

Remark 2.4(c) of Koike (2026) claims that a probability distribution μ has a convex support if it has a Stein kernel, based on the fact that Theorem 1.1 in Döbler (2025) implies that any one-dimensional marginal of μ is supported by an interval in this case. However, the latter does not imply that the support of μ is convex. In fact, we can easily construct a probability distribution on \mathbb{R}^2 which has a Stein kernel but is supported on a non-convex set: Let Z and ϵ be independent random variables such that $Z \sim N(0, 1)$ and $P(\epsilon = 0) = 1 - P(\epsilon = 1) = 1/2$. Set $X := (\epsilon Z, (1 - \epsilon)Z)$. The support of the law of X is $\{(x, 0) : x \in \mathbb{R}\} \cup \{(0, y) : y \in \mathbb{R}\}$ and thus it is non-convex. In the meantime, X has a Stein kernel given by

$$\tau(x) = \mathbb{E} \left[\begin{pmatrix} \epsilon & 0 \\ 0 & 1 - \epsilon \end{pmatrix} \mid X = x \right], \quad x \in \mathbb{R}^2.$$

In fact, for any $h \in C_b^2(\mathbb{R}^d)$,

$$\begin{aligned} \mathbb{E}[X \cdot \nabla h(X)] &= \mathbb{E}[Z \partial_1 h(Z, 0); \epsilon = 1] + \mathbb{E}[Z \partial_2 h(0, Z); \epsilon = 0] \\ &= \mathbb{E}[\partial_{11} h(Z, 0); \epsilon = 1] + \mathbb{E}[\partial_{22} h(0, Z); \epsilon = 0] = \mathbb{E}[\langle \tau(X), \nabla^2 h(X) \rangle], \end{aligned}$$

where the second equality follows by the univariate Stein identity and the independence between Z and ϵ .

Another more interesting example is the uniform distribution on a star-shaped compact set whose kernel has a non-zero Lebesgue measure. To be precise, for $S \subset \mathbb{R}^d$, we define its kernel as $K_S := \{x \in S : (1 - t)x + ty \in S \text{ for all } y \in S \text{ and } t \in [0, 1]\}$. S is called *star-shaped* if $K_S \neq \emptyset$. Now, suppose that $S \subset \mathbb{R}^d$ is compact and star-shaped such that $|K_S| > 0$, where $|\cdot|$ denotes the Lebesgue measure of a set. Let μ be the uniform distribution on S . Then, Theorem 1.1 in Chandrasekaran *et al.* (2010) implies that μ satisfies Cheeger's isoperimetric inequality with constant $\kappa = \frac{|K_S|}{4D|S|}$, where D is the diameter of S . Therefore, μ also satisfies a Poincaré inequality with constant C/κ^2 for some universal constant $C > 0$ (see e.g. Theorems 1.1 and 1.8 in Alonso-Gutiérrez & Bastero (2015)). Consequently, μ has a Stein kernel by Theorem 2.4 in Courtade *et al.* (2019).

References

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