HIGHER HOLONOMY AND ITERATED INTEGRALS

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Abstract. We develop a method to construct representations of the homotopy n-groupoid of a manifold as an n-category by means of K.-T. Chen's formal homology connections for any positive integer n. We establish a higher holonomy functor from the homotopy n-groupoid to a category obtained from the tensor algebra over the homology of the manifold.

1. Introduction

The purpose of this article is to give a systematic treatment of n-holonomy functors from the homotopy n-groupoid of a manifold by means of K.-T. Chen's formal homology connections for any positive integer n. The n-categories play an important role in higher gauge theory (see Baez and Huerta [2]). In particular, the 2-holonomy maps have been investigated in the framework of 2-connections with values in 2-Lie algebras and their generalization to higher holonomy has been studied by several authors (see [1] and [14]).

On the other hand, the notion of formal homology connections was developed by K.-T. Chen in the theory of iterated integrals of differential forms in order to describe the homology group of the loop space of a manifold M by the chain complex formed by the tensor algebra of the homology group of M (see [3] and [4]). We apply such method to construct n-holonomy functors. The case $n = 2$ was described in the author's former works [11] and [12] and an essential idea contained in these articles. In this paper we develop a more general framework for any positive integer n.

We describe some historical background concerning these subjects. The notion of crossed modules which are closely related to 2-groups appeared in the work of J. H. C. Whitehead ([16] and [17]) in the study of homotopy 2-types of spaces. We should mention that there is important progress concerning the notion of ∞-groupoids developed by A. Grothendieck in "Pursuing Stacks" (see [13]).

We mention some motivations and related developments concerning higher holonomy functors. It is an important problem to construct a 2-category version of the Knizhnik-Zamolodchikov (KZ) connections. There is a work by L. S. Cirio and J. F. Martins [6] on the categorification of the KZ connections by means of 2-Yang-Baxter operators for $sl_2(\mathbf{C})$. An approach to such problems using iterated integrals was also developed in [12]. One of our aims is to apply such method to braided surfaces in 4-space studied by S. Carter, S. Kamada and M. Saito (see [7], [9]). We investigated an application of 2-holonomy functors to a construction of representations of the 2-category of braid cobordisms (see [11]). Related to such problems, we discuss higher holonomy for the complement of a hyperplane arrangement in this paper.

The paper is organized in the following way. In Section 2 we briefly review K.-T. Chen's iterated integrals and their basic properties. In particular, we recall the formula for the composition of plots. In Section 3 we recall the notion of n-fold homotopy and a globular category. In Section 4 we describe the notion of formal homology connections. We explain 2-connections and 2-curvatures in this framework. In Section 5 we review path n-groupoids and homotopy n-groupoids. In Section 6 we give a construction of higher holonomy functors based on K.-T. Chen's formal homology connections. Finally, we discuss the case of the complement of hyperplane arrangements in Section 7.

2. Preliminaries on K.-T. Chen's iterated integrals

First, we briefly recall the notion of iterated integrals of differential forms due to K.-T. Chen. We refer the reader to [3] and [4] for details. Let M be a smooth manifold and $\omega_1, \cdots, \omega_k$ be differential forms on M. We fix two points $\mathbf{x}_0$ and $\mathbf{x}_1$ in M and consider a smooth path $\gamma : [0,1] \to M$ with $\gamma(0) = \mathbf{x}_0$ and $\gamma(1) = \mathbf{x}_1$. We suppose that for a sufficiently small ε such that $0 < \varepsilon < 1-\varepsilon < 1$ we have $\gamma(t) = \mathbf{x}_0$ for $0 \leq t \leq \varepsilon$ and $\gamma(t) = \mathbf{x}_1$ for $1-\varepsilon \leq t \leq 1$. Namely, we assume that the path γ is constant in a neighborhood of $t=0$ and $t=1$. We denote by $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ the set of the smooth paths satisfying the above properties. The above assumption is needed to define the composition of smooth paths as a smooth path. It is also possible to give a formulation by piecewise smooth paths, but in this paper we adopt this convention.

In particular, in the case $\mathbf{x}_0 = \mathbf{x}_1$ the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ is called the based loop space of M and is denoted by $\Omega_{\mathbf{x}_0} M$. In the following we suppose that the differential forms $\omega_1, \cdots, \omega_k$ are of positive degrees. We denote by

$$p_j : \underbrace{M \times \cdots \times M}_{k} \longrightarrow M, \ 1 \leq j \leq k$$

the projection to the j-th factor and set

$$\omega_1 \times \cdots \times \omega_k = p_1^* \omega_1 \wedge \cdots \wedge p_k^* \omega_k.$$

We consider the simplex

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^k \ ; \ 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$$

and the evaluation map

$$\varphi : \Delta_k \times \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1) \to \underbrace{M \times \cdots \times M}_{k}$$

defined by $\varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k))$. The iterated integral of $\omega_1, \cdots, \omega_k$ is defined as

$$\int \omega_1 \cdots \omega_k = \int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

where the expression

$$\int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

is the integration along the fiber with respect to the projection

$$p : \Delta_k \times \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1) \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1).$$

The above iterated integral is considered as a differential form on the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ with the degree $q_1 + \cdots + q_k - k$, where we set $q_j = \deg \omega_j$. In order to justify differential forms on the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ we use the notion of plots. A plot $\alpha : U \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ is a family of smooth paths in the above sense smoothly parametrized by a compact convex set U in a finite dimensional Euclidean space. Given a plot α we denote the corresponding iterated integral

$$\left(\int \omega_1 \cdots \omega_k \right)_\alpha$$

as a differential form on U obtained by pulling back the iterated integral $\int \omega_1 \cdots \omega_k$ by the plot α. Namely, the above expression stands for

$$\int_{\Delta_k} ((\mathrm{id} \times \alpha) \circ \varphi)^* (\omega_1 \times \cdots \times \omega_k)$$

where we consider the integration along the fiber with respect to the projection $\Delta_k \times U \to U$. We denote by $\Omega^*(\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1))$ the set of such differential forms on the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ obtained as iterated integrals of differential forms of positive degrees on M.

