QUANTUM AND HOMOLOGICAL REPRESENTATIONS OF BRAID GROUPS

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Abstract. By means of a description of the solutions of the KZ equation using hypergeometric integrals we show that the homological representations of the braid groups studied by Lawrence, Krammer and Bigelow are equivalent at generic complex values to the monodromy of the KZ equation with values in the space of null vectors in the tensor product of Verma modules of $sl_2(\mathbf{C})$.

1. Introduction

The purpose of this paper is to clarify the relation between the Lawrence-Krammer-Bigelow (LKB) representations of the braid groups and the monodromy representations of the Knizhnik-Zamolodchikov (KZ) connection.

The LKB representations of the braid groups were studied by Lawrence [13] in relation with Hecke algebra representations of the braid groups and were extensively investigated by Bigelow [3] and Krammer [12].

On the other hand, it was shown by Schectman-Varchenko [16] and others that the solutions of the KZ equation are expressed by hypergeometric integrals. From the expression of the integrals over homology cycles with coefficients in local systems it is clear that the LKB representation can be expressed as the monodromy representation of the KZ equation, however, I think it is worthwhile to state a precise relation between them.

There are two parameters λ and κ, which are related to the highest weight and the KZ connection respectively. We consider the KZ equation with values in the space of null vectors in the tensor product of Verma modules of $sl_2(\mathbf{C})$ and show that a specialization of the LKB representation is equivalent to the monodromy representation of such KZ equation for a generic parameter λ and κ. A complete statement is given in Theorem 6.1. We describe a sufficient condition for the parameters to be generic so that the statement of the theorem holds. This result was announced in [11] and the present paper is a more detailed account for this subject.

There is other approach due to Marin [14] expressing representations of the braid groups and their generalizations such as Artin groups as the monodromy of integrable connections by an infinitesimal method. Our approach depends on integral representations of the solutions of the KZ equation and is different from Marin's method.

In this article we will treat the case when the parameters are generic, but the case of special parameters are important from the viewpoint of conformal field theory (see [7], [17] and [19]). We will deal with this subject in a separate paper.

The paper is organized in the following way. In Section 2 we recall basic definitions for the LKB representations. In Section 3 we deal with the homology of local systems over the complement of a discriminantal arrangement. We recall the definition of the KZ equation in Section 4 and describe its solutions by hypergeometric integrals in Section 5. Section 6 is devoted to the statement of the main theorem and its proof.

2. Lawrence-Krammer-Bigelow representations

We denote by B_n the braid group with n strands. We fix a positive integer n and a set of distinct n points in $\mathbf{R}^2$ as

$$Q = \{(1,0), \cdots, (n,0)\},$$

where we set $p_\ell = (\ell, 0)$, $\ell = 1, \cdots, n$. We take a 2-dimensional disk in $\mathbf{R}^2$ containing Q in the interior. We fix a positive integer m and consider the configuration space of ordered distinct m points in $\Sigma = D \setminus Q$ defined by

$$\mathcal{F}_m(\Sigma) = \{(t_1, \cdots, t_m) \in \Sigma \ ; \ t_i \neq t_j \text{ if } i \neq j\},$$

which is also denoted by $\mathcal{F}_{n,m}(D)$. The symmetric group $\mathfrak{S}_m$ acts freely on $\mathcal{F}_m(\Sigma)$ by the permutations of distinct m points. The quotient space of $\mathcal{F}_m(\Sigma)$ by this action is by definition the configuration space of unordered distinct m points in Σ and is denoted by $\mathcal{C}_m(\Sigma)$. We also denote this configuration space by $\mathcal{C}_{n,m}(D)$.

In the original papers by Bigelow [3], [4] and by Krammer [12] the case $m = 2$ was extensively studied, but for our purpose it is convenient to consider the case when m is an arbitrary positive integer such that $m \geq 2$.

We identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$. The quotient space $\mathbf{C}^m/\mathfrak{S}_m$ defined by the action of $\mathfrak{S}_m$ by the permutations of coordinates is analytically isomorphic to $\mathbf{C}^m$ by means of the elementary symmetric polynomials. Now the image of the hyperplanes defined by $t_i = p_\ell$, $\ell = 1, \cdots, n$, and the diagonal hyperplanes $t_i = t_j$, $1 \leq i \leq j \leq m$, are complex codimension one irreducible subvarieties of the quotient space $D^m/\mathfrak{S}_m$. This allows us to give a description of the first homology group of $\mathcal{C}_{n,m}(D)$ as

$$H_1(\mathcal{C}_{n,m}(D); \mathbf{Z}) \cong \mathbf{Z}^{\oplus n} \oplus \mathbf{Z} \tag{2.1}$$

where the first n components correspond to meridians of the images of hyperplanes $t_i = p_\ell$, $\ell = 1, \cdots, n$, and the last component corresponds to the meridian of the image of the diagonal hyperplanes $t_i = t_j$, $1 \leq i \leq j \leq m$, namely, the discriminant set. We consider the homomorphism

$$\alpha : H_1(\mathcal{C}_{n,m}(D); \mathbf{Z}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \tag{2.2}$$

defined by $\alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y)$. Composing with the abelianization map $\pi_1(\mathcal{C}_{n,m}(D), x_0) \to H_1(\mathcal{C}_{n,m}(D); \mathbf{Z})$, we obtain the homomorphism

$$\beta : \pi_1(\mathcal{C}_{n,m}(D), x_0) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}. \tag{2.3}$$

Let $\pi : \widetilde{\mathcal{C}}_{n,m}(D) \to \mathcal{C}_{n,m}(D)$ be the covering corresponding to $\operatorname{Ker} \beta$. Now the group $\mathbf{Z} \oplus \mathbf{Z}$ acts as the deck transformations of the covering π and the homology group $H_*(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$ is considered to be a $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$-module. Here $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$ stands for the group ring of $\mathbf{Z} \oplus \mathbf{Z}$. We express $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$ as the ring of Laurent polynomials $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$. We consider the homology group

$$H_{n,m} = H_m(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$$

as an R-module by the action of the deck transformations.

