微分同相群と族のゲージ理論

Hokuto Konno (The University of Tokyo)

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Introduction: 4-dimensional special phenomena

Recall: **Dimension 4 is special!**

An instance of special phenomena in dim 4:

There are (many) compact 4-manifolds (e.g. $K3$ surface) that admit **infinitely** many smooth structures.

Two comparisons:

- dim 4 vs. other dim: This infiniteness never happens in $\dim \neq 4$.
- C^0 vs. C^∞: There is a big difference between the topological and smooth categories in dimension 4.

Upshot: The 4-dim smooth category has specially high "complexity"!

How to detect? ← **Gauge theory**

Introduction: Gauge theory

Idea: Use the moduli space of solutions to a certain non-linear PDE on a 4-manifold X^4 to study the topology of X!

e.g.

- Donaldson (1983) ··· Yang-Mills ASD equation $F_A^+ = 0$
- Witten (1994) ···Seiberg-Witten equations $\begin{cases} F_A^+ = \sigma(\Phi, \Phi), \\ D_A\Phi = 0 \end{cases}$

How to use PDE? Typical way: "Count" the the moduli space to get a numerical invariant to distinguish smooth structures.

Introduction: from manifolds to diffeomorphisms

Given a smooth manifold X, the topological group called **diffeomorphism group**

$$\mathrm{Diff}(X) = \{f : X \to X \mid f \text{ is a diffeomorphism}\}$$

is a natural object to study (automorphism group of a manifold).

	Classification of manifolds	**Diffeomorphism group**
dim > 4	Kirby-Siebenmann theory (announced in 1969) based on surgery	Major developments (later) in the last two decades by Galatius, Randal-Williams···
dim = 4	Gauge theory (since Donaldson in 1983)	**Gauge theory for families**

Introduction: Gauge theory for families

Basic idea of gauge theory for families:

1. Given a smooth fiber bundle $E \to B$ with smooth 4-manifold fiber X, consider gauge-theoretic PDEs along fibers.
2. Count the "parameterized moduli space" (i.e. the union of moduli spaces on fibers) over B and get an invariant of E.
3. Distinguish smooth fiber bundles E's by this invariant

→ Give "lower bounds" on complexity of $\mathrm{Diff}(X)$, which is used to define fiber bundles with fiber X.

History: Ruberman (1998) gave the 1st application of gauge theory for families to topology. After that, sporadic for a while: Ruberman (1998–2001), Nakamura (2003, 2010). However, it has been actively studied in recent years.

Introduction: Special phenomena for $\mathrm{Diff}(X^4)$

Question 1: Discover new special phenomena in dimension 4 by studying **diffeomorphism groups** $\mathrm{Diff}(X)$ of 4-manifolds X.

Question 2: Study comparison $\mathrm{Homeo}(X^4)$ vs. $\mathrm{Diff}(X^4)$.

Using gauge theory for families, we give several answers to these questions:

Topics:

(1) Homological instability

(2) Infiniteness of Diff

(3) Exotic diffeomorphisms

Plan of the talk

(1) Homological instability

(2) Infiniteness of Diff

(3) Exotic diffeomorphisms

(4) Other topics and prospects

$B\mathrm{Diff}(X)$: Moduli space of manifolds

For a smooth manifold X, the classifying space $B\mathrm{Diff}(X)$ of $\mathrm{Diff}(X)$ is called the **moduli space of manifolds** (diffeomorphic to X).

Given a ("good") space B,

$$\frac{\mathrm{Map}(B, B\mathrm{Diff}(X))}{\text{(homotopic)}} \overset{1:1}{\leftrightarrow} \{\text{fiber bundles with fiber } X \text{ over } B\}/\cong .$$

An important invariant: $H^*(B\mathrm{Diff}(X)) \overset{1:1}{\leftrightarrow} \{\text{characteristic classes of } X\text{-bundles}\}$

Basic problem:

Compute $H^*(B\mathrm{Diff}(X))$ (or $H_*(B\mathrm{Diff}(X))$)

This is very hard: not solved even when X are surfaces.

But in (even) $\dim \neq 4$, this is solvable after enough **stabilizations**.

Homological stability in $\dim \neq 4$

Stabilization map: For a manifold W^{2n} with $\partial W \neq \varnothing$, set

$\mathrm{Diff}_\partial(W) := \{f \in \mathrm{Diff}(W) \mid f$ is id near $\partial W\}$. We can define a map

$s : \mathrm{Diff}_\partial(W) \hookrightarrow \mathrm{Diff}_\partial(W \# S^n \times S^n)$ (extend by $\mathrm{id}_{S^n \times S^n}$).

