THE VOLUME OF A HYPERBOLIC SIMPLEX AND ITERATED INTEGRALS

TOSHITAKE KOHNO

Abstract. We express the volume of a simplex in spherical or hyperbolic spece by iterated integrals of differential forms following Schläfli and Aomoto. We study analytic properties of the volume function and describe the differential equation satisfied by this function.

1. Introduction

This is an expository note to illustrate how the volume of a simplex in spherical or hyperbolic space can be expressed by means of iterated integrals on the space of shapes of simplices. A main tool for relating volumes and iterated integrals is Schläfli's differential equality ([8], [6]). In [1], Aomoto studied the Schläfli function, the volume function for spherical simplices, and showed that the volume of a simplex is described by the iterated integrals of logarithmic forms. This approach leads us to the analytic continuation of the the Schläfli function on the space of complexified Gram matrices. We shall clarify the relationship between the analytic continuation of the Schläfli function and the volume of a hyperbolic simplex. Moreover, we give an explicit description of the differential equation for such volume functions. It turns out that this differential equation is derived from a nilpotent connection. A motivation for the investigation of this aspect is to control the asymptotic behavior of the volume on the boundary of the space of shapes. This subject will be treated in a separate publication.

2. Schläfli's differential equality

Let S^n be the unit sphere in the Euclidean space $\mathbf{R}^{n+1}$ equipped with the Riemannian metric induced from the Eucllidean metric. The differential form

$$\omega = \frac{1}{r^{n+1}} \sum_{j=1}^{n+1} (-1)^{j-1} x_j \, dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{n+1},$$

where $r = \|\mathbf{x}\|^{\frac{1}{2}}$, is invariant under the scaling transformation $\mathbf{x} \mapsto \lambda \mathbf{x}$, $\lambda > 0$ and the restriction of ω on S^n gives the volume form for the unit sphere.

Let H_j, $1 \le j \le m$, be hyperplanes in $\mathbf{R}^{n+1}$ defined by linear forms $f_j : \mathbf{R}^{n+1} \to \mathbf{R}$ and we define C to be the intersection of the half spaces $f_j \ge 0, 1 \le j \le m$. By means of the identity $r^n dr \wedge \omega = dx_1 \wedge \cdots \wedge dx_{n+1}$ it can be shown that the volume of the spherical polyhedron $P = S^n \cap C$ is expressed as

$$V(P) = \frac{2}{\Gamma\left(\frac{n+1}{2}\right)} \int_C e^{-x_1^2 - \cdots - x_{n+1}^2} \, dx_1 \cdots dx_{n+1} \tag{2.1}$$

A model for the hyperbolic space with constant curvature -1 is described in the following way. We equip $\mathbf{R}^{n+1}$ with the Minkowski metric defined by the bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle_{(n|1)} = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}, \quad \mathbf{x}, \mathbf{y} \in \mathbf{R}^{n+1}.$$

The hyperboloid $\mathcal{H}^n$ defined by $-x_1^2 - \cdots - x_n^2 + x_{n+1}^2 = 1, x_{n+1} > 0$ has a Riemannian metric induced from the Minkowski metric of $\mathbf{R}^{n+1}$. With respect to this metric $\mathcal{H}^n$ is a hyperbolic space with constant curvature -1. As in the spherical case the volume of the hyperbolic polyhedron $P = \mathcal{H}^n \cap C$ is expressed as

$$V(P) = \frac{2}{\Gamma\left(\frac{n+1}{2}\right)} \int_C e^{-(-x_1^2 - \cdots - x_n^2 + x_{n+1}^2)}\, dx_1 \cdots dx_{n+1}. \tag{2.2}$$

