On large-time behavior for birth-spread type nonlinear PDEs

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§1 Introduction:

Birth and Spread type Partial Differential Equations (PDEs)

= Continuum limit of Birth and Spread model.

Let $f \in \mathrm{Lip}\,(\mathbb{R}^2)$: $f \geq 0$ and $\{f > 0\}$ is bounded. Consider

$$\text{(N)} \quad \begin{cases} v_t = f(x) & \text{in } \mathbb{R}^2 \times (0, \infty), \\ v(\cdot, 0) = u_0 & \text{in } \mathbb{R}^2. \end{cases}$$

Define $S_1(t) : L^\infty(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ by

$$S_1(t)[u_0] := v(\cdot, t) = u_0 + f(x)t.$$

Then,

$$(P) \quad V = g(\kappa, x) \quad \text{for every level sets.}$$

$$\Longleftrightarrow \quad \text{(Level set eq)} \quad \begin{cases} w_t = g\left(\operatorname{div}\left(\frac{Dw}{|Dw|}\right), x\right)|Dw| & \text{in } \mathbb{R}^2 \times (0,\infty). \\ w(\cdot, 0) = u_0 & \text{in } \mathbb{R}^2. \end{cases}$$

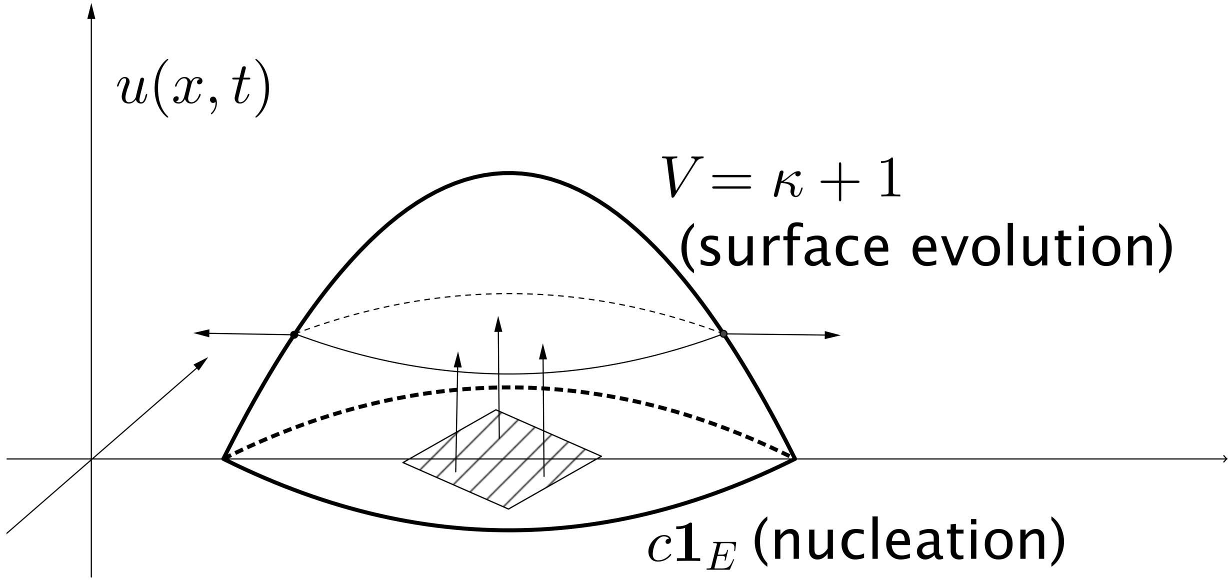
Define $S_2(t) : \mathrm{Lip}\,(\mathbb{R}^2) \to \mathrm{Lip}\,(\mathbb{R}^2)$ by

$$S_2(t)[u_0] := w(\cdot, t).$$

For $x \in \mathbb{R}^n, \tau > 0$ (small), $i \in \mathbb{N}$, set

$$U^\tau(x, i\tau) := S_1(\tau)\big(S_2(\tau)S_1(\tau)\big)^i[u_0].$$

Image picture



Remark.

Two-dimensional nucleation growth theory,

Physics literature: Hillig (1966), Ohara-Reid (1973).

Theorem 1. *Under general assumptions for g, we have*

$$U(x, i\tau) \to u(x,t) \quad \text{as } i \to \infty \text{ with } i\tau = t$$

for a fixed $t > 0$ and locally uniformly for $x \in \mathbb{R}^2$. Moreover, u satisfies

$$\text{(C)} \quad \begin{cases} u_t - g\left(\text{div}\left(\frac{Du}{|Du|}\right), x\right)|Du| = f(x) & \text{in } \mathbb{R}^2 \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^2. \end{cases}$$

Remark.