Theorem (Harer (1985) for $\dim = 2$, Galatius and Randal-Williams (2018) for $\dim > 4$):

Let W be a simply-connected compact smooth manifold of dimension $2n \neq 4$, and let $k \geq 0$. Then the stabilization maps

$$s_* : H_k(B\mathrm{Diff}_\partial(W \#_N S^n \times S^n); \mathbb{Z}) \to H_k(B\mathrm{Diff}_\partial(W \#_{N+1} S^n \times S^n); \mathbb{Z})$$

are **isomorphic** for all $N \gg k$.

Homological stability in $\dim \neq 4$

Theorem (Harer (1985) for $\dim = 2$, Galatius and Randal-Williams (2018) for $\dim > 4$):

Let W be a simply-connected compact smooth manifold of dimension $2n \neq 4$, and let $k \geq 0$. Then the stabilization maps

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are **isomorphic** for all $N \gg k$.

Thus $H_k(B\mathrm{Diff}_\partial(W\#_N S^n \times S^n))$ for $N \gg k$ is identified with the **stable homology**

$\lim_{N\to+\infty} H_k(B\mathrm{Diff}_\partial(W\#_N S^n \times S^n))$, which is computed by homotopy theory:

$\lim_{N\to+\infty} H_k(B\mathrm{Diff}_\partial(W\#_N S^n \times S^n); \mathbb{Q})$ is generated by MMM classes (Mumford conjecture)

(Madsen and Weiss in $\dim = 2$, Galatius and Randal-Williams in other even dimensions)

Homological **instability** in $\dim = 4$

Focus on punctured manifolds: $W = \mathring{X} = X \backslash \mathrm{Int}(D^4)$ for closed 4-manifolds X.

Theorem (K.-Lin (2022)):

Let X be a simply-connected closed smooth 4-manifold, and let $k > 0$. Then there exists a sequence $0 < N_1 < N_2 < \cdots \to +\infty$ such that, for each N_i, the stabilization map

$$s_* : H_k(B\mathrm{Diff}_\partial(\mathring{X} \#_{N_i} S^2 \times S^2); \mathbb{Z}) \to H_k(B\mathrm{Diff}_\partial(\mathring{X} \#_{N_i+1} S^2 \times S^2); \mathbb{Z})$$

is **not isomorphic**.

Thus, unlike $\dim \neq 4$, the homology of the moduli space never stabilizes by $\# S^2 \times S^2$!

The theorem suggests that the computation of $H_k(B\mathrm{Diff}(X^4))$ appears to be beyond the scope of homotopy theory.

Homological instability in $\dim = 4$: proof

Tool: Gauge-theoretic characteristic class:

$$\mathbb{SW}^k(X) \in H^k(B\mathrm{Diff}^+(X^4); \mathbb{Z}/2)$$ (Diff^+: orientation-preserving).

This is a variant of the characteristic class by K. (2018) with the idea of a numerical families Seiberg-Witten invariant by Ruberman (2001).

Idea of definition: Count solutions to families of Seiberg-Witten equations along the universal bundle $E\mathrm{Diff}^+(X) \times_{\mathrm{Diff}^+(X)} X \to B\mathrm{Diff}^+(X)$ over each k-cell of $B\mathrm{Diff}^+(X)$.

Property: $\mathbb{SW}^k(X)$ is an **unstable** characteristic class (i.e. the pull-back of $\mathbb{SW}^k(X)$ under the stabilization map is zero).

⇒ $\mathbb{SW}^k(X)$ captures certain complexity (instability) of $H_*(B\mathrm{Diff}(X))$

Compute $\mathbb{SW}^k(X)$ for a concrete family: Key is a gluing theorem for families by Baraglia-K. (2018) (geometric analysis).

Plan of the talk

(1) Homological instability

(2) Infiniteness of Diff

(3) Exotic diffeomorphisms

(4) Other topics and prospects

Fineteness of Diff in $\dim \neq 4$

For a manifold X, $\pi_k(\mathrm{Diff}(X))$ and $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ are typically infinite groups.

Question: Are $\pi_k(\mathrm{Diff}(X))$ and/or $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ finitely generated?

Conjecture:
Let X be a closed smooth manifold X with finite $\pi_1(X)$. If $\dim X \neq 4$, $\pi_k(\mathrm{Diff}(X))$ and $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ are finitely generated for all degrees k.

Theorem (Bustamante-Krannich-Kupers (2021)):
The above conjecture is true if $\dim X =$ even > 4.

Question:
Is there a counterexample to the 4-dimensional analog of Conjecture?

(Diff-analog of infiniteness of smooth structures on 4-manifolds?)

Infinite generation of $\pi_k(\mathrm{Diff}(X^4))$ for $k > 0$

Theorem (Baraglia (2021)):

$\pi_1(\mathrm{Diff}(K3))$ is not finitely generated.