Let M^n_κ be the spherical, Euclidean or hyperbolic space of constant curvature κ and of dimension $n \geq 2$. Let $\{P\}$ be a family of smoothly parametrized compact n dimensional polyhedra in M^n_κ. Let $V_n(P)$ the n-dimensional volume of P and we regard it as a function on the space of parameters for the family $\{P\}$. For each $(n-2)$-dimensional face F of P let $V_{n-2}(F)$ be the $(n-2)$-dimensional volume of F and θ_F the dihedral angle of the two $(n-1)$-dimensional faces meeting at F. In the case $n = 2$, $V_0(F)$ stands for the number of points in the finite set F. Then the formula

$$\kappa\, dV_n(P) = \frac{1}{n-1} \sum_F V_{n-2}(F)\, d\theta_F \tag{2.3}$$

holds, where the sum is taken over all $(n-2)$-dimensional faces F. The above formula was first shown in the spherical case by Schläfli and is called Schläfli's differential equality. We refer the readers to [5] and [6] for the complete proof.

In the following we consider Δ, an n-dimensional simplex in M^n_κ. Let $E_1, \cdots, E_{n+1}$ be $(n-1)$-dimensional faces of Δ and θ_{ij} the dihedral angle between E_i and E_j. The Gram matrix $A = (a_{ij})$ of the simplex Δ is the $(n+1) \times (n+1)$ matrix defined by $a_{ij} = -\cos\theta_{ij}$. Here all diagonal entries a_{ii} are equal to 1. A matrix with this property is called unidiagonal. We denote by $X_{n+1}(\mathbf{R})$ the set of all symmetric unidiagonal $(n+1) \times (n+1)$ matirices, which is an affine space of dimension $n(n+1)/2$. The shape of a simplex is determined by its Gram matrix. The Gram matrix A for a simplex Δ lies in spherical, Euclidean or hyperbolic space according as $\det A$ is positive, zero or negative. We denote by C_n^+, C_n^0 or C_n^- the set of all possible Gram matrices for spherical, Euclidean or hyperbolic simplices. It is known that the union $C_n = C_n^+ \cup C_n^0 \cup C_n^-$ is a convex open set in $X_{n+1}(\mathbf{R})$ and that the codimension one Euclidean locus C_n^0 is a topological cell which cuts C_n into two open cells C_n^+ and C_n^- (see [6]).

3. Iterated integrals

A main subject of this note is to express the volume of a spherical or a hyperbolic simplex in terms of iterated integrals of 1-forms. Let us first recall the notion of iterated integrals of 1-forms. Let $\omega_1, \cdots, \omega_k$ be differential 1-forms on a smooth manifold M. For a smooth path $\gamma : [0,1] \to M$ we express the pull-back as $\gamma^*\omega_i = f_i(t)dt$, $1 \leq i \leq k$. Now the iterated line integral of the 1-forms $\omega_1, \cdots, \omega_k$ is defined as

$$\int_\gamma \omega_1 \omega_2 \cdots \omega_k = \int_{0 \leq t_1 \leq \cdots \leq t_k \leq 1} f_1(t_1) f_2(t_2) \cdots f_k(t_k)\; dt_1 dt_2 \cdots dt_k. \tag{3.1}$$

For differential forms $\omega_1, \cdots, \omega_k$ on M of arbitrary degrees $p_1, \cdots, p_k$, the iterated integral is defined in the following way. Let $\mathcal{P}M$ be the space of smooth paths on M. We set

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^k \ ; \ 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}.$$

There is an evaluation map

$$\varphi : \Delta_k \times \mathcal{P}M \to \underbrace{M \times \cdots \times M}_{k}$$

defined by $\varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k))$. The iterated integral of $\omega_1, \cdots, \omega_k$ is defined as

$$\int \omega_1 \cdots \omega_k = \int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

where the right hand side is the integration along fiber with respect to the projection $p : \Delta_k \times \mathcal{P}M \to \mathcal{P}M$. The above iterated integral is considered to be a differential form on the path space $\mathcal{P}M$ with degree $p_1 + \cdots + p_k - k$. In particular, in the case $\omega_1, \cdots, \omega_k$ are 1-forms the iterated integral gives a function on the path space.