As is explained in the case of $m = 2$ in [3] it can be shown that $H_{n,m}$ is a free R-module of rank

$$d_{n,m} = \begin{pmatrix} m+n-2 \\ m \end{pmatrix}. \tag{2.4}$$

A basis of $H_{n,m}$ as a free R-module is discussed in relation with the homology of local systems in the next sections. Let $\mathcal{M}(D, Q)$ denote the mapping class group of the pair (D, Q), which consists of the isotopy classes of homeomorphisms of D fixing Q setwise and fixing the boundary ∂D pointwise. The braid group B_n is naturally isomorphic to the mapping class group $\mathcal{M}(D, Q)$. Now a homeomorphism f representing a class in $\mathcal{M}(D, Q)$ induces a homeomorphism $\tilde{f} : \mathcal{C}_{n,m}(D) \to \mathcal{C}_{n,m}(D)$, which is uniquely lifted to a homeomorphism of $\widetilde{\mathcal{C}}_{n,m}(D)$. This homeomorpshim commutes with the deck transformations.

Therefore, for $m \geq 2$ we obtain a representation of the braid group

$$\rho_{n,m} : B_n \longrightarrow \operatorname{Aut}_R H_{n,m} \tag{2.5}$$

which is called the homological representation of the braid group or the Lawrence-Krammer-Bigelow (LKB) representation. Let us remark that in the case $m = 1$ the above construction gives the reduced Burau representation over $\mathbf{Z}[q^{\pm 1}]$.

3. Discriminantal arrangements

First, we recall some basic definition for local systems. Let M be a smooth manifold and V a complex vector space. Given a linear representation of the fundamental group

$$r : \pi_1(M, x_0) \longrightarrow GL(V)$$

there is an associated flat vector bundle E over M. The local system $\mathcal{L}$ associated to the representation r is the sheaf of horizontal sections of the flat bundle E. Let $\pi : \widetilde{M} \to M$ be the universal covering. We denote by $\mathbf{Z}\pi_1$ the group ring of the fundamental group $\pi_1(M, x_0)$. We consider the chain complex

$$C_*(\widetilde{M}) \otimes_{\mathbf{Z}\pi_1} V$$

with the boundary map defined by $\partial(c \otimes v) = \partial c \otimes v$. Here $\mathbf{Z}\pi_1$ acts on $C_*(\widetilde{M})$ via the deck transformations and on V via the representation r. The homology of this chain complex is called the homology of M with coefficients in the local system $\mathcal{L}$ and is denoted by $H_*(M, \mathcal{L})$.

Let $\mathcal{A} = \{H_1, \cdots, H_N\}$ be a set of affine hyperplanes in the complex vector space $\mathbf{C}^n$. We call the set $\mathcal{A}$ a complex hyperplane arrangement. We consider the complement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H.$$

Let $\mathcal{L}$ be a complex rank one local system over $M(\mathcal{A})$ associated with a representation of the fundamental group

$$r : \pi_1(M(\mathcal{A}), x_0) \longrightarrow \mathbf{C}^*.$$

We shall investigate the homology of $M(\mathcal{A})$ with coefficients in the local system $\mathcal{L}$. For our purpose the homology of locally finite chains $H_*^{lf}(M(\mathcal{A}), \mathcal{L})$ also plays an important role.

We briefly summarize basic properties of the above homology groups. For a complex hyperplane arrangement $\mathcal{A}$ we choose a smooth compactification $i : M(\mathcal{A}) \longrightarrow X$ with normal crossing divisors. We shall say that the local system $\mathcal{L}$ is generic if and only if there is an isomorphism

$$(3.1) \qquad i_*\mathcal{L} \cong i_!\mathcal{L}$$

holds, where i_* is the direct image and $i_!$ is the extension by 0. This means that the monodromy of $\mathcal{L}$ along any divisor at infinity is not equal to 1. The following theorem was shown in [9].

Theorem 3.1. *If the local system $\mathcal{L}$ is generic in the above sense, then there is an isomorphism*

$$H_*(M(\mathcal{A}), \mathcal{L}) \cong H_*^{lf}(M(\mathcal{A}), \mathcal{L}).$$

Moreover, we have $H_k(M(\mathcal{A}), \mathcal{L}) = 0$ for any $k \neq n$.

Proof. In general we have isomorphisms

$$H^*(X, i_*\mathcal{L}) \cong H^*(M(\mathcal{A}), \mathcal{L}), \quad H^*(X, i_!\mathcal{L}) \cong H_c^*(M(\mathcal{A}), \mathcal{L})$$

where H_c denotes cohomology with compact supports.