Later, Lin generalized this result to many 4-manifolds, including all elliptic surfaces and complete intersections.

Theorem (Auckly-Ruberman (2025)):

For every $k > 0$, $\exists$ closed smooth 4-manifold X^4 such that $\pi_k(\mathrm{Diff}(X))$ is not finitely generated.

Anything else?

We shall consider:

(1) $\pi_0(\mathrm{Diff}(X))$ (mapping class group)

(2) $H_k(B\mathrm{Diff}(X))$ for all $k > 0$.

Finete generation of $\pi_0(\mathrm{Diff}(X))$ in $\dim \neq 4$

Theorem (Sullivan (1977)):
Let X be a simply-connected closed smooth manifold of $\dim > 4$. Then $\pi_0(\mathrm{Diff}(X))$ is finitely generated.

Remark: One cannot drop the simple-connectivity.
e.g. $\pi_0(\mathrm{Diff}(T^n))$ is not finitely generated for $n > 4$ (Hatcher (1978)).

Also in dimension 4, it is known $\exists X^4$ with $\pi_1(X) \neq 1$ s.t. $\pi_0(\mathrm{Diff}(X))$ is not finitely generated (Budney-Gabai (2019)/Watanabe (2020)).

Infinite generation of $\pi_0(\mathrm{Diff}(X^4))$

Summary in dim $\neq 4$: $\pi_0(\mathrm{Diff}(X))$ is finitely generated for every simply-connected X if $\dim X \neq 4$.

Theorem (K. (2023)/Baraglia (2023)):
There exist simply-connected closed smooth 4-manifolds X such that $\pi_0(\mathrm{Diff}(X))$ are not finitely generated.

e.g. For $X = E(n) \# S^2 \times S^2$ with $n \geq 2$, $\pi_0(\mathrm{Diff}(X))$ is not finitely generated.

Infinite generation of $\pi_0(\mathrm{Diff}(X^4))$

Theorem (K. (2023)/Baraglia (2023)):
There exist simply-connected closed smooth 4-manifolds X such that $\pi_0(\mathrm{Diff}(X))$ are not finitely generated.

Remark: Freedman and Quinn's result (with a recent correction by Gabai-Gay-Hartman-Krushkal-Powell), $\pi_0(\mathrm{Homeo}(X))$ is finitely generated for a simply-connected closed topological 4-manifold X.

→ Infinite generation of mapping class group (under $\pi_1 = 1$) is a special phenomenon of the 4-dim & smooth category.

Infinite generation of $H_k(B\mathrm{Diff}(X^4))$

Theorem (K. (2023)): For every $k \geq 1$, there exist simply-connected closed smooth 4-manifolds X such that $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ are **not** finitely generated.

i.e. Families of 4-manifolds can have particularly many characteristic classes.

Remark: Theorem for $k=1$ implies that $\pi_0(\mathrm{Diff}(X))$ is infinitely generated.

e.g. For $X = E(n)\#_k S^2 \times S^2$ with $n \geq 2$, $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ is not finitely generated.

Infinite generation of $H_k(B\mathrm{Diff}(X^4))$: Proof

Theorem (K. (2023)): For every $k \geq 1$, there exist simply-connected closed smooth 4-manifolds X such that $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ are **not** finitely generated.

Gauge-theoretic characteristic class again:

1. Define infinitely many characteristic classes

$$\mathbb{SW}^k(X,\mathcal{S}) \in H^k(B\mathrm{Diff}^+(X);\mathbb{Z}/2),$$

indexed by $\mathcal{S}$ (which related to spin^c structures on X).

2. Show the linear independence of $\mathbb{SW}^k(X,\mathcal{S})$'s by evaluating infinitely many (concrete) fiber bundles with fiber X.

⇒ $\mathbb{SW}^k(X,\mathcal{S})$ captures certain complexity (infiniteness) of $H_*(B\mathrm{Diff}(X))$.

Plan of the talk

(1) Homological instability

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Exotic diffeomorphism

Definition: Given a smooth manifold X, a diffeomorphism $f : X \to X$ is called an exotic diffeomorphism if f is topologically isotopic to the identity but smoothly not.

In other words, f is exotic if f gives a non-trivial element of

$$\ker(\pi_0(\mathrm{Diff}(X)) \to \pi_0(\mathrm{Homeo}(X))).$$

Remark: If $\dim X \leq 3$, $\mathrm{Diff}(X) \hookrightarrow \mathrm{Homeo}(X)$ is a weak homotopy equivalence.

So there is no exotic diffeomorphism of X.

First examples in 4D···Ruberman (1998):

This is the first topological application of gauge theory for families too.

Dehn twists on 4-manifolds

A 4-dimensional analog of **Dehn twists** often turns out to give a natural example of an exotic diffeomorphism.