The theory of iterated integrals was developed by K. T. Chen in relation with the cohomology of loop spaces. We refer the reader to [2] for details concerning this aspect. The following Proposition due to Chen plays a fundamental role in this note to show the integrability of the volume function.

Proposition 3.1. *As a differential form on the space of paths fixing endpoints the iterated integral satisfies*

$$\begin{aligned} & d \int \omega_1 \cdots \omega_k \\ &= \sum_{j=1}^{k} (-1)^{\nu_{j-1}+1} \int \omega_1 \cdots \omega_{j-1} d\omega_j \ \omega_{j+1} \cdots \omega_k \\ &\quad + \sum_{j=1}^{k-1} (-1)^{\nu_j+1} \int \omega_1 \cdots \omega_{j-1} (\omega_j \wedge \omega_{j+1}) \omega_{j+2} \cdots \omega_k \end{aligned}$$

where we set $\nu_j = \deg \omega_1 + \cdots + \deg \omega_j - j, \quad 1 \leq j \leq k.$

4. Volumes of spherical simplices

Let Δ be an n-dimensional spherical simplex with $(n-1)$-dimensional faces $E_0, \cdots, E_n$. For a positive integer m with $2m \leq n+1$ let

$$I_0 \subset I_1 \subset \cdots \subset I_k \subset \cdots \subset I_m$$

be an increasing sequence of subsets of $I = \{1, 2, \cdots, n+1\}$ such that $|I_k| = 2k$. We denote by $\mathcal{F}_m[n]$ the set of all such sequences $(I_0 \cdots I_m)$. We put

$$\Delta(I_k) = \bigcap_{j \in I_k} E_j, \quad k = 1, 2, \cdots$$

Then, $\Delta(I_k)$ is an $(n-2k)$-dimensional face of Δ. Writing I_k as $I_k = \{a_1, b_1, \cdots, a_k, b_k\}, k = 1, 2, \cdots$, we denote by $\theta(I_{k-1}, I_k)$ the dihedral angle between the faces $\Delta(I_{k-1}) \cap E_{a_k}$ and $\Delta(I_{k-1}) \cap E_{b_k}$. We consider $\theta(I_{k-1}, I_k)$ as a function on C_n^+ and put

$$\omega(I_{k-1}, I_k) = d\theta(I_{k-1}, I_k),$$

which is a 1-form on C_n^+. For $1 \leq m \leq \frac{n+1}{2}$ we put

$$c_{n,m} = \frac{(n-2m)!!}{(n-1)!!} \cdot \frac{V(S^{n-2m})}{2^{n-2m+1}}$$

where $V(S^{-1})$ is formally set to be 1. The following volume formula for a spherical simplex by iterated integrals is due to Aomoto [1].

Proposition 4.1. *The volume of the spherical simplex $\Delta(A) \subset S^n$ with a Gram matrix A is given by*

$$\begin{aligned} &V(\Delta(A)) \\ &= c_{n,0} + \sum_{1 \leq m \leq \left[\frac{n+1}{2}\right]} \left(\sum_{(I_0 \cdots I_m) \in \mathcal{F}_m[n]} c_{n,m} \int_E^A \omega(I_{m-1}, I_m) \cdots \omega(I_0, I_1) \right) \end{aligned}$$

where the iterated integral is for a path from the unit matrix E to A in C_n^+.