We take an extra point $\mathbf{x}_2$ in M and consider the plots

$$\alpha : U \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1), \quad \beta : U \longrightarrow \mathcal{P}(M; \mathbf{x}_1, \mathbf{x}_2).$$

The composition of the plots α and β

$$\alpha\beta : \ U \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_2)$$

is defined by

$$\alpha\beta(x)(t) = \begin{cases} \alpha(x)(2t), & 0 \leq t \leq \frac{1}{2} \\ \beta(x)(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

for $x \in U$. As is shown by K.-T. Chen, we have the following rule for the composition of plots.

Proposition 2.1. *The relation*

$$\left(\int \omega_1 \cdots \omega_k \right)_{\alpha\beta} = \sum_{0 \leq i \leq k} \left(\int \omega_1 \cdots \omega_i \right)_\alpha \wedge \left(\int \omega_{i+1} \cdots \omega_k \right)_\beta$$

holds

For a path α we define its inverse path α^{-1} by

$$\alpha^{-1}(t) = \alpha(1-t).$$

For the composition $\alpha\alpha^{-1}$ we have

$$\left(\int \omega_1 \cdots \omega_i \right)_{\alpha\alpha^{-1}} = 0.$$

As a differential form on the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ we have the following.

Proposition 2.2. *For the iterated integral* $\int \omega_1 \cdots \omega_k$ *we have*

$$\begin{aligned} & d\int \omega_1 \cdots \omega_k \\ =& \sum_{j=1}^{k} (-1)^{\nu_{j-1}+1} \int \omega_1 \cdots \omega_{j-1} d\omega_j \ \omega_{j+1} \cdots \omega_k \\ &+ \sum_{j=1}^{k-1} (-1)^{\nu_j+1} \int \omega_1 \cdots \omega_{j-1} (\omega_j \wedge \omega_{j+1}) \omega_{j+2} \cdots \omega_k \end{aligned}$$

where we put $\nu_j = \deg \omega_1 + \cdots + \deg \omega_j - j$ *for* $j \geq 1$ *and* $\nu_0 = 0$.

Thus we obtain the complex $\Omega^*(\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1))$ with the differential

$$d : \Omega^q(\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)) \longrightarrow \Omega^{q+1}(\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1))$$

explicitly given as in Proposition 2.2.

Let σ be a permutation of $k + \ell$ letters $\{1, 2, \cdots, k + \ell\}$. We say that σ is a (k, ℓ)-shuffle if

$$\begin{gathered} \sigma^{-1}(1) < \cdots < \sigma^{-1}(k) \\ \sigma^{-1}(k+1) < \cdots < \sigma^{-1}(k+\ell) \end{gathered}$$

holds. We denote by $S_{k,\ell}$ the set of (k, ℓ)-shuffles. Let $\omega_1, \cdots, \omega_{k+\ell}$ be differential forms on M. We put $\deg \omega_j = p_j$, $1 \leq j \leq k + \ell$. For $\sigma \in S_{k,\ell}$ we have

$$\omega_{\sigma(1)} \wedge \cdots \wedge \omega_{\sigma(k+\ell)} = \epsilon \ \omega_1 \wedge \cdots \wedge \omega_{k+\ell},$$

with $\epsilon = \pm 1$. We denote such ϵ by $\epsilon(\sigma; \ p_1, \cdots, p_{k+\ell})$. With this notation we have the following.

Proposition 2.3. *The wedge product*

$$\left(\int \omega_1 \cdots \omega_k \right) \wedge \left(\int \omega_{k+1} \cdots \omega_{k+\ell} \right)$$

is expressed as

$$\sum_{\sigma \in S_{k,\ell}} \epsilon(\sigma; \ p_1 - 1, \cdots, p_{k+\ell} - 1) \int \omega_{\sigma(1)} \cdots \omega_{\sigma(k+\ell)}.$$

3. Smooth homotopy and n-fold smooth homotopy

Let $\gamma_0 : [0, 1] \to M$ and $\gamma_1 : [0, 1] \to M$ be smooth paths with $\gamma_i(0) = \mathbf{x}_0$ $\gamma_i(1) = \mathbf{x}_1$ for $i = 0, 1$ in the sense of the previous section. A smooth homotopy between γ_0 and γ_1 is a smooth map

$$H : [0, 1]^2 \longrightarrow M$$

such that the conditions

$$\begin{gathered} H(0, s) = \mathbf{x}_0, \ H(1, s) = \mathbf{x}_1 \\ H(t, 0) = \gamma_0(t), \ H(t, 1) = \gamma_1(t) \end{gathered}$$

are satisfied. Here we suppose that $H(t, s)$ is constant in a neighborhood of $t = 0$ and $t = 1$ and that $H(t, s)$ is independent of s in a neighborhood of $s = 0$ and $s = 1$.

We denote the unit interval $[0,1]$ by I. Let D^2 denote the 2-dimensional disc consisting of $\mathbf{x} \in \mathbf{R}^2$ with $\|\mathbf{x}\| \leq 1$. By means of the identification

$$D^2 \cong I^2/(\partial I) \times I$$

we see that the above homotopy H factors through the quotient map $\pi : I^2 \longrightarrow D^2$.

A smooth homotopy $H : I^2 \to M$ is called a thin homotopy if it sweeps out a surface with zero area. More precisely, a thin homotopy is a smooth homotopy H such that the rank of the differential dH_p is less than 2 at every point $p \in I^2$. If two smooth paths differ by a smooth reparametrization, they are thinly homotopic. For a smooth path γ the composite $\gamma^{-1}\gamma$ is thinly homotopic to the constant path.