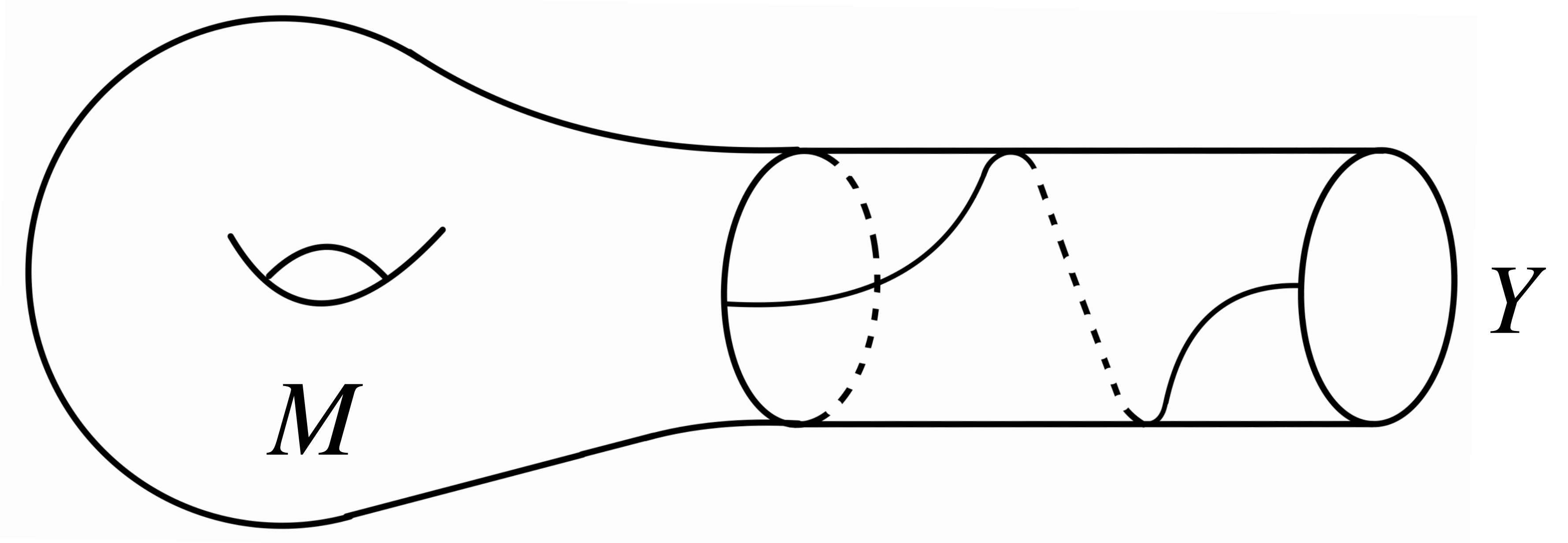
Dehn twist:

For a (3-)manifold Y with a loop $\phi : S^1 \to \mathrm{Diff}(Y)$ based at id_Y, we define a diffeomorphism rel boundary called the Dehn twist on $Y \times [0,1]$:

$$Y \times [0,1] \to Y \times [0,1] \quad ; \quad (y, t) \mapsto (\phi(t) \cdot y, t).$$

When $Y = \partial M^4$, we call the Dehn twist near the boundary ($\in \mathrm{Diff}_\partial(M)$) the **boundary Dehn twist.**

e.g. Y^3 is said to be a **Seifert fibered** 3-manifold if Y admits an S^1-action without fixed-point. Then we can consider the Dehn twist along Y.