Proof. By Schläfli's differential equality we have

$$V(\Delta(A)) = \frac{V(S^n)}{2^{n+1}} + \frac{1}{n-1} \sum_{I_1} \int_E^A V_{n-2}(\Delta(I_1))\, \omega(I_0, I_1)$$

where the sum is for all $I_1 \subset I$ with $|I_1| = 2$. Applying Schläfli's differential equality for $V_{n-2}(\Delta(I_1))$ we have

$$\begin{aligned} V(\Delta(A)) = \ & c_{n,0} + \sum_{I_1} c_{n,1} \int_E^A \omega(I_0, I_1) \\ & + \frac{1}{(n-1)(n-3)} \sum_{I_1 \subset I_2} \int_E^A V_{n-4}(\Delta(I_2))\, \omega(I_1, I_2) \omega(I_0, I_1). \end{aligned}$$

Repeating inductively this procedure we obtain our Proposition. □

The formula in Proposition 4.1 can be simplified by taking a base point on the boundary of C_n^+ in the following way. We take a sequence of Gram matrices A_k in C_n^+ converging to a point $\mathbf{x}_0$ in the Euclidean locus C_n^0. Then, we have $\lim_{k\to\infty} V(\Delta(A_k)) = 0$, which leads us to the following Proposition.

Proposition 4.2. *We put $m = \left[\frac{n+1}{2}\right]$. The volume of the n-dimensional spherical simplex $\Delta(A) \subset S^n$ with a Gram matrix A is expressed as*

$$V(\Delta(A)) = \sum_{(I_0 \cdots I_m) \in \mathcal{F}_m[n]} \frac{1}{(n-1)!!} \int_{\mathbf{x}_0}^A \omega(I_{m-1}, I_m) \cdots \omega(I_0, I_1).$$

For $I_k = \{a_1, b_1, \cdots, a_k, b_k\}, k = 1, 2, \cdots$, we denote by $D(I_k)$ the small determinant of the Gram matrix A with rows and columns indexed by I_k. We set $D(I_{k-1}, I_k)$ to be the small determinant of A with rows and and columns indexed by $I_{k-1} \cup \{a_k\}$ and $I_{k-1} \cup \{b_k\}$ respectively.

In [1] Aomoto showed that the the 1-forms $\omega(I_{k-1}, I_k)$ are expressed as logarithmic forms as follows.

Proposition 4.3. *The 1-form $\omega(I_{k-1}, I_k) = d\theta(I_{k-1}, I_k)$ is expressed by means of small determinants of the Gram matrix as*

$$\begin{aligned}\omega(I_{k-1}, I_k) =& \frac{1}{2i}\, d\log\left(\frac{-D(I_{k-1}, I_k) + i\sqrt{D(I_{k-1})D(I_k)}}{-D(I_{k-1}, I_k) - i\sqrt{D(I_{k-1})D(I_k)}}\right)\\ =& d\arctan\left(-\frac{\sqrt{D(I_{k-1})D(I_k)}}{D(I_{k-1}, I_k)}\right).\end{aligned}$$

example. Let us describe the volume of a 3-dimensional spherical simplex in terms of θ_{ij}, $1 \leq i < j \leq 4$, the dihedral angles between the 2-dimensional faces. The volume of a 3-dimensional spherical simplex for a Gram matrix A satisfies.

$$dV(\Delta(A)) = \frac{1}{2}\sum_{1\leq i<j\leq 4} \arctan\left(-\frac{\sin\theta_{ij}\sqrt{\det A}}{D_{ij}}\right) d\theta_{ij} \tag{4.1}$$

where D_{ij} stands for the small determinant $D(\{i,j\}, \{1,2,3,4\})$. From the formula (4.1) we recover the equalities for the volume of a 3-dimensional spherical orthosimplex obtained by Coxeter in [4]. We refer the reader to [3] and [7] for more recent developments on the volume of a 3-dimensional spherical or hyperbolic simplex.