We inductively define an n-fold smooth homotopy for $n \geq 2$. A 2-fold smooth homotopy $H : I^2 \to M$ is a smooth homotopy between the paths γ_0 and γ_1 in the above sense. For $k \geq 2$ suppose that k-fold smooth homotopies $H_0, H_1 : I^k \to M$ are defined. A $(k+1)$-fold smooth homotopy between H_0 and H_1 is a smooth map

$$H : I^{k+1} \longrightarrow M$$

such that the conditions

$$H(t_1, \cdots, t_k, 0) = H_0(t_1, \cdots, t_k), \; H(t_1, \cdots, t_k, 1) = H_1(t_1, \cdots, t_k)$$

are satisfied. Here we assume that H is constant in a neighborhood of $t_1 = 0$ and $t_1 = 1$ and that H is independent of t_i in a neighborhood of $t_i = 0$ and $t_i = 1$ for $2 \leq i \leq k+1$. An n-fold smooth homotopy $H : I^n \to M$ is called an n-fold thin homotopy if the rank of the differential dH_p is less than n at every point $n \in I^n$.

We denote by D^n the n-dimensional disc defined by

$$D^n = \{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x}\| \leq 1\}.$$

By means of the identification $D^n \cong I^n/(\partial I^{n-1}) \times I$ an n-fold smooth homotopy $H : I^n \to M$ factors through the quotient map $\pi : I^n \to D^n$. The map H defines an $(n-1)$-parameter family of smooth paths connecting $\mathbf{x}_0$ and $\mathbf{x}_1$ by putting

$$\gamma_{t_1 \cdots, t_{n-1}}(t) = H(t, t_1, \cdots, t_{n-1}).$$

This gives a cubical $(n-1)$-chain of the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ by defining

$$c : I^{n-1} \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$$

as $c(t_1, \cdots, t_{n-1})(t) = \gamma_{t_1 \cdots, t_{n-1}}(t)$.

We define the inclusion maps σ_i and τ_i from D^{i-1} to D^i by

$$\sigma_i(\mathbf{x}) = (\mathbf{x}, -\sqrt{1 - \|\mathbf{x}\|^2}), \quad \tau_i(\mathbf{x}) = (\mathbf{x}, -\sqrt{1 - \|\mathbf{x}\|^2})$$

for $i > 0$. Here we set D^0 a set consisting of one point p and define $\sigma_1(p) = -1, \tau_1(p) = 1$. The above n-fold smooth homotopy H is interpreted as a homotopy $H : D^n \to M$ between $H_0 : \sigma_n(D^{n-1}) \to M$ and $H_1 : \tau_n(D^{n-1}) \to M$.

The inclusion maps σ_i and τ_i, $i > 0$, satisfy the relations:

$$\sigma_{i+1}\sigma_i = \tau_{i+1}\sigma_i, \quad \sigma_{i+1}\tau_i = \tau_{i+1}\tau_i.$$

For $i, j \geq 0$ with $i < j$ we set

$$\sigma_i^j = \sigma_j \cdots \sigma_{i+2}\sigma_{i+1}, \quad \tau_i^j = \tau_j \cdots \tau_{i+2}\tau_{i+1}.$$

We define the category $\mathbf{G}$ as follows. The set objects of $\mathbf{G}$ consists of

$$D^0, D^1, \cdots, D^i, \cdots.$$

and the set of morphisms is defined as

$$\mathrm{Hom}_{\mathbf{G}}(D^i, D^j) = \begin{cases} \{\sigma_i^j, \tau_i^j\} & \text{if } i < j, \\ \{1_{D_i}\} \text{ if } & i = j, \\ \emptyset & \text{otherwise.} \end{cases}$$

The category $\mathbf{G}$ is called a globular category.

The above map σ_n defined an n-fold smooth homotopy on $S^n = \partial D^{n+1}$ in the following way. By composing with the projection map $\pi : I^n \to I^n/(\partial I^{n-1}) \times I \cong D^n$, we put $f_n = \pi \circ \sigma_n$. In a similar way we obtain an an n-fold smooth homotopy $g_n = \pi \circ \tau_n$. We observe that f_n and g_n are not $(n+1)$-fold homotopic. We will discuss this from the point of view of n-holonomy in the next sections.

4. Formal homology connections

Let M be a connected smooth manifold. We denote by $H_q(M; \mathbf{R})$ the q-th homology group of M with coefficients in $\mathbf{R}$. We put

$$H_+(M) = \bigoplus_{q>0} H_q(M; \mathbf{R})$$

and consider the tensor algebra

$$TH_+(M) = \bigoplus_{k \geq 0} \left(\bigotimes^k H_+(M) \right).$$

In the following we suppose that $\dim H_+(M)$ is finite. We denote by $\Omega^*(M)$ the algebra of differential forms on M and consider the tensor product $\Omega^*(M) \otimes TH_+(M)$. We define $d : \Omega^q(M) \otimes TH_+(M) \longrightarrow \Omega^{q+1}(M) \otimes TH_+(M)$ by

$$d(\omega \otimes X) = d\omega \otimes X,\ \omega \in \Omega^*(M),\ X \in TH_+(M; \mathbf{R}).$$

When $H_+(M)$ has a basis $X_1, \cdots, X_m$ as a vector space over $\mathbf{R}$, the algebra $\Omega^*(M) \otimes TH_+(M)$ is identified with the ring of non-commutative polynomials

$$\Omega^*(M)[X_1, \cdots, X_m]$$

over $\Omega^*(M)$. For $X_i \in H_{p_i}(M)$ we assign the degree of X_i as

$$\deg X_i = p_i - 1.$$

For the product of homogeneous elements we extend the degree of $X_{i_1} \cdots X_{i_k}$ as

$$\deg X_{i_1} \cdots X_{i_k} = \sum_{p=1}^{k} \deg X_{i_p}.$$

In this way we regard $TH_+(M)$ as a graded algebra. For homogeneous elements X, Y in $\Omega^*(M)[X_1, \cdots, X_m]$ we define the graded Lie bracket by

$$[X, Y] = XY - (-1)^{pq} YX$$

where $\deg X = p$ and $\deg Y = q$.

