HIGHER HOLONOMY OF FORMAL HOMOLOGY CONNECTIONS AND BRAID COBORDISMS

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Abstract. We construct a representation of the homotopy 2-groupoid of a manifold by means of K.-T. Chen's formal homology connections. By using the idea of this 2-holonomy map, we describe a method to obtain a representation of the category of braid cobordisms.

1. Introduction

The notion of formal homology connections was developed by K.-T. Chen in the framework of the theory of iterated integrals of differential forms. The original motivation of K.-T. Chen was to describe the homology group of the loop space of a manifold M by the chain complex formed by the tensor algebra of the homology group of M equipped with a derivation appearing in the formal homology connection (see [6], [7] and [8]). By means of the formal homology connection we obtain a chain map from the singular chain complex of the loop space to the above complex obtained from the tensor algebra of the homology group of M of positive degrees.

The formal homology connection can be used to construct a holonomy map from the homotopy path groupoid. In particular, we obtain representations of fundamental groups. This was used to describe the holonomy of KZ connection in [16] and [17]. The purpose of this paper is to show that the notion of the holonomy can be extended to a 2-holonomy map from the homotopy 2-groupoid by means of formal homology connections. In order to formulate the 2-holonomy we employ the notion of 2-categories. We refer the reader to [4] for an introduction to 2-categories from the viewpoint of higher gauge theory.

Then we apply such method to construct a holonomy representation of the category of braid cobordisms. There is a work by Cirio and Martins [12] on the categorification of the KZ connection by means of 2-Yang-Baxter operator for $sl_2(\mathbf{C})$ (see also [10], [11] and [21]). In this paper we propose a universal construction based on the formal homology connections.

The paper is organized in the following way. In Section 1 we briefly review K.-T. Chen's iterated integrals and their basic properties. In Section 2 we describe the notion of formal homology connections. In Section 3 we construct representations of homotopy 2-groupoids by means of the formal homology connection. In Section 4 is we describe a method to construct a representation of the category of braid cobordisms.

2. Preliminaries on K.-T. Chen's iterated integrals

First, we briefly recall the notion of iterated integrals of differential forms due to K.-T. Chen. We refer the reader to [6], [7] and [8] for details. Let M be a smooth manifold and $\omega_1, \cdots, \omega_k$ be differential forms on M. We fix two points $\mathbf{x}_0$ and $\mathbf{x}_1$ in

M and consider the space of piecewise smooth paths $\gamma : [0,1] \to M$ with $\gamma(0) = \mathbf{x}_0$ and $\gamma(1) = \mathbf{x}_1$. We denote by $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ the above space of paths. In particular, in the case $\mathbf{x}_0 = \mathbf{x}_1$ the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ is called the based loop space of M. We consider the simplex

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^k \ ; \ 0 \le t_1 \le \cdots \le t_k \le 1\}$$

and the evaluation map

$$\varphi : \Delta_k \times \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1) \to \underbrace{M \times \cdots \times M}_{k}$$

defined by $\varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k))$. The iterated integral of $\omega_1, \cdots, \omega_k$ is defined as

$$\int \omega_1 \cdots \omega_k = \int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

where the expression

$$\int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

is the integration along the fiber with respect to the projection

$$p : \Delta_k \times \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1) \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1).$$

The above iterated integral is considered as a differential form on the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ with degree $p_1 + \cdots + p_k - k$, where we set $p_j = \deg \omega_j$. To justify differential forms on the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ we use the notion of plots. A plot $\alpha : U \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ is a family of piecewise linear paths smoothly parametrized by a convex open set U in a finite dimensional Euclidean space. Given a plot α we denote the corresponding iterated integral

$$\left(\int \omega_1 \cdots \omega_k \right)_\alpha$$

as a differential form on U obtained by pulling back by the iterated integral $\int \omega_1 \cdots \omega_k$ by the plot α. In particular, in the case $\omega_1, \cdots, \omega_k$ are 1-forms, the iterated integral $\int \omega_1 \cdots \omega_k$ is a function on the path space and its value on a path $\gamma : [0,1] \to M$ is the iterated line integral

$$\int_\gamma \omega_1 \cdots \omega_k = \int_{\Delta_k} f_1(t_1) \cdots f_k(t_k) \ dt_1 \cdots dt_k$$

where $\gamma^* \omega_j = f_j(t) \ dt$, $1 \le j \le k$.