5. Analytic continuation to hyperbolic volumes

In this section, we consider the volume of a spherical simplex

$$S(A) = \sum_{(I_0\cdots I_m)\in\mathcal{F}_m[n]} \frac{1}{(n-1)!!}\int_{\mathbf{x}_0}^{A} \omega(I_{m-1}, I_m)\cdots\omega(I_0, I_1)$$

obtained in Proposition 4.2 as a function in $A \in C_n^+$, which we shall call the Schläfli function. We denote by $X_{n+1}(\mathbf{C})$ the set of $(n+1)\times(n+1)$ symmetric unidiagonal complex matrices. For a subset J of $I = \{1, 2, \cdots, n+1\}$ we define $Z(J)$ to be the set consisting of $A \in X_{n+1}(\mathbf{C})$ such that the small determinant $D(J)$ of A vanishes. We set

$$\mathcal{Z} = \bigcup_{J\subset I,\ |J|\equiv 1 \bmod 2} Z(J).$$

Then we have the following Lemma.

Lemma 5.1. *The differential form $\omega(I_{k-1}, I_k)$ is holomorphic on the set $X_{n+1}(\mathbf{C}) \setminus \mathcal{Z}$.*

Proof. We set $I_k = \{a_1, b_1, \cdots, a_k, b_k\}$ as in the previous section. By Proposition 4.3 the differential form $\omega(I_{k-1}, I_k)$ is possibly singular only in the case the equality

$$D(I_{k-1}, I_k)^2 + D(I_{k-1})D(I_k) = 0$$

holds. It follow from the Jacobi determinant identity that

$$D(I_{k-1}, I_k)^2 + D(I_{k-1})D(I_k) = D(I_{k-1}\cup\{a_k\})D(I_{k-1}\cup\{b_k\}),$$

which implies that $\omega(I_{k-1}, I_k)$ is holomorphic on the set

$$X_{n+1}(\mathbf{C}) \setminus \{Z(I_{k-1}\cup\{a_k\}) \cup Z(I_{k-1}\cup\{b_k\})\}.$$

This completes the proof. $\square$

Lemma 5.2. *For* $(I_0, \cdots, I_m) \in \mathcal{F}_m[n]$ *there is a relation*

$$\sum_K \omega(I_k, K) \wedge \omega(K, I_{k+2}) = 0$$

where the sum is taken for any K *with* $|K| = 2k+2$ *and* $I_k \subset K \subset I_{k+2}$, $k = 0, 1, \cdots, m-2$.

Proof. We express I_{k+2} as $I_{k+2} = I_k \cup \{j_1, \cdots, j_4\}$ and put $E'_{j_p} = \Delta(I_k) \cap E_{j_p}$, $j = 1, \cdots, 4$. Let P_k be the polyhedron with faces E'_{j_p}, $j = 1, \cdots, 4$. It follows from Schläfli's differential equality that

$$dV_{n-2k}(P_k) = \frac{1}{n-2k-1} \sum_K V_{n-2k-2}(\Delta(K)) d\theta(I_k, K) \tag{5.1}$$

where the sum is taken for any K with $|K| = 2k+2$ and $I_k \subset K \subset I_{k+2}$. Here the volume $V_{n-2k-2}(\Delta(K))$ is proportional to the dihedral angle $\theta(K, I_{k+2})$. Therefore, by taking the exterior derivative of the equation (5.1), we obtain the desired relation. □

Let $\gamma : [0,1] \to X_{n+1}(\mathbf{C})$ be a smooth path such that $\gamma(0) = \mathbf{x}_0$, $\gamma(1) = A$ and $\gamma(t) \in X_{n+1}(\mathbf{C}) \setminus \mathcal{Z}$ for $t > 0$. We fix $I_p, I_q \subset I$ with $I_p \subset I_q$, $|I_p| = 2p$ and $|I_q| = 2q$. Then we have the following Theorem.

Theorem 5.1. *The iterated integral*

$$\mathcal{I}_\gamma(A) = \sum_{I_p \subset I_{p+1} \subset \cdots \subset I_q} \int_\gamma \omega(I_{q-1}, I_q) \cdots \omega(I_p, I_{p+1})$$

is invariant under the homotopy of a path γ *fixing the endpoints* $\mathbf{x}_0$ *and* A.