The augmentation map

$$\epsilon : \mathbf{R}[X_1, \cdots, X_m] \longrightarrow \mathbf{R}$$

is the homomorphism of algebras defined by $\epsilon(X_k) = 0,\ 1 \leq k \leq m$. We denote by J the kernel of the augmentation map ϵ, which is the 2-sided ideal of $TH_+(M)$

generated by $X_1, \cdots, X_m$. We consider the completion of $TH_+(M)$ with respect to the powers of the augmentation ideal as

$$\widehat{TH_+(M)} = \lim_{\leftarrow} TH_+(M)/J^k.$$

The tensor product $\Omega^*(M) \otimes \widehat{TH_+(M)}$ is identified with the ring of non-commutative formal power series

$$\Omega^*(M)\langle\langle X_1, \cdots, X_m \rangle\rangle$$

over $\Omega^*(M)$. We denote by $\widehat{TH_+(M)}_q$ the degree q part of $\widehat{TH_+(M)}$ with respect to the above degrees.

For a differential form ω we define the parity operator ε as $\varepsilon(\omega) = \omega$ when ω is of even degree and $\varepsilon(\omega) = -\omega$ when ω is of odd degree. For $\omega \otimes X \in \Omega^*(M) \otimes \widehat{TH_+(M)}$ we set $\varepsilon(\omega \otimes X) = \omega \otimes X$ if ω is a differential form of even degree and $\varepsilon(\omega \otimes X) = -\omega \otimes X$ if ω is a differential form of odd degree. Extending the above map linearly we obtain the operator ε on $\Omega^*(M) \otimes \widehat{TH_+(M)}$.

We extend the wedge product and iterated integrals on $\Omega^*(M) \otimes \widehat{TH_+(M)}$ as

$$(\omega \otimes X) \wedge (\varphi \otimes Y) = (\omega \wedge \varphi) \otimes XY,$$
$$\int (\varphi_1 \otimes Z_1) \cdots (\varphi_k \otimes Z_k) = \left(\int \varphi_1 \cdots \varphi_k \right) \otimes Z_1 \cdots Z_k.$$

We say that a linear map

$$\delta : \widehat{TH_+(M)}_q \longrightarrow \widehat{TH_+(M)}_{q-1}$$

is a derivation of degree -1 if it satisfies the Leibniz rule

$$\delta(uv) = (\delta u)v + (-1)^{\deg u} u(\delta v).$$

Following K.-T. Chen [3], we recall the definition of a formal homology connection. A formal homology connection

$$\omega \in \Omega^*(M) \otimes \widehat{TH_+(M)}$$

is by definition an expression written as

$$\omega = \sum_{i=1}^{m} \omega_i X_i + \cdots + \sum_{i_1 \cdots i_k} \omega_{i_1 \cdots i_k} X_{i_1} \cdots X_{i_k} + \cdots$$

with differential forms of positive degrees $\omega_{i_1 \cdots i_k}$ satisfying the following properties.

- $[\omega_i]$, $1 \leq i \leq m$, is a dual basis of X_i, $1 \leq i \leq m$.
- $\delta\omega + d\omega - \varepsilon(\omega) \wedge \omega = 0$.
- $\deg \omega_{i_1 \cdots i_k} = \deg X_{i_1} \cdots X_{i_k} + 1$
- δ is a derivation of degree -1.
- $\delta X_j \in \widehat{J}^2$ where $\widehat{J}$ is the augmentation ideal of $\widehat{TH_+(M)}$.

For a formal homology connection ω we define the generalized curvature κ by

$$\kappa = d\omega - \varepsilon(\omega) \wedge \omega.$$

From the above conditions it can be shown that $\delta \circ \delta = 0$ and $(\widehat{TH_+(M)}, \delta)$ forms a complex. The formal homology connection can be written in the sum

$$\omega = \omega^{(1)} + \omega^{(2)} + \cdots + \omega^{(p)} + \cdots$$

with the p-form part

$$\omega^{(p)} \in \Omega^p(M) \otimes \widehat{TH_+(M)}_{p-1}.$$

The 2-form part of κ is written as

$$\kappa^{(2)} = d\omega^{(1)} + \omega^{(1)} \wedge \omega^{(1)}$$

which coincides with the usual curvature form for $\omega^{(1)}$. From the equation $\delta\omega + \kappa = 0$ we have the equation

$$\delta\omega^{(2)} + d\omega^{(1)} + \omega^{(1)} \wedge \omega^{(1)} = 0.$$

Although the formal homology connection is not uniquely determined, we can construct it inductively starting from the initial term $\sum_{i=1}^m \omega_i X_i$. We refer the readers to [4] for details about the construction of a formal homology connections. Here are some examples.

Examples : (1) Let $\mathbf{C}P^n$ denote the complex n-dimensional projective space. and τ the Kähler form. For $k = 0, 1, \cdots, n$ the cohomology group $H_{2k}(\mathbf{C}P^n; \mathbf{R})$ is isomorphic to $\mathbf{R}$ and has a basis $[\tau^k]$. Let X_k denote the dual basis of $[\tau^k]$ in the homology group $H_{2k}(\mathbf{C}P^n; \mathbf{R})$. We put

$$\omega = \tau \otimes X_1 + \tau^2 \otimes X_2 + \cdots + \tau^n \otimes X_n$$

Then we have

$$\kappa = -\omega \wedge \omega = -\sum_{i,j \geq 1} \tau^{i+j} \otimes X_i X_j.$$

By defining

$$\begin{aligned} \delta X_1 &= 0 \\ \delta X_k &= \sum_{1 \leq i \leq k-1} X_i X_{k-i}, \quad 2 \leq k \leq n, \end{aligned}$$

we get the condition $\delta\omega + \kappa = 0$. The above (ω, δ) is a formal homology connection for $\mathbf{C}P^n$.

