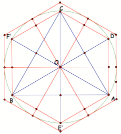
Dynamics of a degenerate PDE model of epitaxial crystal growth

Jian-Guo Liu, Duke University and Duke Kunshan University

祖冲之, 429–500, 昆山
Zu Chongzhi, Kunshan



- Archimedes, 287 BC–212 BC

$3 \times 2^5 = 96$-sided polygon

- Liu, Hui, 225–295

$3 \times 2^6 = 192$-sided polygon

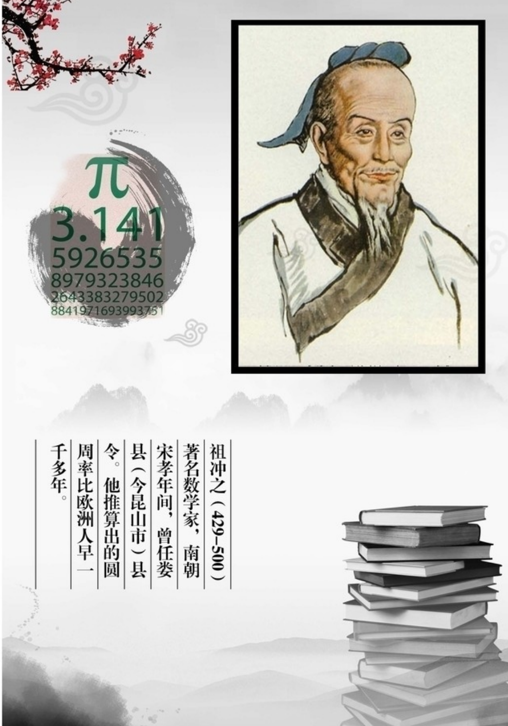
- Zu, Chongzhi, 429–500

$3 \times 2^{13} = 24,576$-sided polygon

$3.1415926 < \pi \sim \frac{355}{113} < 3.1415927$

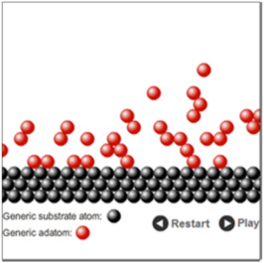
- François Viéte, 1540–1603

$3 \times 2^{17} = 393,216$-sided polygon

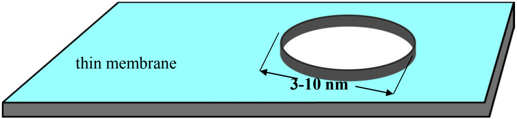


Epitaxial growth in industry and scientific research

- Epitaxial growth is an important physical process for forming solid films or other nano-structures.
- It is the only affordable method of high quality crystal growth for many semiconductor materials. *from Wiki*
- It is also an important tool to produce some single-layer films to perform experimental researches, highlighted by the recent breakthrough experiments on *the quantum anomalous Hall effect* and *superconductivity* above 100 K leaded by Qikun Xue '13, '15.

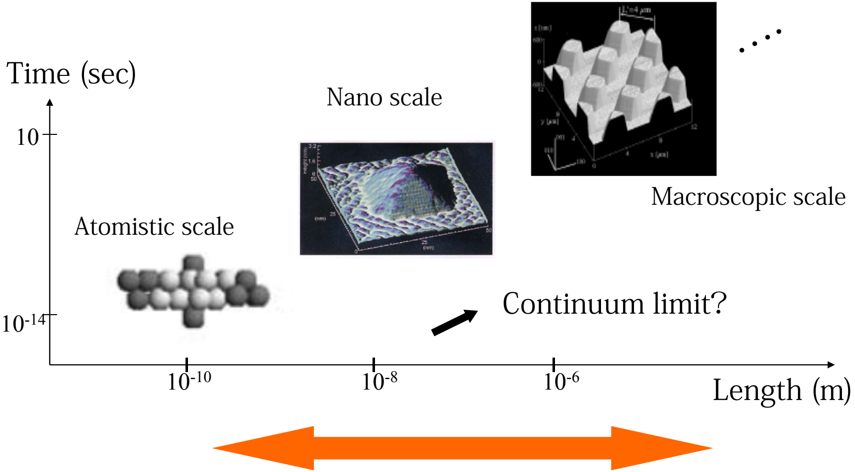


Nanopore-based molecular level detection



[Li *et al.*, 2001]

Multi-scale Modeling: length, time, temperature



Understanding and prediction of epitaxial growth.