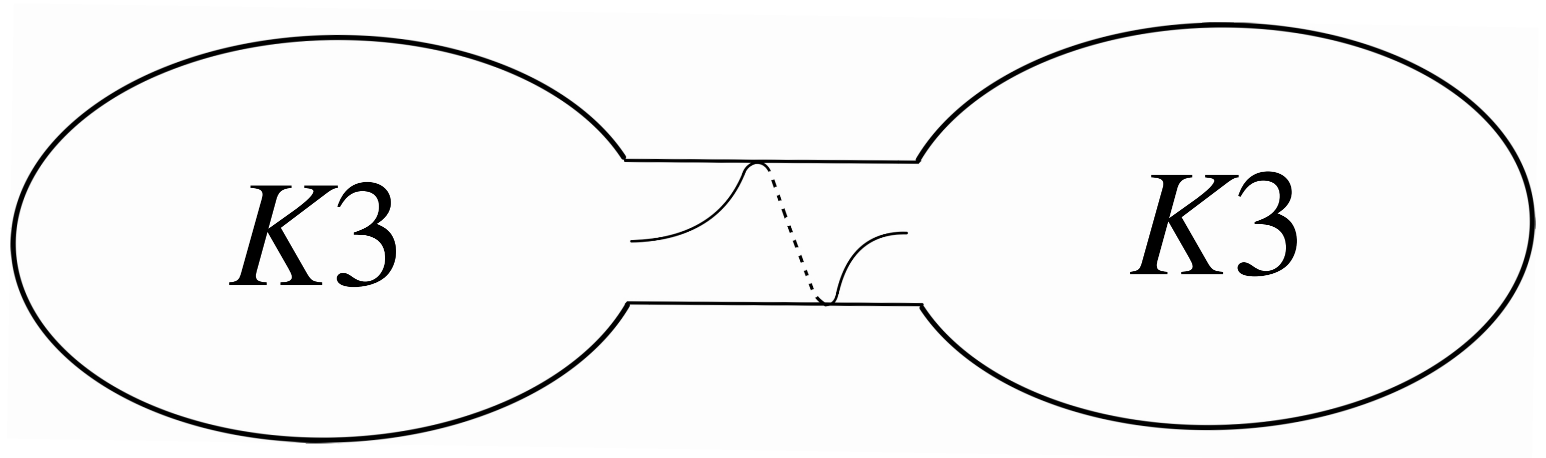


Dehn twists on 4-manifolds

Many approaches to 4-dimensional Dehn twists from families Seiberg-Witten theory appeared since 2020. They often give examples of exotic diffeomorphisms.

- Twists along S^3:

  Kronheimer-Mrowka ($K3 \# K3$), Lin···



- Twists along Seifert fibered 3-manifolds:

  K.-Mallick-Taniguchi, K.-Lin-Mukherjee-Muñoz-Echániz, Kang-Park-Taniguchi, Miyazawa···

Further: The study of Dehn twists also gives 4-dimensional special phenomena.

$\mathrm{Diff}(D^n)$ vs. $\mathrm{Diff}(\mathrm{contractible})$

Given an embedding of a disk $D^n \hookrightarrow W^n$ into an n-manifold W, it induces a map

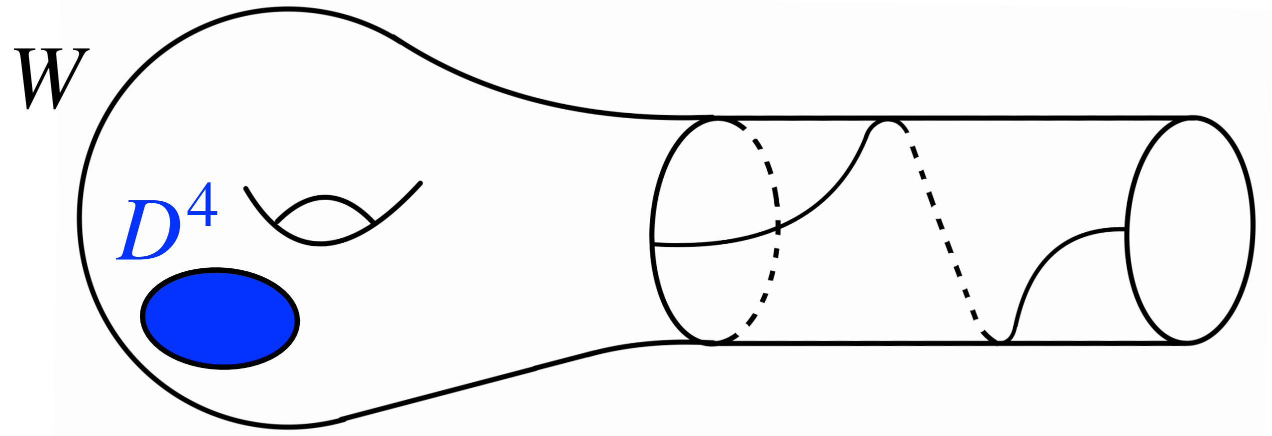
$i : \mathrm{Diff}_\partial(D^n) \hookrightarrow \mathrm{Diff}_\partial(W)$ (extend by id).

Theorem (Galatius—Randal-Williams (2023)/Krannich—Kupers (2024)):
Let W be a contractible compact smooth manifold of $\dim = n \geq 5$. Then, for any $D^n \hookrightarrow W$, the map $i : \mathrm{Diff}_\partial(D^n) \hookrightarrow \mathrm{Diff}_\partial(W)$ is a weak homotopy equivalence.

Namely, $\mathrm{Diff}_\partial(\mathrm{contractible})$ is as simple as $\mathrm{Diff}_\partial(D^n)$. This fails in 4D:

Theorem (Krushkal—Mukherjee—Powell—Warren (2024)/K.—Lin—Mukherjee—Muñoz-Echániz (2024)):
There exists a contractible compact smooth 4-manifold W such that $i_* : \pi_0(\mathrm{Diff}_\partial(D^4)) \to \pi_0(\mathrm{Diff}_\partial(W))$ is not surjective for any $D^4 \hookrightarrow W$.