(2) Let G be the unipotent Lie group consisting of the matrices

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbf{R}$$

and $G_{\mathbf{Z}}$ its subgroup consisting of the above matrices with $x, y, z \in \mathbf{Z}$. We denote by M the quotient space of G by the left action of $G_{\mathbf{Z}}$. We see that M has a structure of a compact smooth 3-dimensional manifold. We put

$$\omega_1 = dx, \ \omega_2 = dy, \ \omega_{12} = -xdy + dz.$$

We observe that $H^1(M)$ has a basis represented by ω_1, ω_2 and $H^2(M)$ has a basis represented by $\omega_1 \wedge \omega_{12}, \omega_2 \wedge \omega_{12}$. These are typical examples of non-trivial Massey product. We denote by $X_1, X_2 \in H_1(M)$ the dual basis of $[\omega_1], [\omega_2]$ and by $Y_1, Y_2 \in H_2(M)$ the dual basis of $[\omega_1 \wedge \omega_{12}], [\omega_2 \wedge \omega_{12}]$. We obtain that the derivation δ is given by

$$\delta(X_1) = 0, \ \delta(X_2) = 0, \ \delta(Y_1) = [[X_1, X_2], X_1], \ \delta(Y_2) = [[X_1, X_2], X_2].$$

In the above example (1) the derivations δ are quadratic, which reflects the fact that the corresponding spaces are formal. On the other hand in the example (2) there are non-trivial Massey products and the derivations are not quadratic. We recall celebrated theorem of Deligne, Griffiths, Morgan and Sullivan [8] that a compact Kähler manifold is formal. Consequently, the derivation for the formal homology connection is quadratic in this case.

For the formal homology connection ω we define its transport by

$$T = 1 + \sum_{k=1}^{\infty} \int \underbrace{\omega \cdots \omega}_{k}.$$

The following proposition plays a key role for the construction of higher holonomy functors.

Proposition 4.1. *Given a formal homology connection (ω, δ) for a manifold M the transport T satisfies $dT = \delta T$.*

Proof. By Proposition 2.2 we have

$$\begin{aligned} dT &= -\int \kappa + \left(-\int \kappa\omega + \int \varepsilon(\omega)\kappa \right) + \cdots \\ &= \sum_{k=0}^{\infty} \sum_{i=0}^{k} (-1)^{i+1} \int \underbrace{\varepsilon(\omega) \cdots \varepsilon(\omega)}_{i} \ \kappa \ \underbrace{\omega \cdots \omega}_{k-i-1}. \end{aligned}$$

Substituting $\kappa = -\delta\omega$ in the above equation and applying the Leibniz rule for δ, we obtain the equation $dT = \delta T$. □

5. Path n-groupoids and homotopy n-groupoids

First we recall the notion of path groupoids and path 2-groupoids. We refer the reader to [2] for more details including the notion of 2-categories. A groupoid is a category such that all the morphisms are invertible. For a connected smooth manifold M we define the path groupoid $\mathcal{P}_1(M)$. We take $\mathbf{x}_0, \mathbf{x}_1 \in M$ and let γ_0 and γ_1 be smooth paths $\gamma_i : [0,1] \to M$, $i = 0, 1$, such that $\gamma_i(0) = \mathbf{x}_0$ and $\gamma_i(1) = \mathbf{x}_1$.

The path groupoid $\mathcal{P}_1(M)$ is a category whose objects are points in M and whose morphisms are smooth paths between points up to a thin homotopy in the sense of Section 3. Namely, for $\mathbf{x}_0, \mathbf{x}_1 \in M$ the set of morphisms between them is

$$\mathrm{Hom}(\mathbf{x}_0, \mathbf{x}_1) = \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)/\sim$$

where the paths $\gamma_0, \gamma_1 \in \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ satisfy the equivalence relation $\gamma_0 \sim \gamma_1$ if and only if the one is obtained from the other by a thin homotopy. We denote by $1_{\mathbf{x}}$ the thin homotopy class of the constant path at $\mathbf{x}$. We observe that in $\mathcal{P}_1(M)$ the composition of morphisms is associative and there exists an identity morphism $1_{\mathbf{x}}$. Each morphism γ has its inverse γ^{-1}. Therefore $\mathcal{P}_1(M)$ has a structure of a groupoid.

The homotopy path groupoid $\Pi_1(M)$ is defined as follows. The set objects of $\Pi_1(M)$ consists of points in M and the set of morphisms between two points $\mathbf{x}_0$ and $\mathbf{x}_1$ consists of smooth homotopy classes of smooth paths connecting $\mathbf{x}_0$ and $\mathbf{x}_1$. We see that $\Pi_1(M)$ has a structure of a groupoid and call it the homotopy path groupoid of M. We also call $\Pi_1(M)$ the fundamental groupoid of M.

Now we explain the notion of 2-categories. In general, a 2-category consists of objects, 1-morphisms and 2-morphisms, which are morphisms between morphisms. There are two kinds of compositions for 2-morphisms, horizontal compositions and vertical compositions and there are several coherency conditions among them.

The path 2-groupoid $\mathcal{P}_2(M)$ is a 2-category defined as follows. The objects are points in M and the 1-morphisms are smooth paths between points up to a thin homotopy. A 2-morphism between the thin homotopy classes of paths γ_0 and γ_1 is a smooth homotopy $H : [0,1]^2 \to M$ spanning the paths γ_0 and γ_1 considered up to a 3-fold thin homotopy in the sense of Section 3. Putting $c(s)(t) = H(t,s)$, we obtain a family of paths

$$c : [0,1] \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1).$$

We represent a 2-morphism between the paths γ_0 and γ_1 as such family of paths.

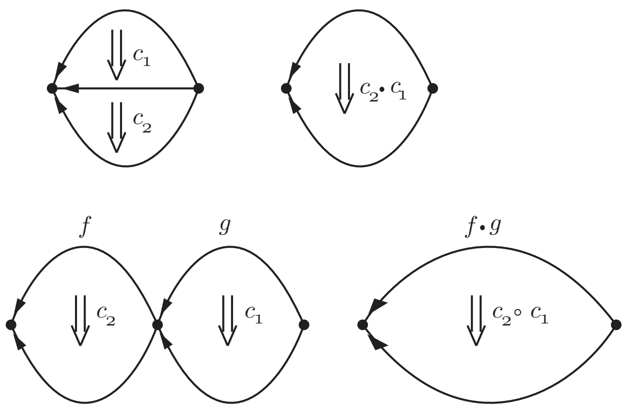

FIGURE 1. vertical and horizontal compositions

Let γ_0, γ_1 and γ_2 be smooth paths connecting $\mathbf{x}_0$ and $\mathbf{x}_1$. For a 2-morphism c_1 between γ_0 and γ_1 and a 2-morphism c_2 between γ_1 and γ_2 we define their vertical composition $c_2 \cdot c_1$ by the family of paths given by

$$(c_2 \cdot c_1)(s)(t) = \begin{cases} c_1(2s)(t), \ 0 \leq s \leq \frac{1}{2} \\ c_2(2s-1)(t), \ \frac{1}{2} \leq s \leq 1 \end{cases}$$

as depicted in Figure 1. For the 2-morphisms c_1 and c_2 respectively represented by the plots

$$\alpha_1 : I \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1), \quad \alpha_2 : I \longrightarrow \mathcal{P}(M; \mathbf{x}_1, \mathbf{x}_2).$$