- Example of g: $g = \kappa + 1$, or $1/\chi(\kappa)$.
- Viscosity solution.
- A continuum limit of Birth and Spread model.
- A general framework by Barles-Souganidis (1991).

Goal: Asymptotic analysis on (C) from PDE point of view.

1. Existence of the asymptotic speed of u, that is,

$$\lim_{t\to\infty} \frac{u(x,t)}{t} (=: c_f) \in \mathbb{R} \text{ as } t \to \infty \text{ in } C(\mathbb{R}^2).$$

2. Qualitative analysis of c_f.

3. Large time behavior: $u(x,t) - c_f t \to v(x)$.

§2 Main theorems.

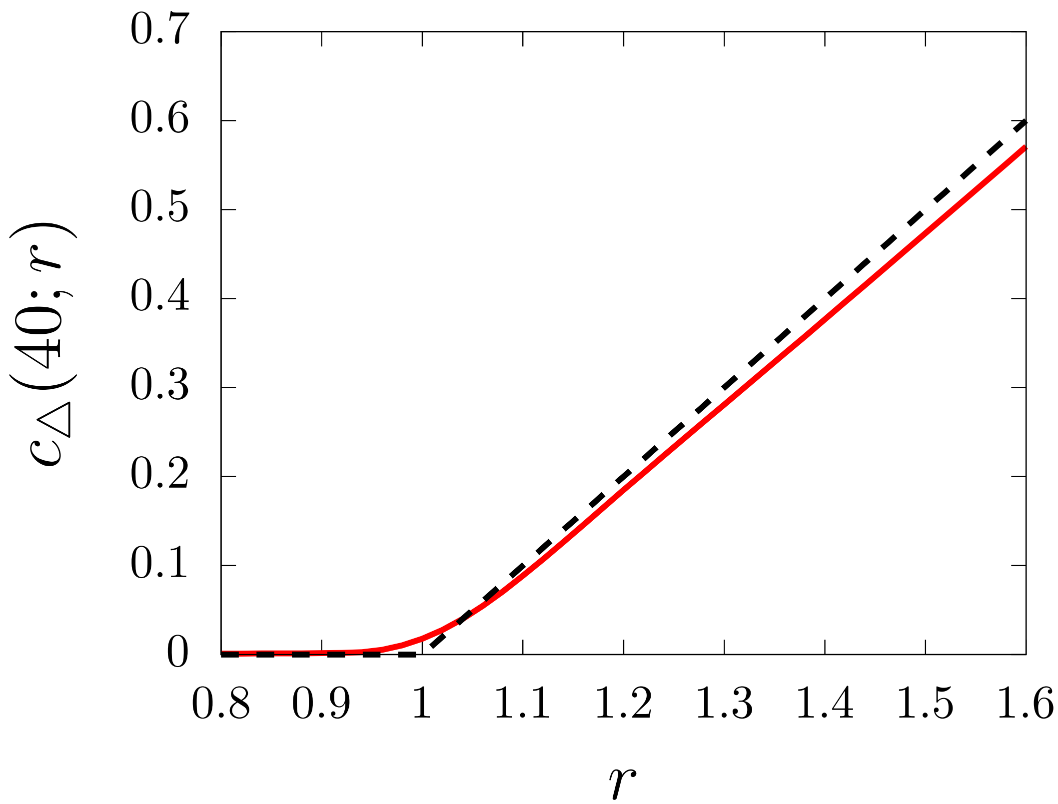
Theorem 2. *Under general assumptions for g, there exists the asymptotic speed.*

Proposition 3. *We have several qualitative results on the asymptotic speed (Analytical/Numerical).*

Theorem 4. *Convergence of solution (itself) under some condition.*

Qualitative results on the asymptotic speed.

Example 1: $V = \kappa + 1$, $f_r(x) = \max\{r - |x|, 0\}$.