A simple broken-bond models for crystals – SOS model

Crystals consists of height columns $h_i, i = 1, \dots N$ with screw-periodic BC

$$h_{i+N} = h_i + \alpha a N, \quad \alpha \text{ average slope }, \quad a \text{ side length }.$$

h_i derived into h_i/a boxes with an atom in center, connect to the nearest neighbor atoms with a bond from up, down, left and right. These bonds contain almost all the energy of the system.

$$E(h) = -\gamma \cdot (\# \text{ of bonds}),$$

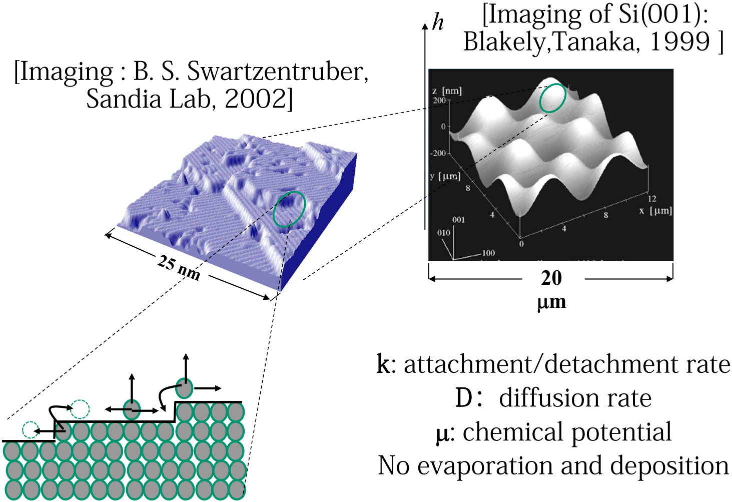
γ energy per bond. Negative sign represents atoms prefer to stay together. Simple algebra shows $E = E_b + E_s$, bulk and surface energy

$E_b = -\frac{2\gamma}{a} \sum_{j=1}^{N} h_i + \frac{\gamma\alpha}{2} N$ independent of time, drop from energy computatio

$$E_s = \frac{\gamma}{2a} \sum_{i=1}^{N} |h_i - h_{i-1}| \,.$$

$$E(h) = \frac{1}{p} \int |\nabla h|^p dx, \text{ or some linear combinations of those}$$

Mesoscale step dynamics: Discrete Burton, Cabrera, Frank (BCF) 1951



k: attachment/detachment rate

D: diffusion rate

μ: chemical potential

No evaporation and deposition

[Lu, Liu, Margetis, Physical Review E 2015]: derive BCF step dynamics from an atomistic broken-bound model.

Mesoscale mobility: Diffusion Limit (DL) regime vs Attachment Detachment Limit(ADL) regime

[Kohn, 2012] linear Gibbs-Thompson relation, $\rho = \rho^0 e^{\beta\mu} \approx \rho^0(1+\beta\mu)$

$$h_t + \nabla \cdot J = 0, \quad \text{Flux } J = -M\nabla\mu = -M\nabla\frac{\delta E}{\delta h}.$$

Dynamic equation for the surface height $h(t,x)$

$$\frac{\partial h}{\partial t} = \nabla \cdot \left(M(\nabla h)\nabla\left(\frac{\delta E}{\delta h}\right)\right),$$

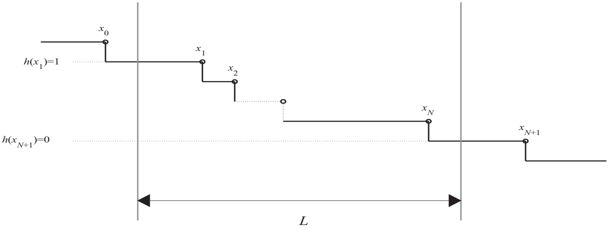
with

$$E(h) = \int \alpha|\nabla h| + \tfrac{1}{2}|\nabla h|^3 \, \mathrm{d}x.$$

$$h_t = -\nabla \cdot \left\{ M(\nabla h)\nabla\left[\nabla \cdot \left(\alpha\frac{\nabla h}{|\nabla h|} + \frac{3}{2}|\nabla h|\nabla h\right)\right]\right\}.$$

In DL regime, $M(\nabla h) = 1$. [Giga, Kohn, 2011]
In ADL regime, $M(\nabla h) = |\nabla h|^{-1}$, [Kohn, Abel Symposium 2010]: open question.

Continuum Limit of step dynamics: $N \to \infty$



[Gao-Liu-Lu, J. Nonlinear Sci. '17]
Step velocity: $v_i = \frac{\mathrm{d}x_i}{\mathrm{d}t} = J_{i-1} - J_i$.
By quasi-static approximation, $D\rho_{xx} = \frac{\partial \rho}{\partial t} = 0$, then on i-th terrace, J_i **is a constant**

$$
\begin{aligned}
J_i &= -D\rho_i' = -D\frac{\rho_i(x_{i+1}) - \rho_i(x_i)}{x_{i+1} - x_i} \\
&= \begin{cases} -k(\rho_i(x_{i_+}) - \rho_i^{eq}) \text{ for BC at } x = x_i; \\ k(\rho_i(x_{i+1_-}) - \rho_{i+1}^{eq}) \text{ for BC at } x = x_{i+1}. \end{cases} \\
&= \frac{k}{2}(\rho_i(x_{i+1}) - \rho_i(x_i)) - \frac{k}{2}(\rho_{i+1}^{eq} - \rho_i^{eq}) \\
&= -D\frac{\rho_{i+1}^{eq} - \rho_i^{eq}}{x_{i+1} - x_i + \frac{2D}{k}}.
\end{aligned}
$$