There are Poincaré duality isomorphisms:

$$H_k^{lf}(M(\mathcal{A}), \mathcal{L}) \cong H^{2n-k}(M(\mathcal{A}), \mathcal{L})$$
$$H_k(M(\mathcal{A}), \mathcal{L}) \cong H_c^{2n-k}(M(\mathcal{A}), \mathcal{L}).$$

By the hypothesis $i_*\mathcal{L} \cong i_!\mathcal{L}$ we obtain an isomorphism

$$H_k^{lf}(M(\mathcal{A}), \mathcal{L}) \cong H_k(M(\mathcal{A}), \mathcal{L}).$$

It follows from the above Poincaré duality isomorpshims and the fact that $M(\mathcal{A})$ has a homotopy type of a CW complex of dimension at most n we have

$$H_k^{lf}(M(\mathcal{A}), \mathcal{L}) \cong 0, \quad k < n$$
$$H_k(M(\mathcal{A}), \mathcal{L}) \cong 0, \quad k > n.$$

Therefore we obtain $H_k(M(\mathcal{A}), \mathcal{L}) = 0$ for any $k \neq n$. □

Let us consider the configuration space of ordered distinct n points in the complex plane defined by

$$(3.2) \qquad X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j \text{ if } i \neq j\}.$$

The fundamental group of X_n is the pure braid group with n strands denoted by P_n. For a positive integer m we consider the projection map

$$(3.3) \qquad \pi_{n,m} : X_{n+m} \longrightarrow X_n$$

given by $\pi_{n,m}(z_1, \cdots, z_n, t_1, \cdots, t_m) = (z_1, \cdots, z_n)$, which defines a fiber bundle over X_n. For $p \in X_n$ the fiber $\pi_{n,m}^{-1}(p)$ is denoted by $X_{n,m}$. We denote by $(p_1, \cdots, p_n)$ the coordinates for p. Then, $X_{n,m}$ is the complement of hyperplanes defined by

$$(3.4) \qquad t_i = p_\ell, \quad 1 \leq i \leq m, \ 1 \leq \ell \leq n, \quad t_i = t_j, \quad 1 \leq i < j \leq m.$$

Such arrangement of hyperplanes is called a discriminantal arrangement. The symmetric group $\mathfrak{S}_m$ acts on $X_{n,m}$ by the permutations of the coordinates functions $t_1, \cdots, t_m$. We put $Y_{n,m} = X_{n,m}/\mathfrak{S}_m$.

Identifying $\mathbf{R}^2$ with the complex plane $\mathbf{C}$, we have the inclusion map

$$\iota : \mathcal{F}_{n,m}(D) \longrightarrow X_{n,m}, \tag{3.5}$$

which is a homotopy equivalence. By taking the quotient by the action of the symmetric group $\mathfrak{S}_m$, we have the inclusion map

$$\bar{\iota} : \mathcal{C}_{n,m}(D) \longrightarrow Y_{n,m}, \tag{3.6}$$

which is also a homotopy equivalence.

We take $p = (1, 2, \cdots, n)$ as a base point. We consider a local system over $X_{n,m}$ defined in the following way. Let $\xi_{i\ell}$ and η_{ij} be normal loops around the hyperplanes $t_i = p_\ell$ and $t_i = t_j$ respectively. We fix complex numbers α_ℓ, $1 \leq \ell \leq n$, and γ and by the correspondence

$$\xi_{i\ell} \mapsto e^{2\pi\sqrt{-1}\alpha_\ell}, \quad \eta_{ij} \mapsto e^{4\pi\sqrt{-1}\gamma}$$

we obtain the representation

$$r : \pi_1(X_{n,m}, x_0) \longrightarrow \mathbf{C}^*.$$

We denote by $\mathcal{L}$ the associated rank one local system on $X_{n,m}$.

Let us consider the embedding

$$i_0 : X_{n,m} \longrightarrow (\mathbf{C}P^1)^m = \underbrace{\mathbf{C}P^1 \times \cdots \times \mathbf{C}P^1}_{m}. \tag{3.7}$$

Then we take blowing-ups at multiple points $\pi : \widehat{(\mathbf{C}P^1)}^m \longrightarrow (\mathbf{C}P^1)^m$ and obtain a smooth compactification $i : X_{n,m} \to \widehat{(\mathbf{C}P^1)}^m$ with normal crossing divisors. We are able to write down the condition $i_*\mathcal{L} \cong i_!\mathcal{L}$ explicitly by computing the monodromy of the local system $\mathcal{L}$ along divisors at infinity.

Examples. (1) In the case $m = 1$ the local system $\mathcal{L}$ is generic if and only if

$$\alpha_\ell \notin \mathbf{Z}, \quad 1 \leq \ell \leq n, \quad \alpha_1 + \cdots + \alpha_n \notin \mathbf{Z}.$$

(2) In the case $m = 2$ the local system $\mathcal{L}$ is generic if and only if

$$\begin{aligned}
&\alpha_\ell \notin \mathbf{Z}, \quad 1 \leq \ell \leq n, \quad 2\gamma \notin \mathbf{Z}, \\
&2(\alpha_\ell + \gamma) \notin \mathbf{Z}, \quad 1 \leq \ell \leq n, \\
&2(\alpha_1 + \cdots + \alpha_n + \gamma) \notin \mathbf{Z}.
\end{aligned}$$

The local system $\mathcal{L}$ on $X_{n,m}$ is invariant under the action of the symmetric group $\mathfrak{S}_m$ and induces the local system $\overline{\mathcal{L}}$ on $Y_{n,m}$.

We will deal with the case $\alpha_1 = \cdots = \alpha_\ell = \alpha$. In this case we have the following proposition.

