Mathematical and numerical analysis of the Hamilton–Jacobi equations on an evolving surface

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Hamilton–Jacobi equation on an evolving surface

We consider the following first-order Hamilton–Jacobi equation:

$$\begin{cases} \partial^\bullet u + H(x,t,\nabla_\Gamma u(x,t)) = 0 & \text{on} \quad S_T = \bigcup_{t\in(0,T)} \Gamma(t)\times\{t\}, \\ u(\cdot,0) = u_0 & \text{on} \quad \Gamma(0). \end{cases}$$

- $\Gamma(t)$: closed, connected, oriented evolving surface in $\mathbb{R}^3$
  - $v_\Gamma(\cdot,t)$: total velocity field of $\Gamma(t)$
  - $\nu(\cdot,t)$: unit outward normal vector field to $\Gamma(t)$
- $\partial^\bullet u = u_t + v_\Gamma \cdot \nabla u$: material derivative
- $\nabla_\Gamma u = (I_3 - \nu\otimes\nu)\nabla u$: tangential gradient
- $H(x,t,p)$: Hamiltonian ($H\colon S_T\times\mathbb{R}^3\to\mathbb{R}$)

Outline

Our aim in this talk is to give

- a motivating example and derivation of (HJ),
- approximation based on a finite volume scheme,
- an existence result of a viscosity solution to (HJ),
- an error estimate between numerical and viscosity solutions.

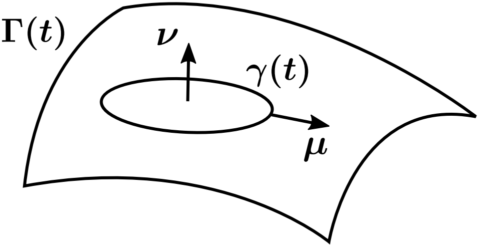
$$(HJ) \begin{cases} \partial^\bullet u + H(x,t,\nabla_\Gamma u(x,t)) = 0 & \text{on} \quad S_T, \\ u(\cdot,0) = u_0 & \text{on} \quad \Gamma(0). \end{cases}$$

Motivating example, Derivation of HJ equation

Consider the motion of a closed curve $\gamma(t)$ on $\Gamma(t)$ given by

$$\boldsymbol{V_\mu}(\boldsymbol{x},t) = \boldsymbol{F}(\boldsymbol{x},t) + \boldsymbol{\beta}(\boldsymbol{x},t)\cdot\boldsymbol{\mu}(\boldsymbol{x},t), \quad \boldsymbol{x}\in\gamma(t),\ t\in(0,T).$$

- $\boldsymbol{\mu}(\cdot,t)$: conormal vector field to $\gamma(t)$ ($\boldsymbol{\mu}\cdot\boldsymbol{\nu}=0$)
- $V_{\mu}(\cdot,t)$: velocity of $\gamma(t)$ in the direction of $\boldsymbol{\mu}$
- $F\colon S_T\to\mathbb{R}$, $\boldsymbol{\beta}\colon S_T\to\mathbb{R}^3$: given functions



Let N_T be an open neighborhood of S_T in $\mathbb{R}^4$ and assume that

$$\gamma(t) = \{(x,t) \in S_T \mid u(x,t) = r\} \quad \text{for some} \quad r \in \mathbb{R}$$

with a function $u\colon N_T \to \mathbb{R}$ satisfying $\nabla_\Gamma u(\cdot,t) \neq 0$ on $\gamma(t)$.
Taking a parametrization $\varphi(\cdot,t)\colon S^1 \to \mathbb{R}^3$ of $\gamma(t)$ we have

$$u(\varphi(s,t),t) = r, \quad \forall s \in S^1,\ \forall t \in (0,T).$$

Then we differentiate both sides with respect to t to get

$$u_t(\varphi(s,t),t) + \varphi_t(s,t) \cdot \nabla u(\varphi(s,t),t) = 0,$$

or equivalently for $x = \varphi(s,t) \in \gamma(t)$ (since $\partial^\bullet u = u_t + v_\Gamma \cdot \nabla u$)

$$\partial^\bullet u(x,t) + \big(\varphi_t(s,t) - v_\Gamma(x,t)\big) \cdot \nabla u(x,t) = 0.$$