[Gao-Liu-Lu, J. Nonlinear Sci. '17]
From Gibbs-Thompson relation

$$\rho_i^{eq} = \rho_s e^{\frac{\mu_i}{k_B T}} \approx \rho_s \Big(1 + \frac{\mu_i}{k_B T}\Big).$$

Non-dimensionalized BCF model [Burton, Cabrera, Frank, 1951]:

$$\dot{x}_i = \frac{D}{ka^2}\left(\frac{\mu_{i+1} - \mu_i}{x_{i+1} - x_i + \frac{2D}{k}} - \frac{\mu_i - \mu_{i-1}}{x_i - x_{i-1} + \frac{2D}{k}}\right) \text{ for } 1 \le i \le N.$$

Diffusion Limit (DL regime), $\frac{D}{k} << x_{i+1} - x_i$,

$$\dot{x}_i = \frac{D}{ka^2}\left(\frac{\mu_{i+1} - \mu_i}{x_{i+1} - x_i} - \frac{\mu_i - \mu_{i-1}}{x_i - x_{i-1}}\right) \text{ for } 1 \le i \le N.$$

Attachment Detachment Limit (ADL regime), $\frac{D}{k} >> x_{i+1} - x_i$,

$$\dot{x}_i = \frac{1}{a^2}(\mu_{i+1} - 2\mu_i + \mu_{i-1}) \text{ for } 1 \le i \le N.$$

Continuum limit $N \to \infty$, $a = 1/N \to 0$?

Continuum Limit of Discrete Model: $N \to \infty$

Free energy and chemical potential:

$$E_N = a \sum_{i=0}^{N-1} f\left(\frac{x_{i+1}-x_i}{a}\right), \quad \mu_i = \frac{1}{a}\frac{\partial E}{\partial x_i} = \frac{1}{a}\left[-f'\left(\frac{x_{i+1}-x_i}{a}\right) + f'\left(\frac{x_i - x_{i-1}}{a}\right)\right]$$

For the DL regime, [Yang Xiang-Weinan E '02,'04] consider

$$E_N = a \sum_{i=0}^{N-1} f_1\left(\frac{x_{i+1}-x_i}{a}\right) + a^2 \sum_{i=0}^{N-1} \sum_{j=0, j\neq i}^{N-1} f_2\left(\frac{x_j - x_i}{a}\right),$$

where $f_1(r) = \frac{1}{2r^2}$ (elastic-dipole interactions), $f_2(r) = \ln|r|$.

[Theorem, Gao-Liu-Lu, J. Nonlinear Sci. '17] continuum limit for this step dynamics

$$h_t = \left(-H(h_x) - \left(\frac{1}{h_x} + 3h_x\right)h_{xx}\right)_{xx},$$

[Dal Maso-Fonseca-Leoni, Arch. Ration. Mech. Anal. '14] variational inequality solution
[Dal Maso et al., Comm. Partial Differential Equations '15] global weak solution.

[Theorem, Gao-Liu-Lu, SIMA '17] Global weak solution in ADL regime
(partially solved this problem proposed by Kohn).

Exponential PDE $h_t = \Delta\, e^{-\Delta_p h}$

Fick's law for diffusion

$$h_t + \nabla \cdot J = 0, \quad \text{Flux } J = -\nabla \rho,$$

Gibbs-Thompson relation,

$$\text{dilute adatoms } \rho = e^{-\beta(E-\mu)} = \rho^0 e^{\beta\mu} \approx \rho^0(1+\beta\mu), \quad \beta = \rho^0 = 1.$$

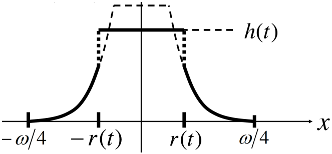
$$E(h) = \int \tfrac{1}{p}|\nabla h|^p \; \mathrm{d}x, \quad \mu = \frac{\delta E}{\delta h} = -\Delta_p h$$

Dynamic equation for the surface height $h(t,x)$

$$\frac{\partial h}{\partial t} = \Delta e^{\mu} = \Delta e^{-\Delta_p h} = -\left(\nabla \cdot M \nabla\right) \Delta_p h, \quad M = e^{-\Delta_p h}$$

Mobility depends on curvature. Asymmetric for concave/convex part.
[Krug-Dobbs '95], [Marzuola-Weare '13]