Proposition 3.1. *There is an open dense subset V in $\mathbf{C}^2$ such that for $(\alpha, \gamma) \in V$ the associated local system $\overline{\mathcal{L}}$ on $Y_{n,m}$ satisfies*

$$H_*(Y_{n,m}, \overline{\mathcal{L}}) \cong H_*^{lf}(Y_{n,m}, \overline{\mathcal{L}})$$

and $H_k(Y_{n,m}, \overline{\mathcal{L}}) = 0$ for any $k \neq m$. Moreover, we have

$$\dim H_m(Y_{n,m}, \overline{\mathcal{L}}^*) = d_{n,m}, \tag{3.8}$$

where we use the same notation as in equation (2.4) for $d_{n,m}$.

Proof. We see that $Y_{n,m}$ is the complement of hypersurfaces in $\mathbf{C}^m$. We consider the embedding

$$i_0 : Y_{n,m} \longrightarrow \mathcal{S}^m\mathbf{C}P^1 \tag{3.9}$$

where $\mathcal{S}^m\mathbf{C}P^1$ is the symmetric product defined as $(\mathbf{C}P^1)^m/\mathfrak{S}_m$. We observe that $\mathcal{S}^m\mathbf{C}P^1$ is a smooth complex manifold. Now by taking blowing-ups we have a smooth compactification

$$i : Y_{n,m} \longrightarrow \widehat{\mathcal{S}^m\mathbf{C}P^1} \tag{3.10}$$

with normal crossing divisors. Let us remark that the argument of the proof of Theorem 3.1 can be applied to this situation and we have an isomorphism $H_*(Y_{n,m}, \overline{\mathcal{L}}) \cong H_*^{lf}(Y_{n,m}, \overline{\mathcal{L}})$ and the vanishing $H_k(Y_{n,m}, \mathcal{L}) = 0$ for $k \neq m$ if the condition $i_*\overline{\mathcal{L}} \cong i_!\overline{\mathcal{L}}$ is satisfied. Actually, by the Lefschetz hyperplane section theorem it is enough to verify the condition for a generic 2 dimensional section. In this case by expressing the monodromy along divisors with normal crossings at infinity by the parameter (α, γ) we can verify that the condition $i_*\overline{\mathcal{L}} \cong i_!\overline{\mathcal{L}}$ is satisfied for $(\alpha, \gamma) \in \mathbf{C}^2$ in an open dense subset of $\mathbf{C}^2$. The dimension formula for $H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$ follows from the calculation of the Euler-Poincaré characteristic of $Y_{n,m}$. □

Remark. It was shown by Andreotti [1] that the above m-fold symmetric product $\mathcal{S}^m\mathbf{C}P^1$ is actually biholomorphically equivalent to $\mathbf{C}P^m$.

For the purpose of describing the homology group $H_m^{lf}(X_{n,m}, \mathcal{L})$ and $H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}})$ we introduce the following notation. We fix the base point $p = (1, \cdots, n)$. For non-negative integers $m_1, \cdots, m_{n-1}$ satisfying

$$m_1 + \cdots + m_{n-1} = m \tag{3.11}$$

we define a bounded chamber $\Delta_{m_1,\cdots,m_{n-1}}$ in $\mathbf{R}^m$ by

$$\begin{aligned}
&1 < t_1 < \cdots < t_{m_1} < 2 \\
&2 < t_{m_1+1} < \cdots < t_{m_1+m_2} < 3 \\
&\cdots \\
&n-1 < t_{m_1+\cdots+m_{n-2}+1} + \cdots + t_m < n.
\end{aligned}$$

We put $M = (m_1, \cdots, m_{n-1})$ and we write Δ_M for $\Delta_{m_1,\cdots,m_{n-1}}$. We denote by $\overline{\Delta}_M$ the image of Δ_M by the projection map $\pi_{n,m}$. The bounded chamber Δ_M defines a homology class $[\Delta_M] \in H_m^{lf}(X_{n,m}, \mathcal{L})$ and its image $\overline{\Delta}_M$ defines a homology class $[\overline{\Delta}_M] \in H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}})$. We shall show in Section 6 that under certain generic conditions $[\overline{\Delta}_M]$ for $M = (m_1, \cdots, m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$ form a basis of $H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}})$.

As we have shown in Theorem 3.1 there is an isomorphism $H_m(X_{n,m}, \mathcal{L}) \cong H_m^{lf}(X_{n,m}, \mathcal{L})$ if the condition $i_*\mathcal{L} \cong i_!\mathcal{L}$ is satisfied. In this situation we denote by $[\widetilde{\Delta}_M]$ the homology class in $H_m(X_{n,m}, \mathcal{L})$ corresponding to $[\Delta_M]$ in the above isomorphism and call $[\widetilde{\Delta}_M]$ the regularized cycle for $[\Delta_M]$.

Example. Let us consider the case $n = 2, m = 1$. The bounded chamber Δ_1 is the open unit interval $(0, 1)$. We suppose the condition $i_*\mathcal{L} \cong i_!\mathcal{L}$ is satisfied. The

Pochhammer double loop Γ as depicted in Figure 1 is related to $[\widetilde{\Delta}_1]$ by

$$[\widetilde{\Delta}_1] = \frac{1}{(1-e^{2\pi\sqrt{-1}\alpha_1})(1-e^{2\pi\sqrt{-1}\alpha_2})}[\Gamma].$$

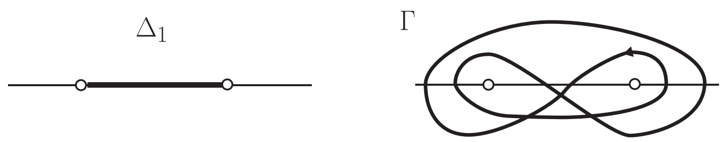

FIGURE 1. Pochhammer double loop

In general regularized cycles can be constructed by means of the boundary of the tubular neighborhood of divisors at infinity. We refer the reader to [2] for more details about this subject.