Let us go back to the iterated integral of differential forms of arbitrary degrees. We take an extra point $\mathbf{x}_2$ in M and consider the plots

$$\alpha : U \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1), \quad \beta : U \longrightarrow \mathcal{P}(M; \mathbf{x}_1, \mathbf{x}_2).$$

The composition of the plots α and β

$$\alpha\beta : \ U \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_2)$$

is defined by

$$\alpha\beta(x)(t) = \begin{cases} \alpha(x)(2t), & 0 \le t \le \frac{1}{2} \\ \beta(x)(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

for $x \in U$. As is shown by K.-T. Chen, we have the following rule for the composition of plots.

Proposition 2.1. *The relation*

$$\left(\int \omega_1 \cdots \omega_k\right)_{\alpha\beta} = \sum_{0\leq i\leq k} \left(\int \omega_1 \cdots \omega_i\right)_{\alpha} \wedge \left(\int \omega_{i+1} \cdots \omega_k\right)_{\beta}$$

holds.

For a path α we define its inverse path α^{-1} by

$$\alpha^{-1}(t) = \alpha(1-t).$$

For the composition $\alpha\alpha^{-1}$ we have

$$\left(\int \omega_1 \cdots \omega_i\right)_{\alpha\alpha^{-1}} = 0.$$

As a differential form on the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ we have the following.

Proposition 2.2. *For the iterated integral $\int \omega_1 \cdots \omega_k$ we have*

$$\begin{aligned} & d \int \omega_1 \cdots \omega_k \\ =& \sum_{j=1}^{k} (-1)^{\nu_{j-1}+1} \int \omega_1 \cdots \omega_{j-1} d\omega_j \ \omega_{j+1} \cdots \omega_k \\ &+ \sum_{j=1}^{k-1} (-1)^{\nu_j+1} \int \omega_1 \cdots \omega_{j-1} (\omega_j \wedge \omega_{j+1}) \omega_{j+2} \cdots \omega_k \end{aligned}$$

where we put $\nu_j = \deg \omega_1 + \cdots + \deg \omega_j - j$.

3. Formal homology connections

For a smooth manifold M we put $H_+(M) = \oplus_{q>0} H_q(M; \mathbf{R})$ and consider the tensor algebra

$$TH_+(M) = \bigoplus_{k\geq 0} \left(\bigotimes^k H_+(M) \right).$$

We suppose that $\dim H_+(M)$ is finite. We denote by $\Omega^*(M)$ the algebra of differential forms on M and consider the tensor product $\Omega^*(M) \otimes TH_+(M)$. We suppose that the differential d acts trivially on $TH_+(M; \mathbf{R})$ and we extend naturally the wedge product and iterated integrals on $\Omega^*(M) \otimes TH_+(M)$. the powers of the augmentation ideal. When $H_+(M)$ has a basis $X_1, \cdots, X_m$, the tensor algebra $TH_+(M)$ is the ring of non-commutative polynomials $\mathbf{R}\langle X_1, \cdots, X_m \rangle$. We denote by J the ideal of the above ring generated by $X_1, \cdots, X_m$, which is called the augmentation ideal. We consider the completion $\widehat{TH_+(M)}$ with respect to the powers of the augmentation ideal J. We see that $\widehat{TH_+(M)}$ is regarded as the ring of non-commutative formal power series $\mathbf{R}\langle\langle X_1, \cdots, X_m \rangle\rangle$. We denote by $\widehat{J}$ the completed augmentation ideal. Namely, $\widehat{J}$ consists of the formal power series of the form

$$\sum_{i=1}^{m} a_i X_i + \cdots + \sum_{i_1 \cdots i_k} a_{i_1 \cdots i_k} X_{i_1} \cdots X_{i_k} + \cdots$$

with zero constant term.