Facet solution for exponential PDE



[Giga-Giga '10], H^{-1} total variation flow

$$h_t = -\partial_{xxx}(h_x/|h_x|).$$

[Liu-Lu-Margetis-Marzuola '18]

$$h_t = \partial_{xx} e^{-\partial_x(h_x/|h_x|)}$$

- facet location and hight: $r(t)$, $h(t)$
- dynamics of top facet

$$\begin{cases} 2\sqrt{1+X^2(t)}\ln\left(X(t)+\sqrt{1+X^2(t)}\right)-2X(t)=\sqrt{\frac{\dot{h}(t)}{2}}, \\ X(t):=r(t)\sqrt{\frac{\dot{h}(t)}{2}} \\ \dot{r}(t)(h_0(r(t))-h(t))=\dot{h}(t)r(t)\ . \end{cases}$$

- dynamics of bottom facet

$$\begin{cases} 2\sqrt{1-X^2(t)}\left(\arctan\frac{\sqrt{1-X^2(t)}}{X(t)}-\frac{\pi}{2}\right)+2X(t)=\sqrt{\frac{\dot{h}(t)}{2}}, \\ X(t):=r(t)\sqrt{\frac{\dot{h}(t)}{2}} \\ \dot{r}(t)(h_0(r(t))-h(t))=\dot{h}(t)r(t)\ . \end{cases}$$

Facet solution for exponential PDE $h_t = \partial_{xx} e^{-\partial_x (h_x/|h_x|)}$

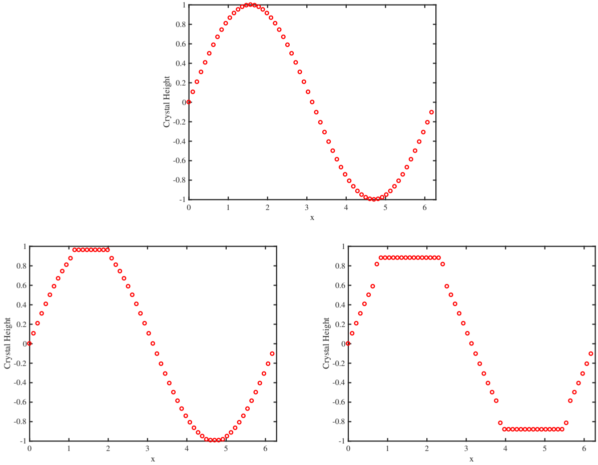


Figure: Snapshots of evolving surface height profile, $h(x, t)$, under initial data $h(0, x) = \sin(x)$ (top panel) by fourth-order total variation flows given by: exponential PDE with regularization parameter $\nu = 10^{-3}$ on a time scale $T = 10^{-4}$ (bottom left panel); and by usual PDE with regularization parameter

Facet solution for exponential PDE $h_t = \partial_{xx} e^{-\partial_x (h_x/|h_x|)}$

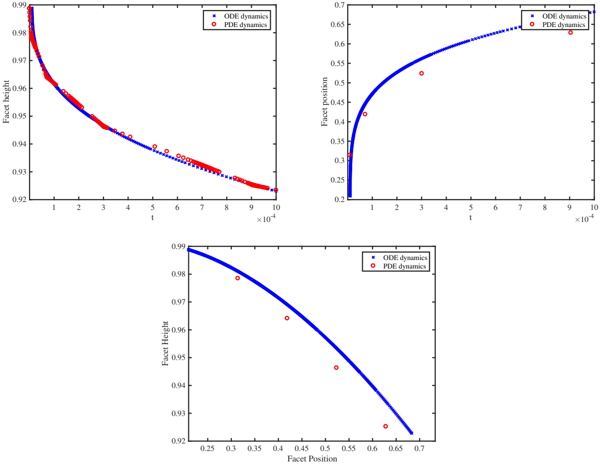


Figure: (Color Online) Plots of facet height $h_f(t)$ versus time t (top left panel), facet position $x_f(t)$ versus t (top right panel) and facet height versus facet position $(x_f(t),\, h_f(t))$ (bottom panel) for exponential PDE.

Global weak solution for $h_t = \Delta e^{-\Delta h}$

[Liu, Xiangsheng Xu (Mississippi State), SIMA '16]

- Recast as
$$\rho_t = -\rho \Delta^2 \rho, \qquad \rho = e^{-\Delta h}$$
- A stationary singular solution. Let $\Omega = (-1, 1)$. Define
$$h(x) = \begin{cases} -(x+1)^2 & \text{if } -1 \le x \le 0 \\ -(x-1)^2 & \text{if } 0 < x \le 1. \end{cases}$$

  An elementary calculation shows that
  $\Delta h = -2 + 4\delta_0, \quad e^{-\Delta h} = e^{-2}$?
- How to make sense of $e^{-\Delta h}$ for $\Delta h \in \mathcal{M}(\Omega)$?
  Decomposition with respect Lebesque measure
$$\Delta h = \Delta h_{\parallel} + \Delta h_{\perp} \tag{1}$$