4. KZ connection

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\{I_\mu\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form. We set $\Omega = \sum_\mu I_\mu \otimes I_\mu$. Let $r_i : \mathfrak{g} \to \mathrm{End}(V_i)$, $1 \leq i \leq n$, be representations of the Lie algebra $\mathfrak{g}$. We denote by Ω_{ij} the action of Ω on the i-th and j-th components of the tensor product $V_1 \otimes \cdots \otimes V_n$. It is known that the Casimir element $c = \sum_\mu I_\mu \cdot I_\mu$ lies in the center of the universal enveloping algebra $U\mathfrak{g}$. Let us denote by $\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ be the coproduct, which is defined to be the algebra homomorphism determined by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$. Since Ω is expressed as $\Omega = \frac{1}{2}(\Delta(c) - c \otimes 1 + 1 \otimes c)$ we have the relation

$$[\Omega, x \otimes 1 + 1 \otimes x] = 0 \tag{4.1}$$

for any $x \in \mathfrak{g}$ in the tensor product $U\mathfrak{g} \otimes U\mathfrak{g}$. By means of the above relation it can be shown that the infinitesimal pure braid relations:

$$[\Omega_{ik}, \Omega_{ij} + \Omega_{jk}] = 0, \quad (i, j, k \text{ distinct}), \tag{4.2}$$

$$[\Omega_{ij}, \Omega_{k\ell}] = 0, \quad (i, j, k, \ell \text{ distinct}) \tag{4.3}$$

hold. Let us briefly explain the reason why we have the above infinitesimal pure braid relations. For the first relation it is enough to show the case $i = 1, j = 3, k = 2$. Since we have

$$[\Omega \otimes 1, (I_\mu \otimes 1 + 1 \otimes I_\mu) \otimes I_\mu] = 0$$

by the equation (4.1) we obtained the desired relation. The equation (4.3) in the infinitesimal pure braid relations is clear from the definition of Ω on the tensor product.

We define the Knizhnik-Zamolodchikov (KZ) connection as the 1-form

$$\omega = \frac{1}{\kappa} \sum_{1 \leq i < j \leq n} \Omega_{ij} d\log(z_i - z_j) \tag{4.4}$$

with values in $\mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ for a non-zero complex parameter κ.

We set $\omega_{ij} = d\log(z_i - z_j)$, $1 \leq i, j \leq n$. It follows from the above infinitesimal pure braid relations among Ω_{ij} together with Arnold's relation

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{k\ell} + \omega_{k\ell} \wedge \omega_{ij} = 0 \tag{4.5}$$

that $\omega \wedge \omega = 0$ holds. This implies that ω defines a flat connection for a trivial vector bundle over the configuration space $X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j \text{ if } i \neq j\}$ with fiber $V_1 \otimes \cdots \otimes V_n$. A horizontal section of the above flat bundle is a solution of the total differential equation

$$d\varphi = \omega\varphi \tag{4.6}$$

for a function $\varphi(z_1, \cdots, z_n)$ with values in $V_1 \otimes \cdots \otimes V_n$. This total differential equation can be expressed as a system of partial differential equations

$$\frac{\partial \varphi}{\partial z_i} = \frac{1}{\kappa} \sum_{j, j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \varphi, \quad 1 \leq i \leq n, \tag{4.7}$$

which is called the KZ equation. The KZ equation was first introduced in [8] as the differential equation satisfied by n-point functions in Wess-Zumino-Witten conformal field theory.

Let $\phi(z_1, \cdots, z_n)$ be the matrix whose columns are linearly independent solutions of the KZ equation. By considering the analytic continuation of the solutions with respect to a loop γ in X_n with base point x_0 we obtain the matrix $\theta(\gamma)$ defined by

$$\phi(z_1, \cdots, z_n) \mapsto \phi(z_1, \cdots, z_n)\theta(\gamma). \tag{4.8}$$

Since the KZ connection ω is flat the matrix $\theta(\gamma)$ depends only on the homotopy class of γ. The fundamental group $\pi_1(X_n, x_0)$ is the pure braid group P_n. As the above holonomy of the connection ω we have a one-parameter family of linear representations of the pure braid group

$$\theta : P_n \to \mathrm{GL}(V_1 \otimes \cdots \otimes V_n). \tag{4.9}$$

The symmetric group $\mathfrak{S}_n$ acts on X_n by the permutations of coordinates. We denote the quotient space $X_n/\mathfrak{S}_n$ by Y_n. The fundamental group of Y_n is the braid group B_n. In the case $V_1 = \cdots = V_n = V$, the symmetric group $\mathfrak{S}_n$ acts diagonally on the trivial vector bundle over X_n with fiber $V^{\otimes n}$ and the connection ω is invariant by this action. Thus we have one-parameter family of linear representations of the braid group

$$\theta : B_n \to \mathrm{GL}(V^{\otimes n}). \tag{4.10}$$

It is known by [6] and [10] that this representation is described by means of quantum groups. We call θ the quantum representation of the braid group.

5. Solutions of KZ equation by hypergeometric integrals

In this section we describe solutions of the KZ equation for the case $\mathfrak{g} = sl_2(\mathbf{C})$ by means of hypergeometric integrals following Schechtman and Varchenko [16]. A description of the solutions of the KZ equation was also given by Date, Jimbo, Matsuo and Miwa [5]. We refer the reader to [2] and [15] for general treatments of hypergeometric integrals.