Since $\gamma(t)$ is on $\Gamma(t)$, the normal components of

- $\varphi_t(s,t)$: velocity of $\gamma(t)$ at $\varphi(s,t)$
- $v_\Gamma(x,t)$: velocity of $\Gamma(t)$ at $x=\varphi(s,t)$

are the same, i.e. $\bigl(\varphi_t(s,t)-v_\Gamma(x,t)\bigr)\cdot\nu(x,t)=0$. Hence

$$\partial^\bullet u(x,t)+\bigl(\varphi_t(s,t)-v_\Gamma(x,t)\bigr)\cdot\nabla_\Gamma u(x,t)=0.$$

By this and $\mu=\nabla_\Gamma u/|\nabla_\Gamma u|$ the conormal velocity V_μ is given by

$$V_\mu(x,t)=\varphi_t(s,t)\cdot\mu(x,t)=\left[-\frac{\partial^\bullet u}{|\nabla_\Gamma u|}+v_\Gamma\cdot\frac{\nabla_\Gamma u}{|\nabla_\Gamma u|}\right](x,t).$$

Combining this with $V_\mu=F+\beta\cdot\mu$ we see that u satisfies

$$\partial^\bullet u+H(x,t,\nabla_\Gamma u(x,t))=0 \quad \text{on} \quad S_T$$

$$\text{with} \quad H(x,t,p)=F(x,t)|p|+\bigl(\beta(x,t)-v_\Gamma(x,t)\bigr)\cdot p.$$

Definition of viscosity solution

Definition

Let u_0 be a function on $\Gamma(0)$. A locally bounded function

$$u \in USC(\overline{S_T}) \quad \left(\text{resp. } u \in LSC(\overline{S_T})\right)$$

is called a viscosity subsolution (resp. supersolution) to (HJ) if

- $u(\cdot,0) \leq u_0$ (resp. $u(\cdot,0) \geq u_0$) on $\Gamma(0)$,
- for any $\varphi \in C^1(\overline{S_T})$, if $u - \varphi$ takes a local maximum (resp. minimum) at $(x_0, t_0) \in \overline{S_T}$ with $t_0 > 0$, then

$$\partial^\bullet \varphi(x_0,t_0) + H(x_0, t_0, \nabla_\Gamma \varphi(x_0,t_0)) \leq 0 \quad (\text{resp. } \geq 0).$$

If u is a sub- and supersolution, then we call u a viscosity solution.

$$(HJ) \begin{cases} \partial^\bullet u + H(x,t,\nabla_\Gamma u(x,t)) = 0 & \text{on} \quad S_T, \\ u(\cdot,0) = u_0 & \text{on} \quad \Gamma(0). \end{cases}$$

Comparison principle

Assuming that

- $\exists L_{H,1} > 0$ s.t. $\forall (x,t), \forall (y,s) \in \overline{S_T}, \forall p \in \mathbb{R}^3$
$$|H(x,t,p) - H(y,s,p)| \leq L_{H,1}(|x-y| + |t-s|)(1+|p|),$$
- $\exists L_{H,2} > 0$ s.t. $\forall (x,t) \in \overline{S_T}, \forall p, \forall q \in \mathbb{R}^3$
$$|H(x,t,p) - H(x,t,q)| \leq L_{H,2}|p-q|,$$

by a standard doubling of variables method we can show

Theorem (Comparison principle)

Let u and v be a subsolution and supersolution to (HJ). Then
$$u(\cdot,0) \leq v(\cdot,0) \ \text{ on } \ \Gamma(0) \implies u \leq v \ \text{ on } \ \overline{S_T}.$$

Triangulation of evolving surface

Setting of triangulation of $\Gamma(t)$:

- $\mathcal{T}_h(t)$: triangulation of $\Gamma(t)$ $(0 < h < h_0)$
- $\Gamma_h(t) = \bigcup_{K(t)\in\mathcal{T}_h(t)} K(t)$: triangulated surface
- $h_{K(t)}$: diameter of a triangle $K(t) \in \mathcal{T}_h(t)$

$$h = \max_{t\in[0,T]} \max_{K(t)\in\mathcal{T}_h(t)} h_{K(t)}$$

- $\rho_{K(t)}$: radius of the inscribed circle of $K(t) \in \mathcal{T}_h(t)$

We assume that there exists $\gamma > 0$ such that

$$h_{K(t)} \leq \gamma\rho_{K(t)}, \quad \forall K(t) \in \mathcal{T}_h(t), \, \forall t \in [0,T], \, \forall h \in (0,h_0).$$

Evolving finite element space

We further assume that

- each vertex of triangles in $\mathcal{T}_h(t)$ moves with velocity v_Γ
  $\Rightarrow$ the number M of vertices of $\mathcal{T}_h(t)$ is fixed in time.

For $i = 1, \dots, M$ we call the i-th vertex just i.

- $x_i(t) \in \Gamma(t) \cap \Gamma_h(t)$: point of the vertex i at time $t \in [0, T]$

For $t \in [0, T]$ we introduce a finite element space

$$V_h(t) = \{u_h \in C(\Gamma_h(t)) \mid u_h|_{K(t)} \text{ is affine for each } K(t) \in \mathcal{T}_h(t)\}$$

- $\chi_1(\cdot, t), \dots, \chi_M(\cdot, t)$: nodal basis of $V_h(t)$, i.e.

$$\chi_i(\cdot, t) \in V_h(t), \quad \chi_i(x_j(t), t) = \delta_{ij}.$$

Finite volume scheme for HJ equation

Our scheme is based on a finite volume scheme for HJ equations in flat stationary domain by Kim and Li (J. Comput. Math., 2015).

For $N \in \mathbb{N}$ let $\tau = T/N$ and

$$t^n = n\tau,\ x_i^n = x_i(t^n),\ V_h^n = V_h(t^n) \quad (n = 0, 1, \dots, N).$$

Consider the viscous approximation of (HJ)

$$\partial^\bullet u + H(x, t, \nabla_\Gamma u(x,t)) = \varepsilon \Delta_\Gamma u \ \text{ on } \ S_T$$
$$(\varepsilon > 0\text{: small, } \Delta_\Gamma u = \nabla_\Gamma \cdot \nabla_\Gamma u).$$

Let $V_i(t) \subset \Gamma(t)$ be a moving set centered at $x_i(t)$ $(i = 1, \dots, M)$. For $t = t^n$ we integrate the approximate equation over $V_i(t^n)$ to get

$$\int_{V_i(t^n)} \partial^\bullet u \, d\mathcal{H}^2 + \int_{V_i(t^n)} H(\cdot, t^n, \nabla_\Gamma u) \, d\mathcal{H}^2 = \varepsilon \int_{V_i(t^n)} \Delta_\Gamma u \, d\mathcal{H}^2.$$

By the transport theorem

$$\frac{d}{dt}\int_{V_i(t)} u\,d\mathcal{H}^2 = \int_{V_i(t)} \left(\partial^\bullet u + (\nabla_\Gamma \cdot v_\Gamma)u\right) d\mathcal{H}^2$$

and $\frac{d}{dt}|V_i(t)| = \int_{V_i(t)} \nabla_\Gamma \cdot v_\Gamma \, d\mathcal{H}^2$ we approximate

$$\begin{cases} \displaystyle\int_{V_i(t^n)} \partial^\bullet u\,d\mathcal{H}^2 \approx -\int_{V_i(t^n)} (\nabla_\Gamma \cdot v_\Gamma)u\,d\mathcal{H}^2 \\ \qquad\qquad\displaystyle + \frac{u(x_i^{n+1}, t^{n+1})|V_i(t^{n+1})| - u(x_i^n, t^n)|V_i(t^n)|}{\tau}, \\ \displaystyle |V_i(t^{n+1})| \approx |V_i(t^n)| + \tau \int_{V_i(t^n)} \nabla_\Gamma \cdot v_\Gamma \, d\mathcal{H}^2. \end{cases}$$

$$\Longrightarrow \int_{V_i(t^n)} \partial^\bullet u\,d\mathcal{H}^2 \approx \frac{u(x_i^{n+1}, t^{n+1}) - u(x_i^n, t^n)}{\tau}|V_i(t^n)|.$$

