Radon transforms of Constructible functions on Grassmann manifolds

Yutaka Matsui

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Introduction

A constructible function ϕ on a real analytic or complex manifold X is a $\mathbb{Z}$-valued function which is constant along a stratification. We can choose a stratification according to the situation, so we work with subanalytic stratifications here.

In [14], P.Schapira defined Radon transforms of constructible functions. It is a kind of integral transformations. We consider the following diagram;

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y. \end{tikzcd}$$

Here X and Y are real analytic or complex manifolds, S is a locally closed subanalytic subset of $X \times Y$, and f and g are real or complex analytic maps, respectively. Then we can define the Radon transform $\mathcal{R}_S(\phi)$ of a constructible function ϕ on X by

$$\mathcal{R}_S(\phi) = \int_g f^*\phi.$$

In [14], P.Schapira obtained a formula for $\mathcal{R}_S$ in the general situation. This formula gives an inversion formula for the Radon transform of constructible functions from a real projective space to its dual in the case when the whole dimension is odd. We can, that is, reconstruct a constructible function ϕ on the projective space from its Radon transform $\mathcal{R}_S(\phi)$. This topological meaning is that we can reconstruct the original subanalytic set from the knowledge of the Euler-Poincaré indices of all its affine slices.

In this paper, we study Radon transforms of constructible functions from $X = F_{n+1}(p)$ to $Y = F_{n+1}(q)$. Here $F_{n+1}(p)$ is the Grassmann manifold, that is, the set of all the p dimensional subspaces in an $n+1$ dimensional vector space. First we construct inversion formulas for Radon transforms. When p is not equal to 1, the hypotheses of Schapira's formula are not satisfied. So the situation that we consider is more complicated in these cases. Second we study the images of Radon transforms of characteristic functions of Schubert cells.

We first review basic properties of Grassmann manifolds, constructible functions and Schapira's formula in the general case.

In Section 2.1, we modify Schapira's formula in the general case under the same hypotheses as Schapira. This gives an inversion formula for the Radon transform $\mathcal{R}_S$. We can apply this formula to the Radon transform $\mathcal{R}_{(n+1;1,q)}$ from $F_{n+1}(1)$ to $F_{n+1}(q)$. Moreover, in Section 2.2 we consider the Radon transform $\mathcal{R}_{(n+1;p,q)}$ from $F_{n+1}(p)$ to $F_{n+1}(q)$ for $p \neq 1$. We obtain an inversion formula for this by modifying the kernel function of this inversion transform under suitable conditions of p and q.

In Section 3, we show that the Radon transform $\mathcal{R}_{(n+1;p,n+1-p)}$ is the non-trivial isomorphism between $CF(F_{n+1}(p))$ and its dual $CF(F_{n+1}(n+1-p))$.

In Section 4, we apply our results to the calculation of indices of $\mathcal{D}$-modules.

In Section 5, we calculate the images of Radon transforms of characteristic functions of Schubert cells. We characterize these images by Young diagrams.

In [7], recently, T.Kakehi constructed an inversion formula for Radon transforms of C^∞-functions on $F_{n+1}(p)$. On the other hand, our definition of Radon transforms is different from them. The meaning of our integration is not usual one but topological one based on the Euler-Poincaré indices of slices. It would be interesting that the condition under which we obtain an inversion formula in both cases coincide with each other in spite of the difference of the meaning of integrals; namely in the real Grassmann case only when $q-p$ is even, we obtain both inversion formulas.

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1 Preliminaries

1.1 A cell decomposition of Grassmann manifolds

We review the notation and well-known results on a cell decomposition of the Grassmann manifold, which is called Schubert decomposition. For more details, we refer to [11, 5].

Definition 1.1. Let E be an $n-$dimensional vector space over $k = \mathbb{R}$ or $\mathbb{C}$, and p and q be integers satisfying $1 \leqq p \leqq q \leqq n$.

(i) $F_n(p) = \{x \mid x \text{ is a linear subspace of } E, \text{ whose dimension is } p.\}$,

(ii) $F_n(p,q) = \{(x,y) \in F_n(p) \times F_n(q) \mid x \subset y \}$,

(iii) $F_n(q,p) = \{(y,x) \in F_n(q) \times F_n(p) \mid y \supset x \}$,

(iv) $\mu_n(p) = \chi(F_n(p))$: the topological Euler-Poincaré index of $F_n(p)$.

We fix a basis $e_1, e_2, \cdots, e_n$ of E. We set $V_i = \mathrm{span}[e_1, e_2, \cdots, e_i]$ for $i = 1, 2, \cdots, n$. Then we have a complete flag of vector spaces;

$$V_1 \subset V_2 \subset \cdots \subset V_n, \quad \dim V_i = i.$$

We review a cell decomposition of $F_n(p)$, which is called Schubert decomposition.

First we review Young diagrams.

Definition 1.2.

(i) Let $\lambda = (a_1, a_2, \cdots, a_p)$ be a sequence of integers such that

$$n - p \geq a_1 \geq a_2 \geq \cdots \geq a_p \geq 0.$$

This sequence corresponds to what is called a Young diagram with at most p rows and $n-p$ columns. We call this sequence a Young diagram, too.