We define their horizontal composition $c_2 \circ c_1$ by the composition of the plots $\alpha_2\alpha_1$. The vertical and horizontal compositions of 2-morphisms obey the interchange law

$$(c_2' \cdot c_2) \circ (c_1' \cdot c_1) = (c_2' \circ c_1') \cdot (c_2 \circ c_1)$$

We define the homotopy 2-groupoid $\Pi_2(M)$ in the following way. The objects consist of points in M and and the 1-morphisms are smooth paths between points up to a thin homotopy. A 2-morphism between the thin homotopy classes of paths γ_0 and γ_1 is a smooth homotopy $H : [0,1]^2 \to M$ spanning the paths γ_0 and γ_1 considered up to a 3-fold homotopy in the sense of Section 3. The homotopy 2-groupoid $\Pi_2(M)$ is also called the fundamental 2-groupoid of M.

We define the path n-groupoid $\mathcal{P}_n(M)$ and the homotopy n-groupoid $\Pi_n(M)$ for any $n \geq 1$ in the following way. First, we define the path n-groupoid $\mathcal{P}_n(M)$. The objects consist of points in M the 1-morphisms are smooth paths between points up to a thin homotopy. For $2 \leq k \leq n$ the k-morphisms are inductively defined as follows. Given $(k-1)$-morphisms represented by $(k-1)$-fold smooth homotopies $f, g : [0,1]^{k-1} \to M$ the k-morphisms from f to g are k-fold smooth homotopies $H : [0,1]^k \to M$ connecting f and g up to a $(k+1)$-fold thin homotopy. To define the homotopy n-groupoid $\Pi_n(M)$ we only modify the set of n-morphisms. Namely, the objects and k morphisms of $\Pi_n(M)$ for $1 \leq k \leq n-1$ are identical to those of $\mathcal{P}_n(M)$. Given $(n-1)$-morphisms represented by $(n-1)$-fold smooth homotopies $f, g : [0,1]^{n-1} \to M$ the n-morphisms from f to g are n-fold smooth homotopies $H : [0,1]^n \to M$ connecting f and g up to an $(n+1)$-fold smooth homotopy.

6. Higher holonomy functors

Now we construct a representation of the path groupoid $\mathcal{P}_1(M)$ by means of the iterated integrals of a formal homology connection. Let ω be a formal homology connection for M with the derivation δ. First, we consider the 1-form part of ω which is denoted by $\omega^{(1)}$. For a smooth path γ in M the holonomy of the connection $\omega^{(1)}$ is given the transport as

$$Hol(\gamma) = \langle T, \gamma \rangle = 1 + \sum_{k=1}^{\infty} \int_\gamma \underbrace{\omega^{(1)} \cdots \omega^{(1)}}_{k}$$

which is an element of $\widehat{TH_+(M)}_0$. Let us notice that the iterated integrals are independent of a thin homotopy of a path and that the above holonomy is well-defined. For the composition of paths we have

$$Hol(\alpha\beta) = Hol(\alpha) Hol(\beta)$$

by Proposition 2.1. Moreover, the relation

$$Hol(\alpha^{-1}) = Hol(\alpha)^{-1}$$

holds. Therefore, we obtain a representation of the path groupoid

$$Hol : \mathcal{P}_1(M) \longrightarrow \widehat{TH_+(M)}_0.$$

We denote by $\widehat{TH_+(M)}_0^\times$ the group of invertible elements in $\widehat{TH_+(M)}_0$. The above Hol is considered to be a functor from the path groupoid $\mathcal{P}_1(M)$ to the group $\widehat{TH_+(M)}_0^\times$. Let us construct a representation of the homotopy path groupoid $\Pi_1(M)$. We consider $\widehat{TH_+(M)}_1$ as a 2-sided module over $\widehat{TH_+(M)}_0$. Let $\mathcal{I}_0$ denote the 2-sided ideal of $\widehat{TH_+(M)}_1$ generated by the image of the derivation

$$\delta : \widehat{TH_+(M)}_1 \longrightarrow \widehat{TH_+(M)}_0.$$

Proposition 6.1. *The above holonomy map induces a well-defined functor.*

$$Hol : \Pi_1(M) \longrightarrow \widehat{TH_+(M)}_0^\times$$

Proof. We consider Hol as a function in $\gamma \in \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$. Then we have

$$d\, Hol(\gamma) = \langle dT, \gamma \rangle = \langle \delta T, \gamma \rangle$$

by Proposition 4.1. Hence we have $d\, Hol(\gamma) = 0$ in $\widehat{TH_+(M)}_0 / \mathcal{I}_0$. This shows that if $\gamma_0, \gamma_1 \in \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ are connected by a homotopy fixing the endpoints $\mathbf{x}_0, \mathbf{x}_1$, then we have $Hol(\gamma_0) = Hol(\gamma_1)$ in $\widehat{TH_+(M)}_0 / \mathcal{I}_0$. This completes the proof. □

Let $\gamma_0 : [0,1] \to M$ and $\gamma_1 : [0,1] \to M$ be smooth paths with $\gamma_i(0) = \mathbf{x}_0$ $\gamma_i(1) = \mathbf{x}_1$ for $i = 0, 1$. For a smooth homotopy between γ_0 and γ_1 denoted by

$$H : [0,1]^2 \longrightarrow M$$

we consider the associated 1-chain

$$c : [0,1] \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1).$$

For the formal homology connection we consider the transport

$$T = 1 + \sum_{k=1}^{\infty} \int \underbrace{\omega \cdots \omega}_{k}.$$

We regard its pullback $c^*T = T_c$ an element of $\Omega^*(I) \otimes \widehat{TH_+(M)}$. We denote by $\langle T, c\rangle$ the integration of the 1-form part of c^*T over the unit interval I. We define the 2-holonomy

$$Hol : \mathcal{P}_2(M) \longrightarrow \widehat{TH_+(M)}_1$$

by $Hol(c) = \langle T, c\rangle$. The symbol $\langle T, c\rangle$ stands for the integration of the 1-form part of T on the 1-chain c.