Then $\Omega^*(M) \otimes \widehat{TH_+(M)}$ is identified with the ring of non-commutative formal power series

$$\Omega^*(M)\langle\langle X_1, \cdots, X_m \rangle\rangle$$

over $\Omega^*(M)$. For a differential operator ω we define the parity operator ε as $\varepsilon(\omega) = \omega$ when ω is of even degree and $\varepsilon(\omega) = -\omega$ when ω is of odd degree. This operator is naturally extended to $\Omega^*(M) \otimes \widehat{TH_+(M)}$. Namely, for a differential form τ and a monomial Z in $X_1, \cdots, X_m$ we set $\varepsilon(\tau Z) = \varepsilon(\tau) Z$. We define a generalized curvature κ by

$$\kappa = d\omega - \varepsilon(\omega) \wedge \omega.$$

According to K.-T. Chen a formal homology connection

$$\omega \in \Omega^*(M) \otimes \widehat{TH_+(M)}$$

is an expression

$$\omega = \sum_{i=1}^m \omega_i X_i + \cdots + \sum_{i_1 \cdots i_k} \omega_{i_1 \cdots i_k} X_{i_1} \cdots X_{i_k} + \cdots$$

with differential forms of positive degrees $\omega_{i_1 \cdots i_k}$ together with a derivation δ satisfying the following properties. We put $\deg x_i = p_i - 1$ for $x_i \in H_{p_i}(M)$.

- $[\omega_i]$, $1 \leq i \leq m$ is the dual basis of X_i, $1 \leq i \leq m$.
- $\deg \omega_{i_1 \cdots i_k} = \deg X_{i_1} \cdots X_{i_k} + 1$.
- $\delta\omega + \kappa = 0$.
- δ is a derivation of degree -1.
- $\delta X_j \in \widehat{J}^2$ where $\widehat{J}$ is the completed augmentation ideal.

Here we suppose that the derivation δ satisfies the Leibniz rule

$$\delta(uv) = (\delta u)v + (-1)^{\deg u} u(\delta v).$$

From the above condition we can show that $\delta \circ \delta = 0$ and $(\widehat{TH_+(M)}, \delta)$ forms a complex. We denote by $\widehat{TH_+(M)}_k$ the degree k part of $\widehat{TH_+(M)}$ with respect to the above degrees. We denote by $\widehat{TH_+(M)}_{\leq k}$ the completed subalgebra of $\widehat{TH_+(M)}$ generated by the homogeneous elements of degree less than or equal to k. For the formal homology connection ω we define its transport by

$$T = 1 + \sum_{k=1}^{\infty} \int \underbrace{\omega \cdots \omega}_{k}.$$

The following proposition plays a key role of for the construction of holonomy maps.

Proposition 3.1. *Given a formal homology connection (ω, δ) for a manifold M the transport T satisfies $dT = \delta T$.*

Proof. By Proposition 2.2 we have

$$dT = -\int \kappa + \left(-\int \kappa\omega + \int \varepsilon(\omega)\kappa \right) + \cdots$$
$$= \sum_{k=0}^{\infty} \sum_{i=0}^{k} (-1)^{i+1} \int \underbrace{\varepsilon(\omega) \cdots \varepsilon(\omega)}_{i} \ \kappa \ \underbrace{\omega \cdots \omega}_{k-i-1}.$$

Substituting $\kappa = -\delta\omega$ in the above equation and applying the Leibniz rule for δ, we obtain the equation $dT = \delta T$. $\square$

Although the formal homology connection ω with the derivation δ is not uniquely determined, we can construct it inductively starting from the initial term $\sum_{i=1}^m \omega_i X_i$. Here are some examples.

Examples : (1) Let $T = S^1 \times S^1$ be the 2-dimensional torus. Let $p_i : S^1 \times S^1$, $i = 1, 2$, the projection to the i-th factor. We denote by v a volume form of S^1. The de Rham cohomology $H^*(T)$ has a basis represented by $p_1^*v, p_2^*v, p_1^*v \wedge p_2^*v$ and we put X_1, X_2, Y its dual basis of the homology. The formal homology connection is given as

$$\omega = p_1^*vX_1 + p_2^*vX_2 + (p_1^*v \wedge p_2^*v)Y$$

with the derivation defined by

$$\delta(X_1) = 0,\ \delta(X_2) = 0,\ \delta(Y) = -[X_1, X_2].$$

(2) Let G be the unipotent Lie group consisting of the matrices

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbf{R}$$

and $G_{\mathbf{Z}}$ its subgroup consisting of the above matrices with $x, y, z \in \mathbf{Z}$. We denote by M the quotient space of G by the left action of $G_{\mathbf{Z}}$. We see that M has a structure of a compact smooth 3-dimensional manifold. The 1-forms