Global weak solution for $h_t = \Delta e^{-\Delta h}$

- A beam type free energy $\Phi(h) := \int_\Omega e^{-\Delta h}\,dx$,

$$\frac{d\Phi(h)}{dt} = -\int_\Omega \left|\Delta e^{-\Delta h}\right|^2\,dx,$$

$$\frac{1}{p!}\|(\Delta h)^-\|_{L^p}^p \le \int_{\Omega_-} e^{(\Delta h)^-}\,dx \le \int_\Omega e^{-(\Delta h)^+ + (\Delta h)^-}\,dx = \Phi(h) \le \Phi(h^0)$$

- Notion of the solution: $\Delta h \in \mathcal{M}(\Omega)$,

$$h_t = \Delta e^{-\Delta h_{\|}}, \quad \text{a.e. on } \Omega \times (0,T).$$

$$\nabla h \cdot \nu = \nabla e^{-\Delta h} \cdot \nu = 0, \quad \text{on } \Gamma$$

- If $h_0 \in W^{2,\infty}(\Omega) \cap W^{4,2}(\Omega)$, then there is a global weak solution.

Global weak solution for $h_t = \Delta e^{-\Delta h}$

Let $T > 0$ be given. We divide $[0, T]$ into N equal subintervals with time step $\tau = \Delta t = \frac{T}{N}$. For $n = 1, \cdots, N$, we solve the following coupled nonlinear elliptic system with some low order regularization,

$$\begin{aligned} \frac{h^{n+1} - h^n}{\tau} &= \Delta e^{\mu^{n+1}} - \tau \mu^{n+1} \quad \text{in } \Omega, \\ \mu^{n+1} &= -\Delta_p h^{n+1} + \tau h^{n+1} \quad \text{in } \Omega, \\ \nabla e^{\mu^{n+1}} \cdot \nu = \nabla h^{n+1} \cdot \nu &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

$$\|\Delta h\|_{\mathcal{M}} \leq 2\phi(h^0), \quad (\Delta h)^- \ll \mathcal{L}^d$$

- using the second order operator to regularize the fourth order equation.
- the $\tau \to 0$ limit is indeed given $h_t = \Delta e^{-\Delta h_\parallel}$.
- I will explain later this notion of weak solution can also be derived by the Legendre transformation in a gradient flow [Gao-Liu-Lu, ESAIM: COCV '18].

A regularized Euler scheme

Denote $\rho = e^{\mu}$, and use linear approximation $\ln \rho^{n+1} \approx \ln \rho^{n} + \frac{\rho^{n+1} - \rho^{n}}{\rho^{n}}$

$$\begin{aligned}
\frac{h^{n+1} - h^{n}}{\tau} &= \Delta \rho^{n+1} - \tau \ln \rho^{n+1} \quad \text{in } \Omega, \\
\ln \rho^{n+1} &= -\nabla \cdot (|h^{n}|^{p-2} \nabla h^{n+1}) + \tau h^{n+1} \quad \text{in } \Omega, \\
\nabla \rho^{n+1} \cdot \nu = \nabla h^{n+1} \cdot \nu &= 0 \quad \text{on } \partial\Omega.
\end{aligned}$$

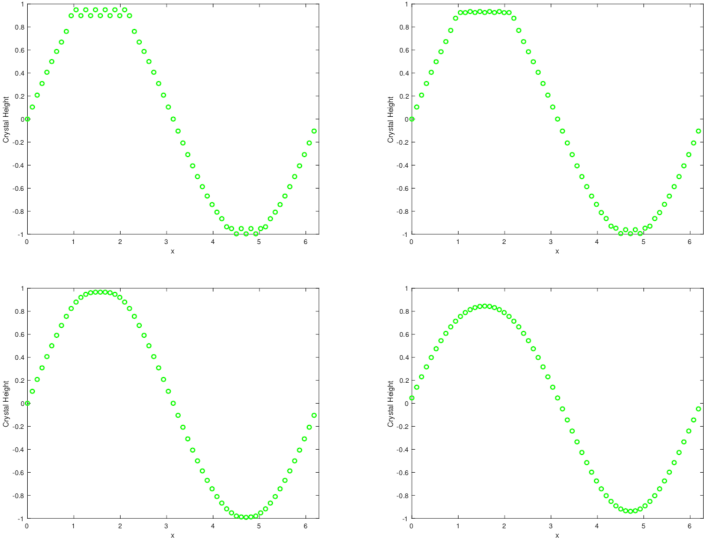


Fig. 1. Snapshots of evolving surface height profile, $h(x, t)$, under initial data $h(x, 0) = \sin(x)$ (top panel) by fourth-order total variation flows given by the method in Section 4 as a p-Laplacian implementation of the Liu-Xu algorith. (Top left) $p = 1.0$, (Top Right) $p = 1.1$, (Bottom Left) $p = 1.5$, (Bottom Right) $p = 1.9$.