Let us recall basic facts about the Lie algebra $sl_2(\mathbf{C})$ and its Verma modules. As a complex vector space the Lie algebra $sl_2(\mathbf{C})$ has a basis H, E and F satisfying the relations:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \tag{5.1}$$

For a complex number λ we denote by M_λ the Verma module of $sl_2(\mathbf{C})$ with highest weight λ. Namely, there is a non-zero vector $v_\lambda \in M_\lambda$ called the highest weight vector satisfying

$$Hv_\lambda = \lambda v_\lambda, \ Ev_\lambda = 0 \tag{5.2}$$

and M_λ is spanned by $F^j v_\lambda$, $j \geq 0$. The elements H, E and F act on this basis as

$$\begin{cases} H \cdot F^j v_\lambda = (\lambda - 2j) F^j v_\lambda \\ E \cdot F^j v_\lambda = j(\lambda - j + 1) F^{j-1} v_\lambda \\ F \cdot F^j v_\lambda = F^{j+1} v_\lambda. \end{cases} \tag{5.3}$$

It is known that if $\lambda \in \mathbf{C}$ is not a non-negative integer, then the Verma module M_λ is irreducible.

For $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n$ we put $|\Lambda| = \lambda_1 + \cdots + \lambda_n$ and consider the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$. For a non-negative integer m we define the space of weight vectors with weight $|\Lambda| - 2m$ by

$$W[|\Lambda| - 2m] = \{x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \ ; \ Hx = (|\Lambda| - 2m)x\} \tag{5.4}$$

and consider the space of null vectors defined by

$$N[|\Lambda| - 2m] = \{x \in W[|\Lambda| - 2m] \ ; \ Ex = 0\}. \tag{5.5}$$

The KZ connection ω commutes with the diagonal action of $\mathfrak{g}$ on $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$, hence it acts on the space of null vectors $N[|\Lambda| - 2m]$.

For parameters κ and λ we consider the multi-valued function

$$\Phi_{n,m} = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{2\kappa}} \prod_{1 \leq i \leq m, 1 \leq \ell \leq n} (t_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}} \prod_{1 \leq i < j \leq m} (t_i - t_j)^{\frac{2}{\kappa}} \tag{5.6}$$

defined over X_{n+m}. Let $\mathcal{L}$ denote the local system associated to the multi-valued function Φ. The restriction of $\mathcal{L}$ on the fiber $X_{n,m}$ is the local system associated with the parameters

$$\alpha_\ell = -\frac{\lambda_\ell}{\kappa}, \quad 1 \leq \ell \leq n, \quad \gamma = \frac{1}{\kappa} \tag{5.7}$$

in the notation of Section 3.

The symmetric group $\mathfrak{S}_m$ acts on $X_{n,m}$ by the permutations of the coordinate functions $t_1, \cdots, t_m$. The function $\Phi_{n,m}$ is invariant by the action of $\mathfrak{S}_m$. The local system $\mathcal{L}$ over $X_{n,m}$ defines a local system on $Y_{n,m}$, which we denote by $\overline{\mathcal{L}}$. The local system dual to $\mathcal{L}$ is denoted by $\mathcal{L}^*$.

We put $v = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}$ and for $J = (j_1, \cdots, j_n)$ set $F^J v = F^{j_1} v_{\lambda_1} \otimes \cdots \otimes F^{j_n} v_{\lambda_n}$, where $j_1, \cdots, j_n$ are non-negative integers. The weight space $W[|\Lambda| - 2m]$ has a basis $F^J v$ for each J with $|J| = j_1 + \cdots + j_n = m$. For the sequence of integers $(i_1, \cdots, i_m) = (\underbrace{1, \cdots, 1}_{j_1}, \cdots, \underbrace{n, \cdots, n}_{j_n})$ we set

$$S_J(z, t) = \frac{1}{(t_1 - z_{i_1}) \cdots (t_m - z_{i_m})} \tag{5.8}$$

and define the rational function $R_J(z, t)$ by

$$R_J(z, t) = \frac{1}{j_1! \cdots j_n!} \sum_{\sigma \in \mathfrak{S}_m} S_J(z_1, \cdots, z_n, t_{\sigma(1)}, \cdots, t_{\sigma(m)}). \tag{5.9}$$

For example, we have

$$R_{(1,0,\cdots,0)}(z,t) = \frac{1}{t_1 - z_1}, \quad R_{(2,0,\cdots,0)}(z,t) = \frac{1}{(t_1 - z_1)(t_2 - z_1)}$$

$$R_{(1,1,0,\cdots,0)}(z,t) = \frac{1}{(t_1 - z_1)(t_2 - z_2)} + \frac{1}{(t_2 - z_1)(t_1 - z_2)}$$

and so on.

Since $\pi_{n,m} : X_{m+n} \to X_m$ is a fiber bundle with fiber $X_{n,m}$ the fundamental group of the base space X_n acts naturally on the homology group $H_m(X_{n,m}, \mathcal{L}^*)$. Thus we obtain a representation of the pure braid group

$$r_{n,m} : P_n \longrightarrow \operatorname{Aut} H_m(X_{n,m}, \mathcal{L}^*) \tag{5.10}$$

which defines a local system on X_n denoted by $\mathcal{H}_{n,m}$. In the case $\lambda_1 = \cdots = \lambda_n$ there is a representation of the braid group

$$r_{n,m} : B_n \longrightarrow \operatorname{Aut} H_m(Y_{n,m}, \overline{\mathcal{L}}^*) \tag{5.11}$$

which defines a local system $\overline{\mathcal{H}}_{n,m}$ on $Y_{n,m}$. For any horizontal section $c(z)$ of the local system $\mathcal{H}_{n,m}$ we consider the hypergeometric type integral

$$\int_{c(z)} \Phi_{n,m} R_J(z,t)\ dt_1 \wedge \cdots \wedge dt_m \tag{5.12}$$

for the above rational function $R_J(z,t)$.