Proof. As a function on the space of paths connecting $\mathbf{x}_0$ and A we have

$$\begin{aligned} & d\int_\gamma \omega(I_{q-1}, I_q) \cdots \omega(I_p, I_{p+1}) \\ = & \sum_{k=2}^{m-1} \omega(I_{q-1}, I_q) \cdots \omega(I_{k-1}, I_k) \wedge \omega(I_{k-2}, I_{k-1}) \cdots \omega(I_p, I_{p+1}) \end{aligned}$$

by Proposition 3.1. Then by applying of Lemma 5.2 we obtain $d\mathcal{I}_\gamma(A) = 0$, which shows the homotopy invariance of $\mathcal{I}_\gamma(A)$. □

Combining the above Theorem with Proposition 4.2 and Lemma 5.1 we obtain the following Theorem.

Theorem 5.2. *The Schläfli function* $S(A)$ *is analytically continued to a multi-valued function on* $X_{n+1}(\mathbf{C}) \setminus \mathcal{Z}$.

In particular, the Schläfli function $S(A)$, $A \in C_n^+$, is continued to an analytic function on the set of Gram matrices $C_n = C_n^+ \cup C_n^0 \cup C_n^-$, which is relevent to the hyperbolic volume in the following way.

Corollary 5.1. *For* $A \in C_n^-$ *the volume of the hyperbolic simplex* $\Delta(A) \subset \mathcal{H}^n$ *is related to the analytic continuation of the Schläfli function by the formula* $i^n V(\Delta(A)) = S(A)$.

Proof. Let $H_1, \cdots, H_{n+1}$ be hyperplanes in $\mathbf{R}^{n+1}$ in general position and

$$f_j(x_1, \cdots, x_{n+1}) = u_{j1}x_1 + \cdots + u_{jn}x_n + u_{j,n+1}x_{n+1}, \quad 1 \le j \le n+1$$

linear forms defining H_j. We express the volume of the spherical simplex $\Delta = S^n \cap C$ as in formula (2.1). Let ξ be a positive real number close to 1 and we deform the linear forms as

$$f_j^{\xi}(x_1, \cdots, x_{n+1}) = u_{j1}x_1 + \cdots + u_{jn}x_n + \xi u_{j,n+1}x_{n+1}, \quad 1 \le j \le n+1.$$

Let C_ξ be the corresponding cone defined by the intersection of the half spaces $f_j^{\xi} \ge 0$ for $1 \le j \le n+1$. The volume of the simplex $\Delta_\xi = S^n \cap C_\xi$ is expressed as

$$V(\Delta_\xi) = \frac{2}{\Gamma\left(\frac{n+1}{2}\right)} \int_{C_\xi} e^{-x_1^2 - \cdots - x_{n+1}^2}\, dx_1 \cdots dx_{n+1}$$

Let A_ξ be the Gram matrix for the simplex Δ_ξ. By the change of variables $x_j = \xi y_j$, $1 \le j \le n$, $x_{n+1} = y_{n+1}$, $S(A_\xi)$ is expressed as

$$S(A_\xi) = \frac{2}{\Gamma\left(\frac{n+1}{2}\right)} \int_{C} \xi^n e^{-\xi^2 y_1^2 - \cdots - \xi^2 y_n^2 - y_{n+1}^2}\, dy_1 \cdots dy_{n+1} \tag{5.2}$$

Now we consider the analytic continuation of the Schläfli function with respect to the path A_ξ, $\xi = e^{i\theta}$, $0 \le \theta \le \frac{\pi}{2}$. Comparing the formula (5.2) with $\xi = i$ and the formula (2.2) we obtained the desired relation. □

The volume corrected curvature $\kappa V(\Delta(A))^{\frac{2}{n}}$ for $A \in C_n$ is scaling invariant and is considered to be a function on C_n. Since it is expressed as $S(A)^{\frac{2}{n}}$ we obtain the following Corollary, which was shown in [6] by an alternative method.