To the integral equality

$$\int_{V_i(t^n)} \partial^\bullet u \, d\mathcal{H}^2 + \int_{V_i(t^n)} H(\cdot, t^n, \nabla_\Gamma u) \, d\mathcal{H}^2 = \varepsilon \int_{V_i(t^n)} \Delta_\Gamma u \, d\mathcal{H}^2$$

we apply the Gauss theorem $\int_{V_i(t^n)} \Delta_\Gamma u \, d\mathcal{H}^2 = \int_{\partial V_i(t^n)} \frac{\partial u}{\partial \mu} \, d\mathcal{H}^1$ and

$$\int_{V_i(t^n)} \partial^\bullet u \, d\mathcal{H}^2 \approx \frac{u(x_i^{n+1}, t^{n+1}) - u(x_i^n, t^n)}{\tau} |V_i(t^n)|.$$

Then we get an approximation formula

$$u(x_i^{n+1}, t^{n+1}) \approx u(x_i^n, t^n) - \frac{\tau}{|V_i(t^n)|} H_i(t^n),$$

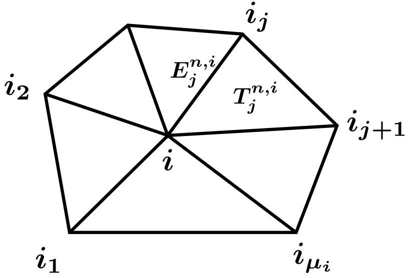
$$H_i(t^n) = \int_{V_i(t^n)} H(\cdot, t^n, \nabla_\Gamma u) \, d\mathcal{H}^2 - \varepsilon \int_{\partial V_i(t^n)} \frac{\partial u}{\partial \mu} \, d\mathcal{H}^1.$$

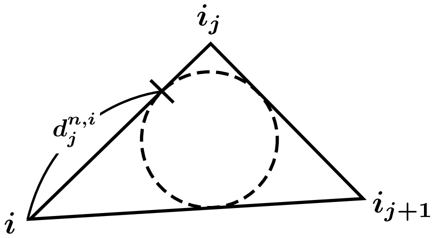
Volume $V^{n,i}$ on $\Gamma_h(t^n)$ centered at vertex i

For each $i = 1, \dots, M$ let

- μ_i: number of triangles with common vertex i,
- $i_1, \dots, i_{\mu_i}$: vertices surrounding i,
- $T_j^{n,i} \in \mathcal{T}_h(t^n)$: triangle with vertices i, i_j, and i_{j+1},
- $E_j^{n,i}$: edge of $T_j^{n,i}$ connecting the vertices i and i_j.

Also, let $d_j^{n,i}$ be the length from i to the contact point on $E_j^{n,i}$ of the inscribed circle of $T_j^{n,i}$ and $d^{n,i} = \min_{j=1,\dots,\mu_i} d_j^{n,i}$.

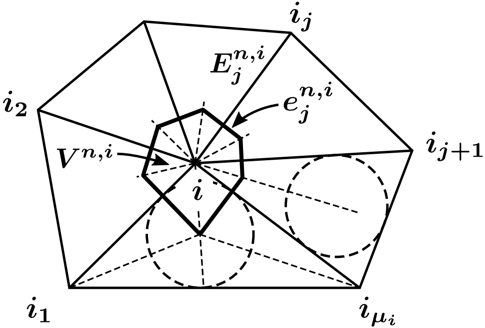


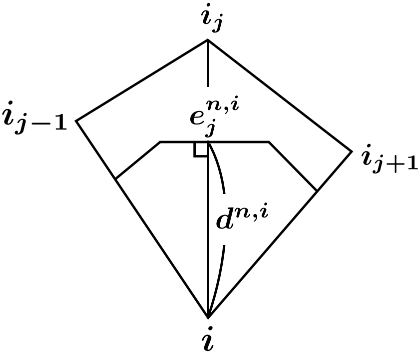


We define the volume $V^{n,i} \subset \Gamma_h(t^n)$ as a polygonal region such that

- $V^{n,i}$ is surrounded by line segments perpendicular to each $E_j^{n,i}$,
- distance from each edge of $V^{n,i}$ to the vertex i is equal to $d^{n,i}$.