According to Schechtman and Varchenko, solutions of the KZ equation are described in the following way.

Theorem 5.1 (Schechtman and Varchenko [16])**.** *The integral*

$$\sum_{|J|=m} \left(\int_{c(z)} \Phi_{n,m} R_J(z,t)\ dt_1 \wedge \cdots \wedge dt_m \right) F^J v$$

lies in the space of null vectors $N[|\Lambda| - 2m]$ and is a solution of the KZ equation.

6. Relation between LKB representation and KZ connection

We fix a complex number λ and consider the space of null vectors

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}$$

by putting $\lambda_1 = \cdots = \lambda_n = \lambda$ in the definition of Section 5. As the monodromy of the KZ connection

$$\omega = \frac{1}{\kappa} \sum_{1 \le i < j \le n} \Omega_{ij} d \log(z_i - z_j)$$

with values in $N[n\lambda - 2m]$ we obtain the linear representation of the braid group

$$\theta_{\lambda,\kappa} : B_n \longrightarrow \operatorname{Aut} N[n\lambda - 2m].$$

The next theorem describes a relationship between a specialization of the Lawrence-Krammer-Bigelow representation $\rho_{n,m}$ and the representation $\theta_{\lambda,\kappa}$.

Theorem 6.1. *There exists an open dense subset U in $(\mathbf{C}^*)^2$ such that for $(\lambda, \kappa) \in U$ the Lawrence-Krammer-Bigelow representation $\rho_{n,m}$ with the specialization*

$$q = e^{-2\pi\sqrt{-1}\lambda/\kappa}, \quad t = e^{2\pi\sqrt{-1}/\kappa}$$

is equivalent to the monodromy representation of the KZ connection $\theta_{\lambda,\kappa}$ with values in the space of null vectors

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}.$$

We assume the conditions $i_*\mathcal{L} \cong i_!\mathcal{L}$ and $i_*\overline{\mathcal{L}} \cong i_!\overline{\mathcal{L}}$ in the following. By means of the argument in Section 3 these conditions are satisfied for (λ, κ) in an open dense subset in $(\mathbf{C}^*)^2$. By the assumption we have an isomorphism $H_m(X_{n,m}, \mathcal{L}) \cong H_m^{lf}(X_{n,m}, \mathcal{L})$ and we can take the regularized cycles $[\widetilde{\Delta}_M] \in H_m(X_{n,m}, \mathcal{L})$ for the bounded chamber Δ_M.

We will consider the integral

$$\sum_{|J|=m} \left(\int_{\Delta_M} \Phi_{n,m} R_J(z,t)\, dt_1 \wedge \cdots \wedge dt_m \right) F^J v$$

in the space of null vectors $N[|\Lambda|-2m]$. In general the above integral is divergent. We replace the integration cycle by the regularized cycle $[\widetilde{\Delta}_M]$ to obtain the convergent integral. This is called the regularized integral. We refer the reader to [2] for details on this aspect.

The rest of this section is devoted to the proof of the above theorem. We first show the following proposition.

Proposition 6.1. *There exists an open dense subset U in $(\mathbf{C}^*)^2$ such that for $(\lambda, \kappa) \in U$ the following properties (1) and (2) are satisfied.*

1. *The integrals in Theorem 5.1 over $[\widetilde{\Delta}_M]$ for $M = (m_1, \cdots, m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$ are linearly independent.*
2. *The homology classes $[\overline{\Delta}_M]$ for $M = (m_1, \cdots, m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$ form a basis of $H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}}^*) \cong H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$.*

Here $m_1, \cdots, m_{n-1}$ are non-negative integers.

Proof. We prepare notation for a basis of $N[|\Lambda| - 2m]$. We suppose that λ_1 is not a non-negative integer. Let us observe that for $\Lambda = (\lambda_1, \cdots, \lambda_n)$ the space of null vectors $N[|\Lambda| - 2m]$ has dimension $d_{n,m}$. This can be shown as follows. First, let us consider the weight space

$$\begin{aligned} &M_{\lambda_2} \otimes \cdots \otimes M_{\lambda_n}[\lambda_2 + \cdots + \lambda_n - 2m] \\ &= \{x \in M_{\lambda_2} \otimes \cdots \otimes M_{\lambda_n} \ ; \ Hx = (\lambda_2 + \cdots + \lambda_n - 2m)x\}. \end{aligned}$$

There is an isomorphism

$$\xi : M_{\lambda_2} \otimes \cdots \otimes M_{\lambda_n}[\lambda_2 + \cdots + \lambda_n - 2m] \longrightarrow N[|\Lambda| - 2m]$$

defined by

$$u \mapsto v_{\lambda_1} \otimes u - \frac{1}{\lambda_1} F v_{\lambda_1} \otimes Eu + \frac{1}{\lambda_1(\lambda_1 - 1)} F^2 v_{\lambda_1} \otimes E^2 u - \cdots$$

This shows that $N[|\Lambda| - 2m]$ has a basis indexed by $J' = (j_1, j_2, \cdots, j_n)$ with $j_1 = 0$ and $j_2 + \cdots + j_n = m$, where $j_2, \cdots, j_n$ are non-negative integers. Let us denote by $S_{n,m}$ the set of such indices J'. The above weight space has a basis $u_{J'}$ indexed by $J' \in S_{n,m}$. We have the corresponding basis $\xi(u_{J'})$ of $N[|\Lambda| - 2m]$.