In our work (KLMME), the boundary Dehn twist along some Seifert 3-manifold gives a mapping class not coming from D^4.



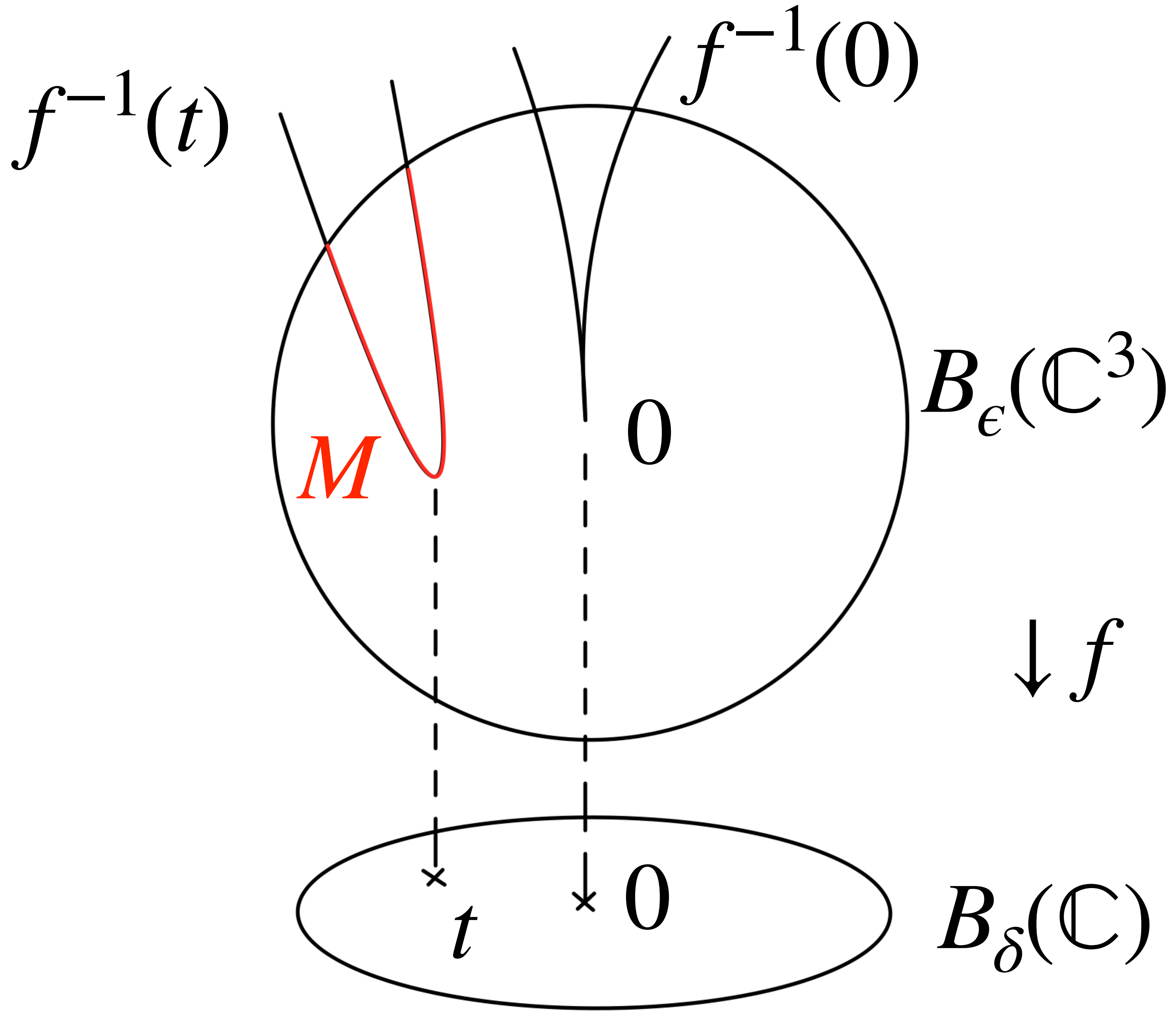
Milnor fibration

Milnor fibration and Milnor fiber: Let $f:\mathbb{C}^3\to\mathbb{C}$ be a polynomial with isolated singularity at $0\in\mathbb{C}^3$ with $f(0)=0$.

$$f:f^{-1}(B_\delta(\mathbb{C})\setminus\{0\})\cap B_\epsilon(\mathbb{C}^3)\to B_\delta(\mathbb{C})\setminus\{0\}\quad(1\gg\epsilon\gg\delta>0)$$

is a fiber bundle with trivialized boundary family (Milnor's fibration theorem). This fibration is called the Milnor fibration and its fiber M is called the Milnor fiber of the singularity.