$$\omega_1 = dx,\ \omega_2 = dy,\ \omega_{12} = -xdy + dz$$

on G are invariant under the left action of $G_{\mathbf{Z}}$ and define 1-forms on M. There is a relation

$$\omega_1 \wedge \omega_2 = -d\omega_{12}.$$

We observe that $H^1(M)$ has a basis represented by ω_1, ω_2 and $H^2(M)$ has a basis represented by $\omega_1 \wedge \omega_{12}, \omega_2 \wedge \omega_{12}$. These are typical examples of non-trivial Massey product. We denote by $X_1, X_2 \in H_1(M)$ the dual basis of $[\omega_1], [\omega_2]$ and by $Y_1, Y_2 \in H_2(M)$ the dual basis of $[\omega_1 \wedge \omega_{12}], [\omega_2 \wedge \omega_{12}]$. By means of the condition $\delta\omega + d\omega = \varepsilon(\omega) \wedge \omega$ we obtain that the derivation δ is given by

$$\delta(X_1) = 0,\ \delta(X_2) = 0,\ \delta(Y_1) = [[X_1, X_2], X_1],\ \delta(Y_2) = [[X_1, X_2], X_2].$$

4. Path groupoids, 2-path groupoids and their representations

We introduce the path groupoid $\mathcal{P}_1(M)$ and its 2-category extension $\mathcal{P}_2(M)$. The path groupoid $\mathcal{P}_1(M)$ is a category whose objects are points in M and whose morphisms are piecewise smooth paths between points up to reparametrization and a thin homotopy. Here a thin homotopy is a homotopy sweeping on the path. We see that $\mathcal{P}_1(M)$ has a structure of a groupoid since there is an associativity and each morphism has its inverse by means of the invariance of iterated integrals under the thin homotopy.

Now we discuss its extension to 2-categories. In general, a 2-category consists of objects, morphims and 2-morphims, which are morphims between morphims. There are two kinds of compositions for 2-morphisms, horizontal compositions and vertical compositions and there are several consistency conditions among them. We do not give here a full definition of a 2-category. We refer the reader to [4] for an introduction to the notion of 2-categories. The path 2-groupoid $\mathcal{P}_2(M)$ is a 2-category whose morphisms are piecewise smooth paths between points up to reparametrization and a thin homotopy and whose 2-morphisms are piecewise smooth discs $[0,1]^2 \to M$ spanning 2 paths up to reparametrization and a thin homotopy. As in the case of the path groupoid, a thin homotopy is a homotopy sweeping on the disc.

The homotopy equivalence classes of the path groupoid $\mathcal{P}_1(M)$ is the homotopy path groupoid denote by $\Pi_1(M)$. In a similar way, we define the homotopy 2-groupoid $\Pi_2(M)$ whose 2-morphisms are relative piecewise smooth homotopy classes of piecewise smooth homotopies between paths. We refer the reader to [13] for a general construction of a homotopy 2-groupoid of a topological space.

Let ω be a formal homology connection for M with the derivation δ. We decompose ω as

$$\omega = \omega^1 + \omega^2 + \cdots + \omega^p + \cdots$$

where ω^p is the sum consisting of p-forms and is called the p-form part of ω. First, we consider the 1-form part ω^1. For a piecewise smooth path γ in M the holonomy of the connection ω^1 is given the transport as

$$Hol(\gamma) = 1 + \sum_{k=1}^{\infty} \int_\gamma \underbrace{\omega^1 \cdots \omega^1}_{k}$$

which is an element of $\widehat{TH_+(M)}_0$. For the composition of paths we have

$$Hol(\alpha\beta) = Hol(\alpha)Hol(\beta)$$

by Proposition 2.1. Moreover, the relation

$$Hol(\alpha^{-1}) = Hol(\alpha)^{-1}$$

holds. Therefore, we obtain a representation of the path groupoid

$$Hol : \mathcal{P}_1(M) \longrightarrow \widehat{TH_+(M)}_0.$$

We denote by $\widehat{TH_+(M)}_0^\times$ the group of invertible elements in $\widehat{TH_+(M)}_0$. The above Hol is considered to be a map of groupoids from $\mathcal{P}_1(M)$ to $\widehat{TH_+(M)}_0^\times$. Here the map Hol is regarded as a functor.