Global classic solution and decay $h_t = \Delta e^{-\Delta h}$ in $\mathbb{R}^d$

[Liu-Strain '18], [Granero-Belinchón & Magliocca '18]

- Global classic solution: For $h_0 \in L^2(\mathbb{R}^d)$, $\|\Delta h_0\|_{\mathbb{A}(\mathbb{R}^d)} < 1/10$. Then there exists a global unique solution, $h(t) \in C\left(0, T; W^{2,\infty}(\mathbb{R}^d)\right)$,

$$\|\Delta h(\cdot, t)\|_{\mathbb{A}(\mathbb{R}^d)} + c\int_0^t \|\Delta^3 h\|_{\mathbb{A}(\mathbb{R}^d)}(\tau)d\tau \leq \|\Delta h_0\|_{\mathbb{A}(\mathbb{R}^d)}$$

  where Wiener Algebra $\|f\|_{\mathbb{A}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\hat{f}(\xi)| \, d\xi$

- Optimal large time decay rates: If $h_0 \in \dot{\mathcal{F}}^s$, $s > -\min(2, d)$, $h_0 \in H^2$ then

$$\|h(t)\|_{\dot{\mathcal{F}}^s} \lesssim (1+t)^{-(s+d)/4}, \quad \|f\|_{\dot{\mathcal{F}}^s} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^s |\hat{f}(\xi)| \, d\xi.$$

- Uniform gain of analyticity: In addition if $\|h_0\|_{-d,\infty} < \infty$, $h_0 \in \dot{\mathcal{F}}^s$ for $s \geq 0$, then there exists a positive increasing function $\nu(t) > 0$ such that $\nu(t) \approx t^{1/4}$ for large $t \gtrsim 1$. $h(t,x)$ is analyticity and decays

$$\|h(t)\|_{\dot{\mathcal{F}}^s_\nu} \lesssim (1+t)^{-(s+d)/4}, \quad \|f\|_{\dot{\mathcal{F}}^s_\nu}(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^s e^{\nu(t)|\xi|} |\hat{f}(\xi, t)| d\xi,$$

Gradient flow $u_t = \Delta e^{-\Delta u} \in -\partial\Phi(u), \quad \Phi = \int e^{-\Delta u}$

[Gao-Liu-Lu, ESAIM: COCV '18]

Let Hilbert space

$$H = \left\{ u \in L^2(\Omega) : \int_\Omega u\,dx = 0 \right\}, \quad \langle u, v\rangle_H$$

Take $d < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Define Banach space

$$\widetilde{V} = \{u \in H;\ \nabla u \in L^q(\Omega),\ \Delta u \in \mathcal{M}(\Omega),\ \int_\Omega \varphi\, d(\Delta u) = -\int_\Omega \nabla u \cdot \nabla\varphi, \forall \varphi \in W^{1,p}(\Omega)\}$$

Endow with the norm

$$\|u\|_{\widetilde{V}} = \|u\|_{L^2(\Omega)} + \|\Delta u\|_{\mathcal{M}(\Omega)}.$$

Decompose Radon measure $\mu = \Delta u$ with respect to the Lebesgue measure,

$$\mu = \mu_\parallel + \mu_\perp$$

$$\Phi(u) := \begin{cases} \int_\Omega e^{-(\Delta u)_\parallel^+ + (\Delta u)^-} x, & \text{if } u \in \widetilde{V} \text{ and } (\Delta u)^- \ll \mathcal{L}^d, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\Psi(u) := \begin{cases} 0 & \text{if } u \in \widetilde{V},\ \|\Delta u\|_{\mathcal{M}(\Omega)} \le C_* = 2\phi(u_0) + 1, \\ +\infty & \text{otherwise.} \end{cases}$$

The main issue is to show the weak-* lower semi-continuity for functional Φ in $\widetilde{V}$.

weak-* lower semi-continuity for functional Φ in $\widetilde{V}$

Let $u_n,\, u \in \widetilde{V}$. If $\Delta u_n \rightharpoonup \Delta u$ in $\mathcal{M}(\Omega)$, we have

$$\liminf_{n\to+\infty} \Phi(u_n) \geq \Phi(u).$$

[Goffman & Serrin, '64], Sublinear functions of measures and variational integrals.
Recall the conjugate convex function of $f(x) := e^{-x}$ for $x \geq 0$ is

$$f^*(y) = \sup_{x\geq 0}(xy - f(x)) = xy - f(x)\big|_{x=-\ln(-y)} = y - y\ln(-y), \quad -1 \leq y \leq 0.$$