Monodromy: The most basic invariant of the Milnor fibration is the **monodromy** $\mu\in\pi_0(\mathrm{Diff}_\partial(M))$. The action of μ on $H_2(M;\mathbb{Z})$ is classically studied, but not much studied as an element of $\pi_0(\mathrm{Diff}_\partial(M))$.

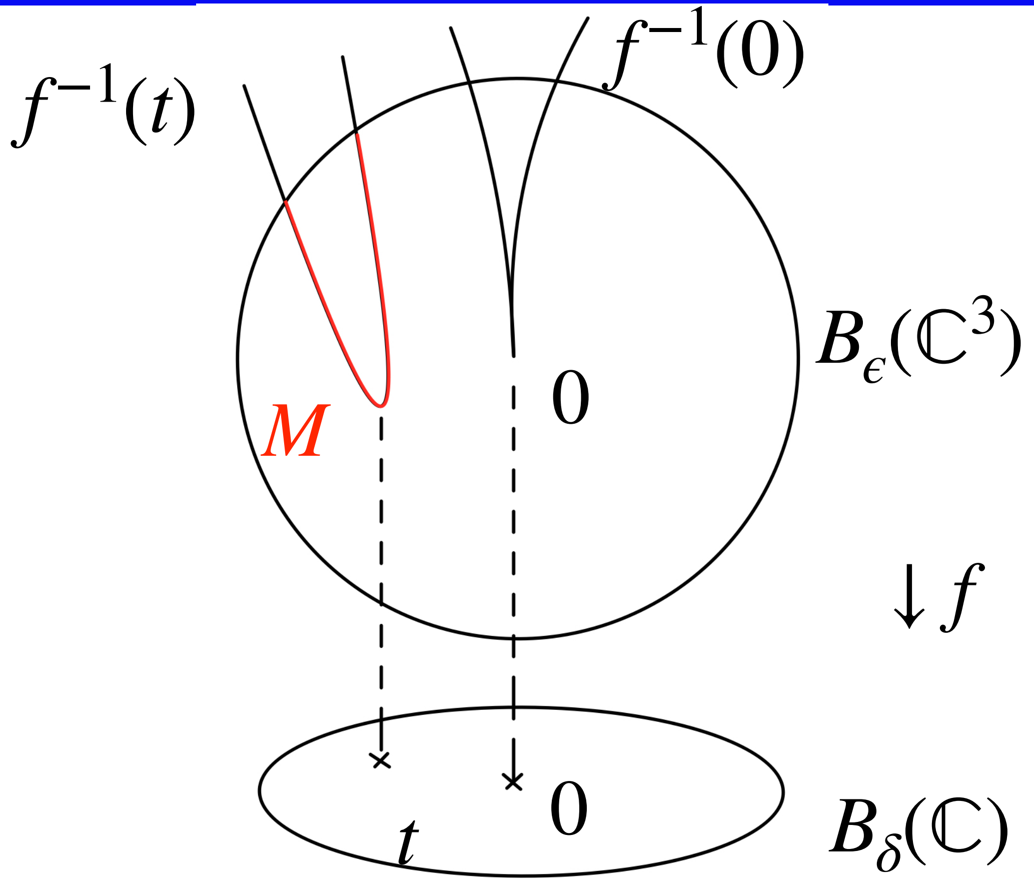


Monodromy of a Milnor fibration

Suppose f is weighted homogeneous, i.e.

$$f(\lambda^{w_1}z_1, \lambda^{w_2}z_2, \lambda^{w_3}z_3) = \lambda^d f(z_1, z_2, z_3) \text{ for } \lambda \in \mathbb{C}^* \ (\exists d, w_1, w_2, w_3 > 0)$$

<u>e.g.</u> Brieskorn singularity, ADE singularity



<u>Theorem (K.—Lin—Mukherjee—Muñoz-Echániz (2024)):</u>
Suppose $f : \mathbb{C}^3 \to \mathbb{C}$ is a weighted homogeneous isolated singularity. Then the monodromy μ of the Milnor fibration for f has finite order in $\pi_0(\mathrm{Diff}_\partial(M))$ if and only if f is an ADE singularity.

"If part" is due to Brieskorn (1971). The theorem follows from that $(\text{monodromy})^d$ = boundary Dehn twist for weighted homogeneous f, and the Dehn twist turns out to be an infinite order (and exotic!)

Monodromy of a Milnor fibration

Theorem (K.—Lin—Mukherjee—Muñoz-Echániz (2024)):
Suppose $f: \mathbb{C}^3 \to \mathbb{C}$ is a weighted homogeneous isolated singularity. Then the monodromy μ of the Milnor fibration for f has finite order in $\pi_0(\mathrm{Diff}_\partial(M))$ if and only if f is an ADE singularity.