Let us consider the homotopy path groupoid $\Pi_1(M)$. In this case we have a holonomy map

$$Hol : \Pi_1(M) \longrightarrow \widehat{TH_+(M)}_0/\mathcal{I}_0$$

where $\mathcal{I}_0$ is the ideal generated by the image of the derivation

$$\delta : \widehat{TH_+(M)}_1 \longrightarrow \widehat{TH_+(M)}_0.$$

This can be verified by means of Proposition 3.1 and the Stokes theorem. Here the curvature of ω^1 is

$$\kappa = d\omega^1 + \omega^1 \wedge \omega^1,$$

which is zero modulo the ideal $\mathcal{I}_0$. The above holonomy functor is a categorical formulation of the holonomy of Chen's formal homology connection. By fixing a base point $\mathbf{x}_0 \in M$ we have a holonomy map

$$Hol : \pi_1(M, \mathbf{x}_0) \longrightarrow \widehat{TH_+(M)}_0$$

and one of the main results due to K.-T. Chen is that the holonomy map induces an isomorphism

$$\mathbf{R}\widehat{\pi_1(M,\mathbf{x}_0)} \cong \widehat{TH_+(M)}_0$$

where $\mathbf{R}\widehat{\pi_1(M,\mathbf{x}_0)}$ is the completion of the group ring $\mathbf{R}\pi_1(M,\mathbf{x}_0)$ with respect to the powers of the augmentation ideal. The algebra $\mathbf{R}\widehat{\pi_1(M,\mathbf{x}_0)}$ is called the Malcev completion of the fundamental group $\pi_1(M,\mathbf{x}_0)$.

Now we construct representations of the homotopy 2-groupoid $\mathcal{P}_2(M)$. For two paths γ_0 and γ_1 in $\mathcal{P}(M;\mathbf{x}_0,\mathbf{x}_1)$ we consider a piecewise smooth disc $F:[0,1]^2\to M$ with

$$F(t,0)=\gamma_0(t),\ F(t,1)=\gamma_1(t)$$
$$F(0,s)=\mathbf{x}_0,\ F(1,s)=\mathbf{x}_1,$$

which is considered to be a 2-morphism between γ_0 and γ_1. Putting $c(s)(t)=F(t,s)$, we obtain a family of paths

$$c:[0,1]\longrightarrow \mathcal{P}(M;\mathbf{x}_0,\mathbf{x}_1),$$

which is considered to be a 1-chain in $\mathcal{P}(M;\mathbf{x}_0,\mathbf{x}_1)$. For the formal homology connection we consider the transport

$$T=1+\sum_{k=1}^{\infty}\int\underbrace{\omega\cdots\omega}_{k}$$

and denote by $\langle T,c\rangle$ its integration on the 1-chain c. We define the 2-holonomy

$$Hol_2:\mathcal{P}_2(M)\longrightarrow \widehat{TH_+(M)}_{\leq 1}$$

by $Hol_2(c)=\langle T,c\rangle$. The above holonomy map is additive with respect to the sum as 1-chains and for the composition of paths we have

$$Hol_2(\alpha\beta)=Hol_2(\alpha)Hol_2(\beta)$$

by means of Proposition 2.1. The above two types of compositions correspond to to horizontal and vertical compositions of 2-morphisms in the 2-category. We obtain that the 2-holonomy map Hol_2 gives a representation of the path 2-groupoid $\mathcal{P}_2(M)$.

Theorem 4.1. *The above 2-holonomy map gives a representation of the homotopy 2-groupoid*

$$Hol_2:\Pi_2(M)\longrightarrow \widehat{TH_+(M)}_{\leq 1}/\mathcal{I}_1$$

where $\mathcal{I}_1$ is the ideal generated by the image of the derivation

$$\delta:\widehat{TH_+(M)}_2\longrightarrow \widehat{TH_+(M)}_1$$

Proof. As is shown in the above argument we have a representation of the path 2-groupoid given by

$$Hol_2:\mathcal{P}_2(M)\longrightarrow \widehat{TH_+(M)}_{\leq 1}.$$

Suppose that for paths γ_0 and γ_1 in $\mathcal{P}(M;\mathbf{x}_0,\mathbf{x}_1)$ piecewise smooth discs $F_j:[0,1]^2\to M$, $j=1,2$ with

$$F_j(t,0)=\gamma_0(t),\ F_j(t,1)=\gamma_1(t)$$
$$F_j(0,s)=\mathbf{x}_0,\ F_j(1,s)=\mathbf{x}_1,$$

