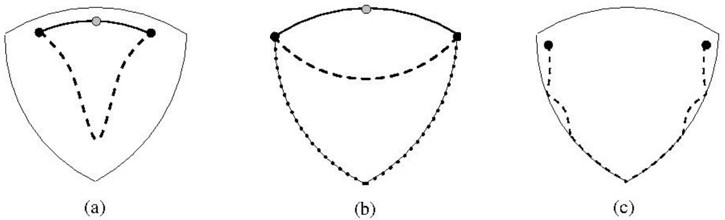
then

$$\boldsymbol{\mu}(t) = \frac{\mathbf{u}(t)^*}{|\mathbf{u}(t)^*|} |D\mathbf{u}(t)|$$

- For the diffuse part one can rely on l.s.c. results about linear growth functionals in S^{N-1} by Alicandro-Corbo-Esposito and Leone '07.
- For the jump part, one can obtain $\mathcal{H}^{m-1}$-a.e $x \in J_{\mathbf{u}(t)}$,

$$\frac{\mathbf{u}^*}{|\mathbf{u}^*|} \cdot \tilde{\boldsymbol{\mu}} \geq \inf \left\{ \int_0^1 \frac{\mathbf{u}^*}{|\mathbf{u}^*|} \cdot \boldsymbol{\gamma}(\tau) |\dot{\boldsymbol{\gamma}}(\tau)| d\tau : \boldsymbol{\gamma} \in W^{1,1}(I; \mathbb{S}_+^{N-1}), \boldsymbol{\gamma}(0) = \mathbf{u}_-, \boldsymbol{\gamma}(1) = \mathbf{u}_+ \right\}$$

Therefore, showing that the above infimum is equal to $|\mathbf{u}_+ - \mathbf{u}_-|$ finishes the proof. In fact, a standard geodesic in the sphere joining $\mathbf{u}_+$ and $\mathbf{u}_-$ yields the optimal bound. Unfortunately, the problem is genuinely non-convex.



The case of a 1-D domain

$\Omega := I =]0,1[$

Definition

Let $\mathbf{u} \in W^{1,2}(0,T;L^2(I,\mathcal{N})) \cap L^\infty(0,T;BV(I,\mathcal{N}))$ be such that $\mathrm{dist}_g(\mathbf{u}_-,\mathbf{u}_+) < inj\mathcal{N}$ on $J_\mathbf{u}$. $\mathbf{u}$ is a solution to the 1-harmonic flow if there exists $\mathbf{Z} \in L^\infty(]0,T[\times I)^n$ such that a.e. $t \in]0,T[$,

$$\mathbf{u}_t = \pi_\mathcal{N}(\mathbf{u})\mathbf{Z}_x^a$$

$$\mathbf{Z} \in T_\mathbf{u}\mathcal{N}, \quad |\mathbf{Z}| \le 1$$

$$Z = \frac{\mathbf{u}_x}{|\mathbf{u}_x|}, \quad |\tilde{\mathbf{u}}_x| - a.e.$$

$$\mathbf{Z}^\pm = T(\mathbf{u}^\pm) \quad \text{on } J_\mathbf{u}$$

$$\mathbf{Z} = 0 \,\text{on}\{0,1\}$$

Theorem

Let $\mathbf{u}_0 \in BV(I, \mathcal{N})$ satisfy $\operatorname{dist}_g((\mathbf{u}_0)_-, (\mathbf{u}_0)_+) < R_*(\mathcal{N})$ on $J_{\mathbf{u}}$. Then, for any $T > 0$, there exists a solution to the 1-harmonic flow starting at $\mathbf{u}_0$.

In case that $\mathcal{K}_{\mathcal{N}} \leq 0$, then the functional is convex. Then, there is a unique abstract solution in the sense of gradient flow given by Ambrosio-Gigli-Savaré's theory. Our solution coincides with this one and it is therefore unique.

Relaxed total variation

Given $\mathbf{u} \in BV(I, \mathcal{N})$,

$$TV_g(\mathbf{u}) = \inf\{\liminf \int_I |\mathbf{u}_x^k| : \mathbf{u}^k \in W^{1,\infty}(I, \mathcal{N}), \mathbf{u}^k \overset{*}{\rightharpoonup} \mathbf{u}\}$$

Then (Giacquinta-Mucci '06),

$$TV_g(\mathbf{u}) = \int_I |\mathbf{u}_x|_g,$$

with

$$|\mathbf{u}_x|_g = |\tilde{\mathbf{u}}_x| \llcorner I \setminus J_{\mathbf{u}} + \mathrm{dist}_g(\mathbf{u}_-, \mathbf{u}_+)\mathcal{H}^0(J_{\mathbf{u}})$$

Sketch of proof.

- Smooth the initial data and obtain a global regular solution u_ε.
- Use the completely local estimate (Giacomelli-Łasica '18):

$$|\mathbf{u}_x(t,\cdot)|_g \leq |(\mathbf{u}_0)_x|_g$$

to obtain uniform bounds and to compute

$$\frac{\mathbf{u}_x}{|(\mathbf{u}_0)_x|}, \quad \frac{|\mathbf{u}_x|}{|(\mathbf{u}_0)_x|}$$

outside $J_{\mathbf{u}_0}$.
- Use chain rule to compute $\frac{\mathbf{u}_x}{|\mathbf{u}_x|}$.

Jump part+uniqueness

- Choose special coordinates on the jump (Fermi coordinates)
- Use lower semicontinuity of the energy
- Show that the energy converges;

$$TV_g(u_\varepsilon) \to TV_g(u)$$

- For uniqueness, show that $\mathbf{u}$ satisifes

$$\frac{1}{2}\frac{d}{dt}d_g^2(\mathbf{u}(t), \mathbf{v}) + TV_g(\mathbf{u}(t)) \leq TV_g(\mathbf{v})$$

for any $\mathbf{v} \in BV(I, \mathcal{N})$.

Future directions

- Uniqueness in case $\mathcal{N} = S_+^{N-1}$
- BV-solutions for smooth manifolds with unique geodesics.
- Non-smooth curves (Wulff shape of a 2-D crystalline norm).
- Non-smooth manifolds.

どうもありがとう