Namely, μ has **infinite order in** $\pi_0(\mathrm{Diff}_\partial(M))$, except for the ADEs.

Remark: Under the weighted homogeneous assumption, the monodromy μ is finite order for the topological category and for the higher-dimensional Milnor fibers under a mild assumption ("the link is a homology sphere").

→ A natural diffeomorphism (monodromy of a Milnor fibration) also gives a special phenomenon of the 4-dim & smooth category!

Exotic diffeomorphism of closed 4-manifolds

Question: Does a Dehn twist give an exotic diffeomorphism of a closed 4-manifold?

Answer (so far): Yes, but not of an "interesting" closed 4-manifold.

Interesting/important closed 4-manifolds:

Typically, Kähler/complex surfaces, symplectic 4-manifolds.

After blowing-down, such a 4-manifold is **irreducible**, i.e. $\nexists$ non-trivial connected sum decomposition ("building block" of 4D topology).

Question: Can an irreducible 4-manifold admit an exotic diffeomorphism?

Theorem (Baraglia-K. (2024)): Yes, many minimal complex surfaces (elliptic surfaces/complete intersections) can admit exotic diffeomorphisms.

Proof: A constraint on smooth families of 4-manifolds from families SW theory (Baraglia-K, 2022) and the families index theorem.

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Constraints on smooth families of 4-manifolds

Big source for application⋯Constraints on smooth families of 4-manifolds from families Seiberg-Witten theory (Kato-K.-Nakamura (2019), Baraglia-K. (2019), Baraglia (2019), K.-Taniguchi (2020)).

e.g. Baraglia's result is a family version of Donaldson's diagonalization

A typical application⋯ Detect a non-smoothable topological family of 4-manifolds, i.e. a fiber bundle with structure group $\mathrm{Homeo}(X^4)$ that does not reduce to $\mathrm{Diff}(X^4)$.

$$\begin{array}{ccc} & \nexists & B\mathrm{Diff}(X) \\ & \nearrow & \downarrow \\ B & \to & B\mathrm{Homeo}(X) \end{array}$$

This idea can also be used to detect non-smoothable group actions on 4-manifolds via the Borel construction (Nakamura (2003, 2010), Baraglia (2019)).

Secondary invariant

Another type of application···given a certain vanishing theorem for solutions to the Seiberg-Witten equations, and we could get a secondary families invariant to study the space of "vanishing reasons".

This type includes:

1. Configurations of surfaces in 4-manifolds (K. (2016, 2022))
2. Exotic embeddings of surfaces/3-manifolds into 4-manifolds (Baraglia (2020), Iida-K.-Mukherjee-Taniguchi (2022), K.-Mukherjee-Taniguchi (2022), Auckly-Ruberman (2025)···)
3. Space of positive scalar curvature metrics of 4-manifolds (Ruberman (2001), K. (2019), Auckly-Ruberman (2025))
4. $\mathrm{Symp}(X^4, \omega)$ vs. $\mathrm{Diff}(X^4)$ (Kronheimer (1997), Smirnov (2020, 2022), Lin (2022))

Kontsevich characteristic classes

Another important recent advance in the study of $\mathrm{Diff}(X^4)$ was initiated by:

Theorem (Watanabe (2018)):

$\mathrm{Diff}(S^4)$ is not homotopy equivalent to $O(5)$.

Watanabe's proof uses Kontsevich characteristic classes based on configuration space integral (totally different method from gauge theory).

Theorem (Lin-Xie (2023)):
Kontsevich characteristic classes are well-defined on the classifying space $\mathscr{M}^{fs}(X)$ of "formally smooth" families of 4-manifolds (i.e. topological families equipped with linear structures on vertical tangent microbundles).

$$B\mathrm{Diff}(X^4) = \mathscr{M}^s(X) \xrightarrow{\text{Detected by gauge theory}} \mathscr{M}^{fs}(X) \xrightarrow{\text{Detected by Kontsevich char.}} \mathscr{M}^t(X) = B\mathrm{Homeo}(X^4)$$

Smooth moduli space — Special to dim=4 — Formally smooth moduli space — Similar to dim>4 — Topological moduli space

Problems and Prospects

1. Ring structure of $H^*(B\mathrm{Diff}(X^4))$?

2. Instability and/or infiniteness of $H_*(B\mathrm{Diff}(X^4); \mathbb{Q})$? (cf. Advances in $\dim \neq 4$)

3. Relation between exotic/complex/symplectic structures on 4-manifolds and their diffeomorphism groups? (cf. Exotic diffeo. of irreducible 4-manifolds)

4. Is the algebraic (as opposed to topological) structure of $\mathrm{Diff}(X^4)$ special?

Thank you for your attention!