are connected by a piecewise smooth homotopy preserving the above boundary conditions. This gives homologous 1-chains c_1 and c_2 in $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ and there is a 2-chain y such that $c_1 - c_2 = \partial y$. We have

$$Hol_2(c_1) - Hol_2(c_2) = Hol_2(\partial y)$$

which is by definition $\langle T, \partial y \rangle$. By Stokes theorem we have

$$\langle T, \partial y \rangle = \langle dT, y \rangle.$$

On the other hand we have $dT = \delta T$ by Proposition 3.1. This shows that $Hol_2(c_1) = Hol_2(c_2)$ in $\widehat{TH_+(M)}_{\leq 1}/\mathcal{I}_1$ and the 2-holonomy map from the homotopy 2-groupoid $\Pi_2(M)$ is well-defined. The fact that this give a representation of the 2-groupoid $\Pi_2(M)$ follows from the corresponding properties such as

$$Hol_2(\alpha\beta) = Hol_2(\alpha) Hol_2(\beta)$$

for the path 2-groupoid $\mathcal{P}_2(M)$. This completes the proof. □

We refer the reader to [1] and [2] for a different approach to higher holonomies based on iterated integrals.

5. Holonomy of braids and its extension to braid cobordisms

We apply a method explained in the previous sections to holonomy of braids and representation of the category of braid cobordisms. We start by recalling basic facts on hyperplane arrangements. Let $\mathcal{A} = \{H_1, \cdots, H_\ell\}$ be a collection of finite number of complex hyperplanes in $\mathbf{C}^n$. We call $\mathcal{A}$ a hyperplane arrangement. Let f_j, $1 \leq j \leq \ell$, be linear forms dining the hyperplanes H_j. We consider the complement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$$

and denote by $\Omega^*(M(\mathcal{A}))$ the algebra of differential forms on $M(\mathcal{A})$ with values in $\mathbf{C}$. The Orlik-Solomon algebra $OS(\mathcal{A})$ is the subalgebra of $\Omega^*(M(\mathcal{A}))$ generated by the logarithmic forms $\omega_j = d \log f_j$, $1 \leq j \leq \ell$. We refer the reader to [22] and [23] for basic properties of the Orlik-Solomon algebra. The fundamental fact is that the inclusion map

$$i : OS(\mathcal{A}) \longrightarrow \Omega^*(M(\mathcal{A}))$$

induces an isomorpshim of cohomology, where the differential on $OS(\mathcal{A})$ is trivial. In particular, we have an isomorpshim of algebras

$$OS(\mathcal{A}) \cong H^*(M(\mathcal{A}); \mathbf{C}).$$

A formal homology connection for $M(\mathcal{A})$ is given as follows. Let $\{Z_j\}$ be a basis of $H_+(M(\mathcal{A}); \mathbf{C})$ and $\{\varphi_j\}$ be its dual basis in the Orlik-Solomon algebra $OS(\mathcal{A})$. Then we can take a formal homology connection given as

$$\omega = \sum_{j=1}^{m} \varphi_j Z_j$$

where the derivation $\delta : \widehat{TH_+(M(\mathcal{A}))}_p \longrightarrow \widehat{TH_+(M(\mathcal{A}))}_{p-1}$ is the dual of the wedge product. More explicitly, as is described in [18], when the wedge product is given by

$$\varepsilon(\varphi_i) \wedge \varphi_j = \sum_k c_{ij}^k \varphi_k$$

the derivation δ is defined as

$$\delta Z_k = \sum_{i,j} c_{ij}^k [Z_i, Z_j].$$

This is a consequence of the formality of $M(\mathcal{A})$ in the sense of rational homotopy theory. There are no non-trivial Massey products and the derivation δ is completely determined by the product structure of the Orlik-Solomon algebra.

We consider the configuration space of ordered distinct n points in the complex plane $\mathbf{C}$. Namely, we put

$$X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j \text{ if } i \neq j\}.$$

The configuration space X_n is the complement of the union of big diagonal hyperplanes H_{ij} defined by $z_i = z_j$ in $\mathbf{C}^n$ for $1 \leq i < j \leq n$. By considering the action of the symmetric group $\mathfrak{S}_n$ by the permutation of coordinates, we set

$$Y_n = X_n / \mathfrak{S}_n.$$

We have a covering map

$$\pi : X_n \longrightarrow Y_n$$

and the fundamental group $\pi_1(Y_n)$ is the braid group of n strings denoted by B_n and $\pi_1(X_n)$ is the pure braid group of n strings denoted by P_n.