Given some positive measure μ, define the convex functional of μ

$$\Phi_1(\mu) := \sup_{-1\leq\varphi\leq 0,\, \varphi\in C_c^\infty(\Omega)} \left\{ \int_\Omega \varphi d\mu - \int_\Omega f^*(\varphi)dx \right\}, \tag{2}$$

Lemma

Assume $\mu \in \mathcal{M}^+(\Omega)$. Then

$$\Phi_1(\mu) = \int_\Omega e^{-\mu_\parallel}\, dx\, .$$

weak-* lower semi-continuity for functional ϕ in $\widetilde{V}$

Given $N > 0$ and a sequence of measures μ_n such that $\mu_n \ll \mathcal{L}^d$, observe that

$$\mu_n = \min\{\mu_n, N\} + \max\{\mu_n, N\} - N.$$

$$\|\min\{\mu, N\}\|^2_{L^2(\Omega)} \leq 4e^N \int_\Omega e^{-\mu} + 2|\Omega|N^2.$$

Lemma

Assume $\mu_n \ll \mathcal{L}^d$, $\varphi(\mu_n) \leq A$, $\mu_n \rightharpoonup \mu$. Then for any $N > 0$, there exist $\mu_{\mathsf{down}}, \mu_{\mathsf{up}} \in \mathcal{M}(\Omega)$ and subsequence μ_n, such that

$$\min\{\mu_n, N\} \rightharpoonup \mu_{\mathsf{down}}, \quad \mu_{\mathsf{down}} \ll \mathcal{L}^d, \quad \mu_{\mathsf{down}} \leq \mu_\|,$$

$$\max\{\mu_n, N\} \rightharpoonup \mu_{\mathsf{up}}, \quad (\mu_{\mathsf{up}})_\| \geq N,$$

$$\int_\Omega e^{-\mu_\|}\, dx \leq \int_\Omega e^{-\mu_{\mathsf{down}}}\, dx.$$

weak-* lower semi-continuity for functional ϕ in $\widetilde{V}$

Denote $f_n := \Delta u_n$ and $f := \Delta u$. $\varphi(v) := \int_\Omega e^{-v} x$ is lower-semicontinuous on $L^1(\Omega)$ and hence $\varphi(v)$ l.s.c on $L^1(\Omega)$ with respect to the weak topology.
"cross convergence":
(i) there are some f_n are positive measures, i.e. $f_{n\perp} \neq 0$, and $f_{n\|} \rightharpoonup g_1 \ll \mathcal{L}^d$, $f_{n\perp} \rightharpoonup g_2 \geq 0$ and $g_1 + g_2 = f_\|$;

$$\liminf_n \Phi(u_n) = \liminf_n \int_\Omega e^{-f_{n\|}} dx \geq \int_\Omega e^{-g_1} dx \geq \int_\Omega e^{-f_\|} dx = \Phi(u)$$

ii) all f_n are absolutely continuous and $f_{n\|} = f_n$ may weakly-* converge to a singular measure.

$$|\Phi(u_n) - \Phi_N(u_n)| \leq e^{-N} \mathcal{L}^d(\{f_n > N\}) \leq e^{-N} |\Omega|.$$

$$\liminf_{n\to+\infty} \int_\Omega e^{-\min\{f_n, N\}} dx \geq \int_\Omega e^{-f_{down}} dx \geq \int_\Omega e^{-f_\|} dx = \Phi(u).$$

$$\begin{aligned}\liminf_{n\to+\infty} \Phi(u_n) &\geq \liminf_{n\to+\infty} \Phi_N(u_n) - e^{-N}|\Omega| \\ &= \liminf_{n\to+\infty} \int_\Omega e^{-\min\{f_n, N\}} dx - e^{-N}|\Omega| \\ &\geq \Phi(u) - e^{-N}|\Omega|,\end{aligned}$$

Global "Strong" solution to $u_t = \Delta e^{-\Delta u}$

[Gao-Liu-Lu, ESAIM: COCV '18]

Given $T > 0$, $u^0 \in L^2(\Omega)$, mean zero, such that $\Phi(u^0) < +\infty$, then there is also a strong solution, i.e.,

$$u_t = \Delta(e^{-(\Delta u)_\parallel})$$

for a.e. $(t, h) \in [0, T] \times \Omega$. Besides, we have

$$u \in L^\infty([0,T]; \widetilde{V}) \cap C^0([0,T]; L^2(\Omega)), \quad u_t \in L^\infty([0,T]; L^2(\Omega))$$

$$\Delta(e^{-(\Delta u)_\parallel}) \in L^\infty([0,T]; L^2(\Omega))$$

and $\Phi(h(\cdot, t))$ decay and

$$E(u(t)) := \frac{1}{2} \int_\Omega \left[\Delta(e^{-(\Delta u)_\parallel})\right]^2 dx \le E(u^0),$$

where $(\Delta u)_\parallel$ is the absolutely continuous part of Δu in the decomposition.