For $j = 1, \ldots, \mu_i$ let $e_j^{n,i}$ be the edge of $V^{n,i}$ perpendicular to $E_j^{n,i}$.





Using the notations on $V^{n,i}$, we interpret the approximation formula

$$u(x_i^{n+1}, t^{n+1}) \approx u(x_i^n, t^n) - \frac{\tau}{|V_i(t^n)|} H_i(t^n),$$

$$H_i(t^n) = \int_{V_i(t^n)} H(\cdot, t^n, \nabla_\Gamma u)\, d\mathcal{H}^2 - \varepsilon \int_{\partial V_i(t^n)} \frac{\partial u}{\partial \mu}\, d\mathcal{H}^1$$

as $u(x_i^n, t^n) \to u_i^n$, $V_i(t^n) \to V^{n,i}$, and

$$\int_{V_i(t^n)} H(\cdot, t^n, \nabla_\Gamma u)\, d\mathcal{H}^2 \to \sum_{j=1}^{\mu_i} H\left(x_i^n, t^n, \nabla_{\Gamma_h} u_h^n|_{T_j^{n,i}}\right) |V^{n,i} \cap T_j^{n,i}|,$$

$$\int_{\partial V_i(t^n)} \frac{\partial u}{\partial \mu}\, d\mathcal{H}^1 \to \sum_{j=1}^{\mu_i} \frac{u_{i_j}^n - u_i^n}{|E_j^{n,i}|} |e_j^{n,i}|$$

for $u_h^n = \sum_{i=1}^{M} u_i^n \chi_i(\cdot, t^n) \in V_h^n$ (note that $\nabla_{\Gamma_h} u_h^n|_{T_j^{n,i}}$ is constant).

Definition of numerical scheme (NS)

- For a given $u_0 : \Gamma(0) \to \mathbb{R}$ set

$$u_h^0 = \sum_{i=1}^{M} u_i^0 \chi_i(\cdot, 0) \in V_h^0, \quad u_i^0 = u_0(x_i^0).$$

- For $n \geq 0$, if $u_h^n = \sum_{i=1}^{M} u_i^n \chi_i(\cdot, t^n) \in V_h^n$ is given, define

$$u_h^{n+1} = S_h^n(u_h^n) = \sum_{i=1}^{M} u_i^{n+1} \chi_i(\cdot, t^{n+1}) \in V_h^{n+1}$$

  by $u_i^{n+1} = u_i^n - \tau H_i^n$ $(i = 1, \dots, M)$, where

$$H_i^n = \sum_{j=1}^{\mu_i} \frac{|V^{n,i} \cap T_j^{n,i}|}{|V^{n,i}|} H\left(x_i^n, t^n, \nabla_{\Gamma_h} u_h^n|_{T_j^{n,i}}\right) - \frac{\varepsilon_i^n}{|V^{n,i}|} \sum_{j=1}^{\mu_i} \frac{u_{i_j}^n - u_i^n}{|E_j^{n,i}|} |e_j^{n,i}|.$$

Properties of numerical scheme

Lemma (Invariance under translation with constants)

For $u_h^n \in V_h^n$ and $c \in \mathbb{R}$ we have

$$S_h^n(u_h^n + c) = S_h^n(u_h^n) + c \text{ on } \Gamma_h(t^{n+1}).$$

Lemma (Monotonicity)

There exists $C_1, C_2 > 0$ such that if

$$(\sharp) \quad \varepsilon_i^n = C_1 \max_j h_{T_j^{n,i}}, \quad \tau \le C_2 \max_{i,j} |E_j^{n,i}| \;\; (\Rightarrow \varepsilon_i^n, \tau \le Ch),$$

then for $u_h^n, v_h^n \in V_h^n$ we have

$$u_h^n \le v_h^n \text{ on } \Gamma_h(t^n) \implies S_h^n(u_h^n) \le S_h^n(v_h^n) \text{ on } \Gamma_h(t^{n+1}).$$