We denote by $OS(X_n)$ the Orlik-Solomon algebra for the above arrangement of hyperplanes $\{H_{ij}\}_{1 \leq i < j \leq n}$. We set

$$\omega_{ij} = d \log(z_i - z_j), \quad 1 \leq i < j \leq n.$$

Then the Orlik-Solomon algebra $OS(X_n)$ is generated by ω_{ij}, $1 \leq i < j \leq n$. We have

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{ij} = 0, \quad 1 \leq i < j < k \leq n$$

and it was shown by Arnol'd [3] that these are actually fundamental relations. Namely, the Orlik-Solomon algebra $OS(X_n)$ is isomorphic to the exterior algebra generated by e_{ij}, $1 \leq i < j \leq n$, modulo the ideal generated by $e_{ij}e_{jk} + e_{jk}e_{ik} + e_{ik}e_{ij}$. It turns out that the degree q part of $OS(X_n)$ has a basis represented by

$$\omega_{i_1 j_1} \wedge \cdots \wedge \omega_{i_q j_q}, \quad j_1 < \cdots < j_q.$$

This is called the normal form of a basis of $OS(X_n)$. We denote by $X_{i_1 j_1, \cdots, i_q j_q}$ its dual basis of the homology $H_q(X_n)$. The formal homology connection is given by

$$\omega = \sum_{j_1 < \cdots < j_q, 1 \leq q \leq n} \omega_{i_1 j_1} \wedge \cdots \wedge \omega_{i_q j_q} X_{i_1 j_1, \cdots, i_q j_q}.$$

Since $d\omega = 0$ the generalized curvature $\kappa = d\omega - \varepsilon(\omega) \wedge \omega$ is decomposed as

$$\kappa = \omega^1 \wedge \omega^1 + (\omega^1 \wedge \omega^2 - \omega^2 \wedge \omega^1) + \cdots$$

according the degrees of differential forms.

The 1-form part of the formal homology connection is

$$\omega^1 = \sum_{i<j} \omega_{ij} X_{ij}$$

where X_{ij}, $1 \leq i < j \leq n$, is a basis of $H_1(X_n; \mathbf{C})$ corresponding to the hyperplanes H_{ij} and the representation of the path groupoid described in the previous section is give as

$$Hol : \mathcal{P}_1(X_n) \longrightarrow \mathbf{C}\langle\langle X_{ij} \rangle\rangle$$

where $\mathbf{C}\langle\langle X_{ij}\rangle\rangle$ is the ring of non-commutative formal power series with indeterminates X_{ij}, $1 \leq i < j \leq n$. This induces the representation of the homotopy path groupoid

$$Hol : \Pi_1(X_n) \longrightarrow \mathbf{C}\langle\langle X_{ij}\rangle\rangle/\mathcal{I}_0.$$

The generators of the ideal $\mathcal{I}_0$ are determined in the following way. We express the 2-form part of κ by the normal form of the basis of $OS(X_n)$ as

$$\omega^1 \wedge \omega^1 = \sum_{j_1<j_2} \omega_{i_1j_1} \wedge \omega_{i_2j_2} Z_{i_1j_1,i_2j_2}.$$

Then by the condition $\delta\omega + \kappa = 0$ we have $\delta(X_{i_1j_1,i_2j_1}) = -Z_{i_1j_1,i_2j_2}$. It turns out that the generators of $\mathcal{I}_0$ are infinitesimal pure braid relations:

$$[X_{ik}, X_{ij} + X_{jk}],\ [X_{ik} + X_{ij}, X_{jk}] \quad (i, j, k \ \text{ distinct}),$$
$$[X_{ij}, X_{k\ell}], \quad (i, j, k, \ell \ \text{ distinct}).$$

In particular, we obtain a holonomy homomorphism

$$Hol : P_n \longrightarrow \mathbf{C}\langle\langle X_{ij}\rangle\rangle/\mathcal{I}_0$$

which is a prototype of the Kontsevich integral [19] for knots and gives a universal finite type invariants for pure braids (see [16], [17] and [9]).