Global solution of exponential SOS model ($p = 1$) with logarithmic correction in ADL regime (Yuan Gao)

SOS free energy ($p = 1$) with logarithmic correction

$$E(h) = \int |\nabla h| \ln |\nabla h| \, \mathrm{d}x, \quad \mu := \frac{\delta E}{\delta h} = -\nabla \cdot \Big(\frac{\nabla h}{|\nabla h|} (\ln |\nabla h| + 1) \Big).$$

ADL mobility

$$\mathcal{M}(h) = \frac{1}{|\nabla h|}$$

$$h_t = \nabla \cdot (\mathcal{M}(h) \nabla e^{\mu}) = \nabla \cdot \Big(\frac{1}{|\nabla h|} \nabla e^{-\nabla \cdot \big(\frac{\nabla h}{|\nabla h|} (\ln |\nabla h| + 1) \big)} \Big).$$

For one dimensional case with monotone initial data, i.e. $\partial_x h_0 > 0$. If we can prove $h_x > 0$ for all the time, then we obtain a mathematical validation for surface hight equation (26), i.e.

$$h_t = \nabla \cdot (\mathcal{M}(h) \nabla e^{\mu}) = \Big(\frac{1}{h_x} (e^{-(\ln h_x)_x})_x \Big)_x$$

with $\mu = \frac{\delta E}{\delta h} = -(\ln h_x)_x$.

Global solution of exponential SOS model ($p = 1$) with logarithmic correction in ADL regime (Yuan Gao)

For $T > 0$, $h^0 \in L^2$, $\Phi(h^0) < +\infty$, then there is a strong solution

$$h_t = \left(\frac{1}{h_x} \left(e^{-((\ln h_x)_x)_\parallel} \right)_x \right)_x$$

for a.e. $(t, h) \in [0, T] \times \mathbb{T}$ and

$$\left(\frac{1}{h_x} \left(e^{-((\ln h_x)_x)_\parallel} \right)_x \right)_x \in L^\infty([0, T]; L^2)$$

$$\Phi(h(t)) = \int_{\mathbb{T}} e^{-((\ln h_x)_x)_\parallel} \, \mathrm{d}x \leq \phi(h^0), \quad t \geq 0.$$

Furthermore, if $E(h^0) := \frac{1}{2} \int_{\mathbb{T}} \left[\left(\frac{1}{h_x} \left(e^{-((\ln h_x)_x)_\parallel} \right)_x \right)_x \right]^2 \mathrm{d}x < \infty$, then

$$E(u(t)) := \frac{1}{2} \int_{\mathbb{T}} \left[\left(\frac{1}{h_x} \left(e^{-((\ln h_x)_x)_\parallel} \right)_x \right)_x \right]^2 \mathrm{d}x \leq E(u^0), \quad t \geq 0,$$

where $((\ln h_x)_x)_\parallel$ is the absolutely continuous part of $(\ln h_x)_x)$ in the decomposition.

Summery: $h_t = \Delta e^{-\Delta_p h}$

- Epitaxial growth is an important physical process for forming solid films or other nano-structures.
- It is also an important tool to produce some single-layer films to perform experimental researches, highlighted by the recent breakthrough experiments on the quantum anomalous Hall effect and superconductivity above 100 K leaded by Qikun Xue '13, '15.
- It has been a focus of research both in mathematics and physics since 50's: BCF step dynamics, KPZ eq, Spohn, Kohn, Giga, E, etc.
- A simple broken-bond counting shows the surface energy

$$E_s = \frac{\gamma}{2a} \sum_{i=1}^{N} |h_i - h_{i-1}|, \text{ general } E(h) = \frac{1}{p} \int |\nabla h|^p dx, \text{ linear combinat}$$

- Fick's law & Gibbs-Thompson lead naturally to an exponential PDE

$$h_t = -\nabla \cdot J = \Delta \rho^0 e^{\beta \mu} = \Delta e^{-\Delta_p h} = -\left(\nabla \cdot M \nabla\right) \Delta_p h, \quad M = e^{-\Delta_p h}$$

Summery: $h_t = \Delta e^{-\Delta_p h}$

- For medium size initial in the Wiener algebra $\|h_0\|_{A(\mathbb{R}^d)} \leq \frac{1}{10}$, global unique classic solution with optimal large time decay rates, and uniform gain of analyticity.
- Numerical shows, for large data, solution develops singularity. The natural solution space is Δh in Radon space. We constructed such steady singularity solution.
- We introduced a new notion of weak solution and proved global existence. We also used the method of gradient flow and maximal monotone operator to show this notation of solution is general, we proved global "strong" solution with energy-dissipation inequality.
- Global solution of exponential SOS model $(p = 1)$ with logarithmic correction in ADL regime in 1-D with monotone initial data,.

Thank You !!!