For the vertical composition of the 1-morphisms

$$\alpha : I \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1), \quad \beta : I \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$$

we have

$$Hol(\alpha \cdot \beta) = Hol(\alpha) + Hol(\beta)$$

since the left hand side is considered to be the integration of over the some of the 1-chains represented by α and β. The horizontal composition of the 1-morphisms

$$c_1 : I \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1), \quad c_2 : I \longrightarrow \mathcal{P}(M; \mathbf{x}_1, \mathbf{x}_2)$$

the 2-holonomy is expressed as

$$Hol(c_2 \circ c_1) = \int T_{c_2} \wedge T_{c_1}$$

by means of Proposition 2.1.

For $n \geq 2$ we set

$$\widehat{TH_+(M)}_{\leq n-1} = \bigoplus_{0 \leq k \leq n-1} \widehat{TH_+(M)}_k$$

and introduce an n-category as follows. The set of objects consists of one point. The 1-morphisms are elements of $\widehat{TH_+(M)}_0^\times$. For $g_1, g_2 \in \widehat{TH_+(M)}_0^\times$ the set of 2-morphisms from g_1 to g_2 is defined as

$$\mathrm{Hom}(g_1, g_2) = \{v \in \widehat{TH_+(M)}_1 \mid \delta(v) = g_2 - g_1\}.$$

We define the set of k-morphisms for $2 \leq k \leq n$ inductively as follows. For $(k-1)$-morphisms $f_1, f_2 \in \widehat{TH_+(M)}_{k-2}$ we define the set of k-morphisms from f_1 to f_2 as

$$\mathrm{Hom}(f_1, f_2) = \{v \in \widehat{TH_+(M)}_{k-1} \mid \delta(v) = f_2 - f_1\}.$$

We regard $\widehat{TH_+(M)}_k$ as a 2-sided module over $\widehat{TH_+(M)}_0$.

Now we construct a higher holonomy functor from the path n-groupoid $\mathcal{P}_n(M)$ to $\widehat{TH_+(M)}_{\leq n-1}$ as follows. For k such that $2 \leq k \leq n$ let f be a k-morphism in $\mathcal{P}_n(M)$ represented by a k-fold homotopy $H : I^k \longrightarrow M$ and we consider the associated $(k-1)$ chain

$$c_f : I^{k-1} \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1).$$

For the transport T we consider its pullback $c_f^* T$ as an element of $\Omega^*(I^{k-1}) \otimes T\widehat{H_+(M)}$. We denote by $\langle T, c_f \rangle$ the integration of the $(k-1)$-form part of $c_f^* T$ over I^{k-1}. For the above k-morphism f we define its k-holonomy by $Hol(f) = \langle T, c_f \rangle$.

To define the n-holonomy functor for the homotopy n-groupoid $\Pi_n(M)$ we modify the target of the n-holonomy functor as follows. Using the derivation

$$\delta_n : T\widehat{H_+(M)}_n \longrightarrow T\widehat{H_+(M)}_{n-1}$$

we consider the two sided ideal in $T\widehat{H_+(M)}_{n-1}$ generated by the image of δ_n as a module over $T\widehat{H_+(M)}_0$ and denote it by $\mathcal{I}_{n-1}$. The holonomy functor is defined as

$$Hol : \Pi_n(M) \longrightarrow T\widehat{H_+(M)}_{\leq n-1}/\mathcal{I}_{n-1}$$

by means of the integration of the transport over the associated chain of $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ as in the case of $\mathcal{P}_n(M)$.

Theorem 6.1. *The above construction gives a well-defined functor*

$$Hol : \Pi_n(M) \longrightarrow T\widehat{H_+(M)}_{\leq n-1}/\mathcal{I}_{n-1}.$$

For a k-morphism f between $(k-1)$-morphisms between g_0 and g_1 we have

$$\delta Hol(f) = Hol(g_1) - Hol(g_0).$$

Proof. For a be a k-morphism f in $\mathcal{P}_n(M)$ represented by a k-fold homotopy $H : I^k \to M$, $Hol(f)$ is defined as $\langle T, c_f \rangle$ where c_f is the $(k-1)$-chain

$$c_f : I^{k-1} \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$$

associated with f and T is the transport of the formal homology connection. Since the iterated integrals are invariant under smooth thin homotopy $Hol(f)$ is well-defined for f in $\mathcal{P}_n(M)$. Let us consider the case of $\Pi_n(M)$. We suppose that there is an $(n+1)$-fold smooth homotopy $H : I^{n+1} \to M$ between n-fold smooth homotopies f_0 and f_1. Let c_0 and c_1 denote the associated $(n-1)$-chains of $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$. We notice that these $(n-1)$-chains factor through $D^{n-1} \cong I^{n-1}/(\partial I^{n-2}) \times I$. The $(n+1)$-fold smooth homotopy H gives an n-chain $y : I^n \to \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ such that $c_1 - c_0 = \partial y$. We have

$$Hol(c_1) - Hol(c_0) = Hol(\partial y)$$

which is by definition $\langle T, \partial y \rangle$. By the Stokes theorem we have

$$\langle T, \partial y \rangle = \langle dT, y \rangle.$$

On the other hand we have $dT = \delta T$ by Proposition 4.1. This shows that $Hol(c_0) = Hol(c_1)$ in $T\widehat{H_+(M)}_{n-1}/\mathcal{I}_{n-1}$ and the n-holonomy functor from the homotopy n-groupoid $\Pi_n(M)$ is well-defined. We have

$$\delta Hol(c) = \langle \delta T, c \rangle.$$

By using the equality $dT = \delta T$ and the Stokes theorem we obtain that

$$\delta Hol(c) = Hol(g_1) - Hol(g_0)$$

holds. This shows that for an n-morpshism c in $\Pi_n(M)$, $Hol(c)$ is an n-morphism as well. This completes the proof. $\square$