Now we consider the 2-holonomy map

$$Hol_2 : \Pi_2(X_n) \longrightarrow \widehat{TH_+(X_n)}_{\leq 1}/\mathcal{I}_1.$$

We deal with the 1-form and the 2-form

$$\omega^1 = \sum_{i<j} \omega_{ij} X_{ij}, \quad \omega^2 = \sum_{j_1<j_2} \omega_{i_1j_1} \wedge \omega_{i_2j_2} X_{i_1j_1,i_2j_2}.$$

In the expression of ω^2 we consider the sum for the normal basis of the degree 2 part of $OS(X_n)$ and our formulation is slightly different from the one by Cirio and Martins ([10], [11] and [12]). Although we do not give an explicit form here, we explain a method to determine the generators of the ideal $\mathcal{I}_1$. We express the 3-form part of the generalized curvature κ by the normal form of a basis of $OS(X_n)$ as

$$\omega^1 \wedge \omega^2 - \omega^2 \wedge \omega^1 = \sum_{j_1<j_2<j_3} \omega_{i_1j_1} \wedge \omega_{i_2j_2} \wedge \omega_{i_3j_3} Z_{i_1j_1,i_2j_2,i_3j_3}$$

Then we have $\delta(X_{i_1j_1,i_2j_2,i_3j_3}) = -Z_{i_1j_1,i_2j_2,i_3j_3}$ and the ideal $\mathcal{I}_1$ is generated by $Z_{i_1j_1,i_2j_2,i_3j_3}$, which are expressed by Lie brackets of X_{ij} and $X_{i_1j_1,i_2j_2}$.

Based on the idea of the construction of the 2-holonomy map we discuss a method to construct a representation of the category of braid cobordisms. First, we describe the notion of the category of braid cobordisms. Let us recall that a braid is an embedding of a 1-manifold which is a disjoint union of closed intervals into $\mathbf{C} \times [0, 1]$ so that the projection onto $[0, 1]$ has no critical points, and the boundary of the 1-manifold is mapped to $2n$ points

$$(1, 0), (2, 0), \cdots, (n, 0), (1, 1), (2, 1), \cdots, (n, 1) \in \mathbf{C} \times [0, 1].$$

The isotopy classes of braids fixing the boundary form the braid group B_n. A braid cobordism between braids g and h is a compact surface S with boundary and corners, smoothly and properly embedded in $\mathbf{C} \times [0, 1]^2$, such that the following conditions are satisfied.

(1) The boundary of S is the union of 1-manifolds

$$
\begin{aligned}
&S \cap (\mathbf{C} \times [0,1] \times \{0\}) = g, \\
&S \cap (\mathbf{C} \times [0,1] \times \{1\}) = h, \\
&S \cap (\mathbf{C} \times \{0\} \times [0,1]) = \{1, 2, \cdots, n\} \times \{0\} \times [0,1], \\
&S \cap (\mathbf{C} \times \{1\} \times [0,1]) = \{1, 2, \cdots, n\} \times \{1\} \times [0,1].
\end{aligned}
$$

(2) The projection of S onto $[0,1]^2$ is a branched covering with simple branch points only.

Considering the set of braids as a category, we can equip the set of braid cobordisms with a structure of a 2-category, which is denoted by $\mathcal{B}C_n$. Here the 2-morphisms are equivalence classes of braid cobordims with the isotopies fixing the boundary. A braid cobordism is also called a braided surface (see [5] and [14]).

To extend the 2-holonomy map to $\mathcal{B}C_n$ we consider the integration of the transport T on one-parameter deformation family of singular braids with double points associated with a braid cobordism. To get a finite value we need to regularize the integral at branched points. This regularization was described in a slightly different setting in [12]. By a regularization we obtain a representation of the category of braid cobordism

$$Hol_2 : \mathcal{B}C_n \longrightarrow T\widehat{H_+(X_n)}_{\leq 1}/\mathcal{I}_1.$$

An approach for a regularization is as follows. In the expression of the transport T an infinite sum of iterated integrals of 1-forms and 2-forms appear, but they are convergent for a one-parameter deformation family of non-singular braids. A possible divergence for a braid with double points for such iterated integrals can be regularized by a method similar to the one used by Le and Murakami [20]. Details of this construction will be discussed in a separate publication. Finally, we refer the reader to Khovanov and Thomas [15] for interesting problems concerning the extension of actions of braids to representations of the category of braid cobordisms.

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