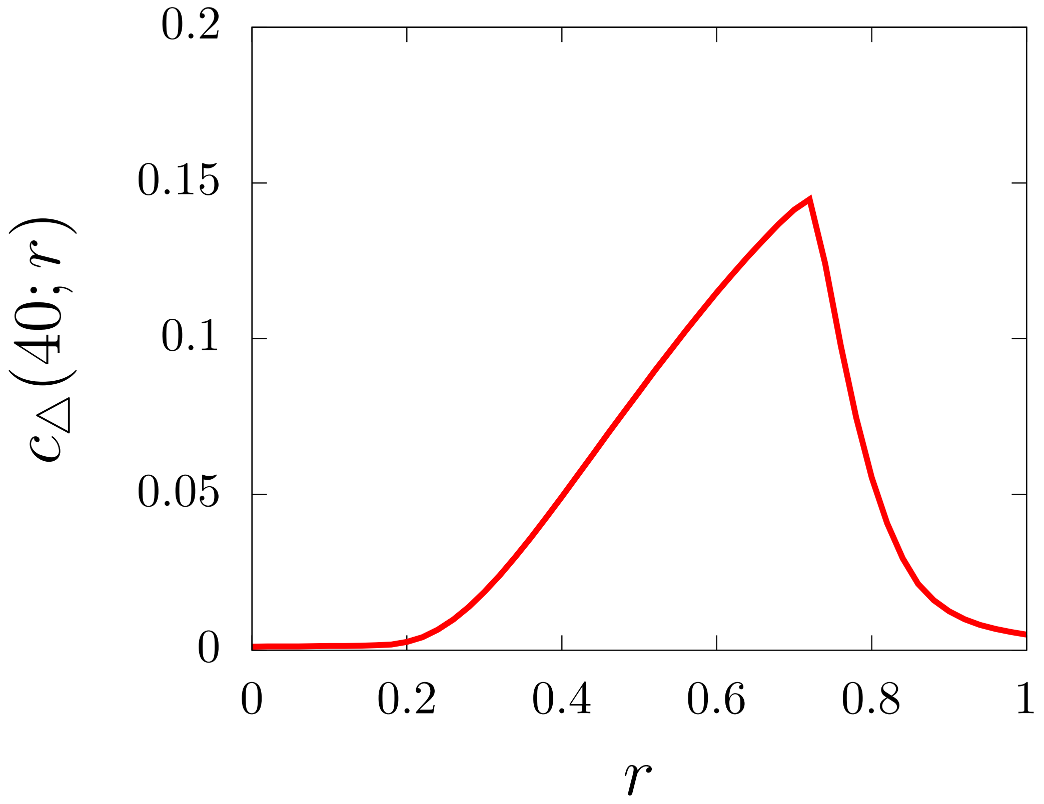


Example 2:

$V = \kappa + 1,$

$f_r(x) = \max\{R_0 - |x - (r,0)|, 0\} + \max\{R_0 - |x + (r,0)|, 0\},$

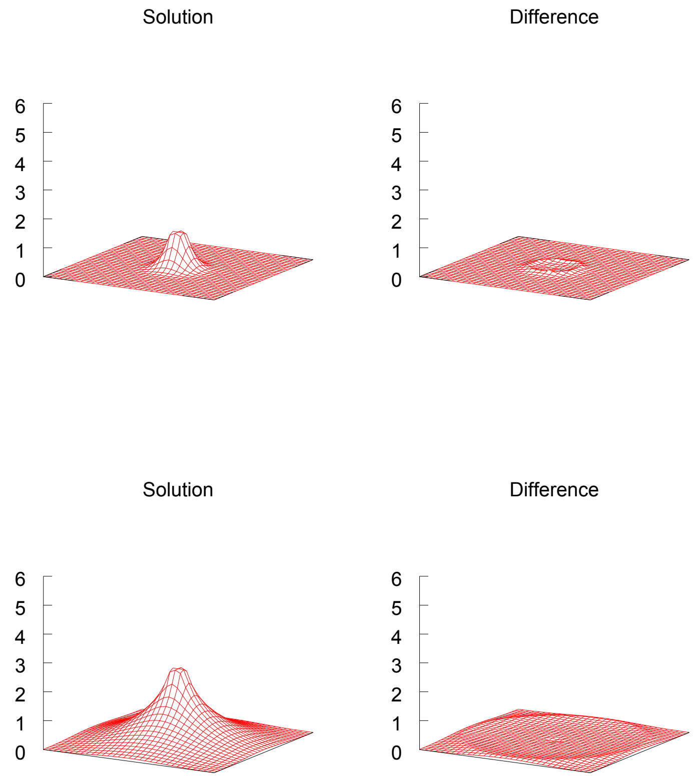
$R_0 = 0.2 \in (0,1).$



Example 3:

$V = 1/\chi(\kappa)$, $\chi(r) :=$ min$\{$max$\{r, \lambda\}, \Lambda\}$ for $0 \le \lambda < \Lambda$,

$f(x) = \mathbf{1}_{B(0,R_0)}$.



Thank you for
your kind attention!