Example : Let us consider the case $M = S^n$ with $n \geq 2$. As we observed at the end of Section 3 there are n-fold smooth homotopies f_n and g_n on S^n. The formal homology connection is given by $\omega = \nu \otimes X$ where ν is the standard volume form of S^n and X is the fundamental homology class in $H_n(S^n; \mathbf{R})$. We denote by c_1 and c_2 the $(n-1)$-chains in $\mathcal{P}(S^n; \mathbf{x}_0, \mathbf{x}_1)$ corresponding to f_n and g_n respectively. We see that $z = c_1 - c_2$ is a $(n-1)$-cycle representing a basis of $H_{n-1}(\mathcal{P}(S^n; \mathbf{x}_0, \mathbf{x}_1); \mathbf{R}) \cong \mathbf{R}$. Applying the n-holonomy functor, we have

$$Hol(f_n) - Hol(g_n) = \langle T, c_1 \rangle - \langle T, c_2 \rangle = \langle T, z \rangle,$$

which is equal to the volume of S^n. This confirms that $Hol(f_n) \neq Hol(g_n)$ and that $Hol(f_n)$ and $Hol(g_n)$ are not $(n+1)$-fold homotopic.

7. Higher holonomy for hyperplane arrangements

We start by recalling basic facts on hyperplane arrangements. Let

$$\mathcal{A} = \{H_1, \cdots, H_\ell\}$$

be a collection of finite number of complex hyperplanes in $\mathbf{C}^n$. We call $\mathcal{A}$ a hyperplane arrangement. Let f_j, $1 \leq j \leq \ell$, be linear forms defining the hyperplanes H_j. We consider the complement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$$

and denote by $\Omega^*(M(\mathcal{A}))$ the algebra of differential forms on $M(\mathcal{A})$ with values in $\mathbf{C}$. The Orlik-Solomon algebra $OS(\mathcal{A})$ is the subalgebra of $\Omega^*(M(\mathcal{A}))$ generated by the logarithmic forms $\omega_j = d \log f_j$, $1 \leq j \leq \ell$. We refer the reader to [15] for basic properties of the Orlik-Solomon algebra. It is known that there is an isomorphism of algebras

$$OS(\mathcal{A}) \cong H^*(M(\mathcal{A}); \mathbf{C}).$$

Let $\{Z_j\}$ be a basis of $H_+(M(\mathcal{A}); \mathbf{C})$ and $\{\varphi_j\}$ be its basis in the Orlik-Solomon algebra $OS(\mathcal{A})$. A formal homology connection for $M(\mathcal{A})$ is given by

$$\omega = \sum_{j=1}^{m} \varphi_j \otimes Z_j$$

with the derivation described as follows. We define

$$\delta : T\widehat{H_+(M(\mathcal{A}))}_p \longrightarrow T\widehat{H_+(M(\mathcal{A}))}_{p-1}$$

as the dual of the wedge product. More explicitly, when the wedge product is given by

$$\varepsilon(\varphi_i) \wedge \varphi_j = \sum_k c_{ij}^k \varphi_k$$

the derivation δ is defined as

$$\delta Z_k = \sum_{i,j} c_{ij}^k Z_i Z_j.$$

We obtain the condition

$$\varepsilon(\omega) \wedge \omega = \delta(\omega)$$

by defining the derivation δ in the above way (see [10] and [12]).

In the following we consider a typical example where we have non-trivial higher holonomies. Let L_j, $1 \leq j \leq 3$, be complex lines in general position in $\mathbf{C}^2$ and consider the complement $M = \mathbf{C}^2 \setminus (L_1 \cup L_2 \cup L_3)$. Let ω_j be the logarithmic form associated with L_j, $1 \leq j \leq 3$. The formal homology connection is of the form

$$\omega = \omega_1 \otimes X_1 + \omega_2 \otimes X_2 + \omega_3 \otimes X_3 + \sum_{1 \leq i < j \leq 3} (\omega_i \wedge \omega_j) \otimes X_{ij}$$

with the derivation defined by

$$\delta(X_j) = 0,\ 1 \leq j \leq 3, \quad \delta(X_{ij}) = -[X_i, X_j],\ 1 \leq i < j \leq 3$$

It can be shown that M has a homotopy type of the 2-skeleton of the 3-torus T^3. The fundamental group $\pi_1(M, *)$ is isomorphic to the rank 3 free abelian group $\mathbf{Z}^{\oplus 3}$. We observe the cell decomposition: 0-cell e^0, 1-cells e^1_j, $1 \leq j \leq 3$, 2-cells e^2_{ij}, $1 \leq i < j \leq 3$. The boundary of the 2-cells is described as

$$\partial e^2_{ij} = [e^1_i, e^1_j],\ 1 \leq i < j \leq 3$$

where $[g, h]$ denotes the path $ghg^{-1}h^{-1}$. The universal covering of M has a homotopy type of the union of the planes in $\mathbf{R}^3$ given by $x = \ell\ y = m,\ z = n$ where ℓ, m and n are arbitrary integers. We observe that the second homotopy group $\pi_2(M)$ is isomorphic to the group ring $\mathbf{Z}[\mathbf{Z}^{\oplus 3}]$. There is a smooth homotopy f between the paths $e^1_1 e^1_2$ and $e^1_2 e^1_1$. In a similar way, we have a smooth homotopy between the paths $e^1_1 e^1_2 e^1_3$ and $e^1_2 e^1_1 e^1_3$. We see that f and g are not 3-fold homotopy equivalent. This can be confirmed by comparing the terms containing X_{12} of the 2-holonomies $Hol(f)$ and $Hol(g)$.

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