We set $\alpha_1, \cdots, \alpha_n$ and γ as in (5.7). We put

$$\widetilde{\Phi}_{n,m} = \prod_{1 \le i \le m, 1 \le \ell \le n} (t_i - z_\ell)^{\alpha_\ell} \prod_{1 \le i < j \le m} (t_i - t_j)^{2\gamma} \tag{6.1}$$

and for $J' \in S_{n,m}$ put

$$\alpha'_{J} = \prod_{k=2}^{n} (j_k)! \alpha_k(\alpha_k+\gamma)\cdots(\alpha_k+(j_k-1)\gamma). \tag{6.2}$$

We assume that $\alpha_1, \cdots, \alpha_n$ and γ are positive. We express the integral in Theorem 5.1 over the cycle Δ_M in the linear combination for the basis $\xi(u_{J'})$ of $N[|\Lambda|-2m]$ and we donote by $\widetilde{R}_{J'}(z,t)$ the corresponding rational function. In [18] Varchenko gave a formula for the determinant

$$\det_{M,J'} \left(\alpha_{J'} \int_{\Delta_M} \widetilde{\Phi}_{n,m} \widetilde{R}_{J'}(z,t) dt_1 \wedge \cdots \wedge dt_m \right), \tag{6.3}$$

where $M=(m_1,\cdots,m_{n-1})$ with $m_1+\cdots+m_{n-1}=m$ and $J' \in S_{n,m}$. According to Varchenko's formula the above determinant is expressed as a non-zero constant times the gamma factor given by

$$\prod_{i=0}^{m-1} \left(\frac{\Gamma((i+1)\gamma+1)^{n-1}}{\Gamma(\gamma+1)^{n-1}} \; \frac{\Gamma(\alpha_1+i\gamma+1)\cdots(\alpha_n+i\gamma+1)}{\Gamma(\alpha_1+\cdots+\alpha_n+(2m-2-i)\gamma+1)} \right)^{\nu_i} \tag{6.4}$$

where ν_i is defined by

$$\nu_i = \begin{pmatrix} m+n-i-3 \\ m-i-1 \end{pmatrix}. \tag{6.5}$$

Since the gamma function does not have zeros and has only poles of order one at non-positive integers, it is clear that the determinant is zero only when the denominator of the gamma factor has a pole. Considering the regularized integrals over the cycles $[\widetilde{\Delta}_M]$ we can analytically continue the determinant formula to complex numbers $\alpha_1, \cdots, \alpha_n$ and γ.

Let us recall that we deal with the case

$$\alpha_\ell = -\frac{\lambda}{\kappa}, \quad 1 \leq \ell \leq n, \quad \gamma = \frac{1}{\kappa}.$$

From the determinant formula we observe that the linearly independence for the solutions of the KZ equation in (1) in the statement of the proposition is satisfied for (λ, κ) in an open dense subset in $(\mathbf{C}^*)^2$. Under the same condition we have the linear independence for the homology classes $[\overline{\Delta}_M]$ for $M=(m_1,\cdots,m_{n-1})$ with $m_1+\cdots+m_{n-1}=m$. Since we have $\dim H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}}^*) = d_{m,n}$ we obtain the property (2). This completes the proof of our proposition. □

Let us consider the specialization map

$$s : R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}] \longrightarrow \mathbf{C} \tag{6.6}$$

defined by the substitutions $q \mapsto e^{-2\pi\sqrt{-1}\lambda/\kappa}$ and $t \mapsto e^{2\pi\sqrt{-1}/\kappa}$. This induces in a natural way a homomorphism

$$H_m(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z}) \longrightarrow H_m(Y_{n,m}, \overline{\mathcal{L}}^*). \tag{6.7}$$

We take a basis $[c_M]$ of $H_m(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$ as the R-module for $M=(m_1,\cdots,m_{n-1})$ with $m_1+\cdots+m_{n-1}=m$ in such a way that $[c_M]$ maps to the regularized cycle for $[\overline{\Delta}_M]$ by the above specialization map. We observe that the LKB representation specialized at $q \mapsto e^{-2\pi\sqrt{-1}\lambda/\kappa}$ and $t \mapsto e^{2\pi\sqrt{-1}/\kappa}$ is identified with the linear representation of the braid group $r_{n,m} : B_n \to \mathrm{Aut}\, H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$.

Since the basis of $N[n\lambda - 2m]$ is indexed by the set $S_{n,m}$ we have an isomorphism

$$H_m(Y_{n,m}, \overline{\mathcal{L}}^*) \cong N[n\lambda - 2m].$$

Now the fundamental solutions of the KZ equation with values in $N[n\lambda - 2m]$ is give by the matrix of the form

$$\left(\int_{\widetilde{\Delta}_M} \omega_{M'} \right)_{M,M'}$$

with $M = (m_1, \cdots, m_{n-1})$ and $M' = (m'_1, \cdots, m'_{n-1})$ such that $m_1 + \cdots + m_{n-1} = m$ and $m'_1 + \cdots + m'_{n-1} = m$. Here $\omega_{M'}$ is a multivalued m-form on $X_{n,m}$. The column vectors of the above matrix form a basis of the solutions of the KZ equation with values in $N[n\lambda - 2m]$. Thus the representation $r_{n,m} : B_n \to \mathrm{Aut}\, H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$ is equivalent to the action of B_n on the solutions of the KZ equation with values in $N[n\lambda - 2m]$. This completes the proof of Theorem 6.1.

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