Lemma (Consistency)

Suppose that (♯) is satisfied. Then there exists $C > 0$ such that

$$\left|\frac{\varphi_i^{n+1} - [S_h^n(I_h^n\varphi)]_i}{\tau} - \left(\partial^\bullet\varphi(x_i^n, t^n) + H(x_i^n, t^n, \nabla_\Gamma\varphi(x_i^n, t^n))\right)\right|$$
$$\leq Ch\left(\|\nabla_\Gamma\varphi\|_{B(\overline{S_T})} + \|\nabla_\Gamma^2\varphi\|_{B(\overline{S_T})} + \|(\partial^\bullet)^2\varphi\|_{B(\overline{S_T})}\right)$$

for all $\varphi \in C^2(\overline{S_T})$, $n = 0, 1, \dots, N-1$, and $i = 1, \dots, M$.

- $I_h^n\varphi$: interpolant of φ on $\Gamma_h(t^n)$, i.e.

$$I_h^n\varphi = \sum_{i=1}^{M} \varphi_i^n \chi_i(\cdot, t^n) \in V_h^n, \quad \varphi_i^n = \varphi(x_i^n, t^n)$$

- $[S_h^n(I_h^n\varphi)]_i = S_h^n(I_h^n\varphi)(x_i^{n+1})$: nodal value at x_i^{n+1}

Convergence of numerical solution to viscosity solution

For $u_h^n = \sum_{i=1}^M u_i^n \chi_i(\cdot, t^n) \in V_h^n$ given by (NS) set

$$u_h^l(x,t) := \sum_{i=1}^{M} u_i^n \chi_i(p_h(x,t),t), \quad t \in [t^n, t^{n+1}),\ x \in \Gamma(t).$$

$\big(p_h(\cdot,t)$: closest point mapping from $\Gamma(t)$ onto $\Gamma_h(t) \big)$

We define $\bar{u}, \underline{u} \colon \overline{S_T} \to \mathbb{R}$ by

$$\left.\begin{aligned} \bar{u}(x,t) &= \limsup_{h\to 0,\, \overline{S_T} \ni (y,s) \to (x,t)} u_h^l(y,s), \\ \underline{u}(x,t) &= \liminf_{h\to 0,\, \overline{S_T} \ni (y,s) \to (x,t)} u_h^l(y,s), \end{aligned}\right. \quad (x,t) \in \overline{S_T}.$$

Lemma

Suppose that $u_0 \in C(\Gamma(0))$ and ($\sharp$) is satisfied. Then

- $\bar{u}(\cdot, 0) = \underline{u}(\cdot, 0) = u_0$ on $\Gamma(0)$,
- $\bar{u}$ (resp. $\underline{u}$) is a subsolution (resp. supersolution) to (HJ).

By the above lemma and the comparison principle, we see that

$$\bar{u} = \underline{u} \quad \text{on} \quad \overline{S_T}.$$

Setting $u = \bar{u} = \underline{u}$ we get a unique viscosity solution u to (HJ).

Theorem (Elliott–Deckelnick–M., in preparation)

For any $u_0 \in C(\Gamma(0))$ there exists a unique viscosity solution to (HJ).

Error bound between numerical and viscosity solutions

Theorem (Elliott–Deckelnick–M., in preparation)

Let u be a viscosity solution to (HJ) with initial data u_0 and

$$u_h^n = \sum_{i=1}^{M} u_i^n \chi_i(\cdot, t^n) \in V_h^n \quad (h > 0,\ n = 0, 1, \dots, N)$$

a finite element function constructed from u_0 by (NS). Suppose that

- $(\sharp)$ holds, i.e. $\varepsilon_i^n = C_1 \max_j h_{T_j^{n,i}}$ and $\tau \leq C_2 \min_{i,j} |E_j^{n,i}|$,
- u is Lipschitz continuous on $\overline{S_T}$.

Then there exist $h_0, C > 0$ such that

$$\max_{1\leq i\leq M,\, 0\leq n\leq N} |u(x_i^n, t^n) - u_i^n| \leq C\sqrt{h}, \ \ \forall h \in (0, h_0).$$