Corollary 5.2. *The volume corrected curvature $\kappa V(\Delta(A))^{\frac{2}{n}}$ is an analytic function on the set of Gram matrices $C_n = C_n^+ \cup C_n^0 \cup C_n^-$.*

6. Nilpotent connections

In this section we describe the differential equation satisfied by iterated integrals appearing in the expression of the Schläfli function. We set $m = \left[\frac{n+1}{2}\right]$ and I_p denotes a subset of I with $|I_p| = 2p$. For an integer $0 \le k \le m$ we define the function $f(I_{m-k}; z)$ by

$$f(I_{m-k}; z) = \sum_{I_{m-k} \subset I_{m-k+1} \subset \cdots \subset I_m} \int_\gamma \omega(I_{m-1}, I_m) \cdots \omega(I_{m-k}, I_{m-k+1})$$

where γ is a path from the base point $\mathbf{x}_0$ to $z \in X_{n+1}(\mathbf{C}) \setminus \mathcal{Z}$. Let W_k be the vector space over $\mathbf{C}$ spanned by $f(I_{m-k}; z)$ for all $I_{m-k} \subset I$. In particular, W_0 is set to be the one dimensional vector space consisting of constant functions. We put $W = W_0 \oplus W_1 \oplus \cdots \oplus W_m$.

Theorem 6.1. *Any $\varphi \in W$ satisfies the differential equation $d\varphi = \varphi\,\omega$ with*

$$\omega = \sum_{I_{p-1} \subset I_p} A(I_{p-1}, I_p)\, \omega(I_{p-1}, I_p)$$

where $A(I_{p-1}, I_p)$ is a nilpotent endomorphism of W.

Proof. Let Ω^1 be the vector space spanned by all $\omega(I_{p-1}, I_p)$ for $I_{p-1} \subset I_p \subset I$. The operator d acts as $d\,W_k \subset W_{k-1} \otimes \Omega^1$ since we have

$$\begin{aligned} & df(I_{m-k}; z) \\ &= \sum_{I_{m-k} \subset I_{m-k+1}} \left(\sum_{I_{m-k+1} \subset \cdots \subset I_{m-1} \subset I_m} f(I_{m-k+1}; z) \right) \omega(I_{m-k}, I_{m-k+1}). \end{aligned}$$

Expressing d by means of linear endomorphisms of W, we obtain the assertion. □

References

[1] K. Aomoto, *Analytic structure of Schläfli function*, Nagoya J. Math., **68** (1977), 1–16.

[2] K. T. Chen, *Iterated path integrals*, Bull. Amer. Math. Soc., **83** (1977), 831–879.

[3] Y. Cho and H. Kim, *On the volume formula for hyperbolic tetrahedra*, Discrete Comput. Geom., **22** (1999), 347–366.

[4] H. S. M. Coxeter, *The functions of Schläfli and Lobatchefsky*, Quart. J. Math., **6** (1935), 13–29.

[5] H. Kneser, *Der Simplexinhalt in der nichteuklidische Geometrie*, Deutsche Math **1** (1936), 337–340.

[6] J. Milnor, *The Schläfli differential equality*, in "John Milnor Collected Papers", **1** (1994), 281–295.

[7] J. Murakami and M. Yano, *On the volume of a hyperbolic and spherical tetrahedron*, Comm. Anal. Geom., **13** (2005), 379–400.

[8] L. Schläfli, *On the multiple integral* $\int^n dxdy \cdots dz$, *whose limits are* $p_1 = a_1x + b_1y + \cdots + h_1z > 0, p_2 > 0, \cdots, p_n > 0$, *and* $x^2 + y^2 + \cdots + z^2 < 1$, Quart. J. Math., **3** (1860), 54–68, 97–108.

Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Meguro-ku, Tokyo 153-8914, Japan

E-mail address: kohno@ms.u-tokyo.ac.jp