(ii) For a Young diagram $\lambda = (a_1, a_2, \cdots, a_p)$, we define its complement $\lambda^c = (b_1, b_2, \cdots, b_p)$ such that

$$b_j = n - p - a_j \quad \text{for } j = 1, 2, \cdots, p.$$

This $\lambda^c = (b_1, b_2, \cdots, b_p)$ satisfies

$$0 \leq b_1 \leq b_2 \leq \cdots \leq b_p \leq n - p.$$

(iii) For a sequence $\lambda = (a_1, a_2, \cdots, a_p)$, we define

$$|\lambda| = \sum_{k=1}^{p} a_k.$$

Let $\lambda^c = (b_1, b_2, \cdots b_p)$ be the above sequence. Then we have

$$V_{b_1+1} \subset V_{b_2+2} \subset \cdots \subset V_{b_p+p} \subset E.$$

Definition 1.3. Let λ be a Young diagram, and $\lambda^c = (b_1, b_2, \cdots, b_p)$ be its complement. Then we define the Schubert cell corresponding to λ;

$$\Omega_\lambda^\circ = \left\{ x \in F_n(p) \;\middle|\; \begin{array}{l} \dim(x \cap V_{b_i+i}) = i, \\ \dim(x \cap V_{b_i+i-1}) = i-1 \end{array} \quad (1 \le i \le p) \right\}.$$

These Schubert cells $\{\Omega_\lambda^\circ\}$ give a cell decomposition of the Grassmann manifold $F_n(p)$.

Proposition 1.4. *Let λ be a Young diagram with p rows and $n-p$ columns. Then we have*

(i) $\Omega_\lambda^\circ \simeq k^{|\lambda^c|} = k^{p(n-p)-|\lambda|}$,

(ii) $F_n(p) = \coprod_\lambda \Omega_\lambda^\circ$ *(disjoint union).*

Definition 1.5. We define the Schubert variety for a Young diagram λ which is the analytic submanifold of $F_n(p)$ by

$$\Omega_\lambda = \{x \in F_n(p) \;\; | \;\; \dim(x \cap V_{b_i+i}) \ge i, \;(1 \le i \le p)\},$$

where $\lambda^c = (b_1, b_2, \cdots, b_p)$.

Remark 1.6. We remark that Ω_λ is the whole Grassmann manifold when $\lambda = (0, 0, \cdots, 0)$.

Proposition 1.7. *Let λ be a Young diagram λ. Then we have*

$$\Omega_\lambda = \coprod_{\lambda \subset \mu} \Omega_\mu^\circ,$$

where μ ranges through Young diagrams containing λ as a subset.

1.2 The calculation of Euler-Poincaré indices of Schubert varieties

We calculate the Euler-Poincaré indices with compact supports of Schubert varieties. In the later section, we apply these results obtained in this section.

First, we extend the definition of $\binom{n}{p}$ by

$$\binom{n}{p} = \begin{cases} \binom{n}{p} & (n \geq p \geq 0), \\ 0 & (\text{otherwise}). \end{cases}$$

Let x be an m-dimensional subspace of E. We calculate the Euler-Poincaré index with compact supports of the following Schubert variety;

$$\Omega^{m,k} := \{y \in F_n(p) \mid \dim(x \cap y) \geq k\},$$

which corresponds to the Young diagram $\lambda = (a_1, a_2, \cdots, a_p)$ with

$$a_j = \begin{cases} n-p-m+k & (1 \leq j \leq k), \\ 0 & (k+1 \leq j \leq p). \end{cases}$$

We denote the Euler-Poincaré index by χ, and the Euler-Poincaré index with compact supports by χ_c.

First we consider complex Grassmann manifolds i.e. $E = \mathbb{C}^{n+1}$.

Proposition 1.8. *We have*

$$\chi_c\left(\Omega^{m,k}\right) = \sum_{l=0}^{m-k} \binom{n-m}{p-k-l}\binom{m}{k+l}.$$

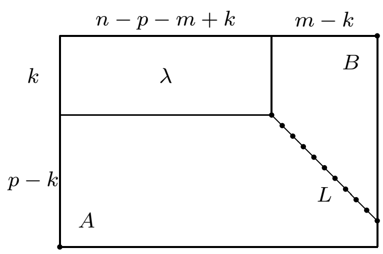
Proof. First, we calculate the Euler-Poincaré index with compact supports of a Schubert cell. Since $\mathbb{C}$ is a real 2-dimensional vector space, we have

$$\begin{aligned} \chi_c(\Omega^\circ_\mu) &= \chi_c(\mathbb{C}^{p(n-p)-|\mu|}) \\ &= \chi(\mathbb{C}^{p(n-p)-|\mu|}) \\ &= 1. \end{aligned}$$

By the additivity of χ_c, we have for a Schubert variety

$$\begin{aligned}\chi_c(\Omega^{m,k}) &= \chi_c\left(\coprod_{\lambda\subset\mu}\Omega_\mu^\circ\right)\\ &= \sum_{\lambda\subset\mu}\chi_c(\Omega_\mu^\circ)\\ &= \sum_{\lambda\subset\mu}1\\ &= \sharp\{\mu \mid \lambda\subset\mu\}.\end{aligned}$$

We should count the number of the ways which it goes from A to B through each point on the L (see figure 1).



(figure 1)

$\square$

Next, we consider real Grassmann manifolds i.e. $E=\mathbb{R}^{n+1}$.

First, we calculate the Euler-Poincaré index with compact supports of a Schubert cell. By the Poincaré duality we have

$$\begin{aligned}\chi_c(\Omega_\mu^\circ) &= \chi_c(\mathbb{R}^{p(n-p)-|\mu|})\\ &= (-1)^{p(n-p)-|\mu|}.\end{aligned}$$

By the additivity of χ_c, we have for a Schubert variety

$$\begin{aligned}\chi_c(\Omega_\lambda) &= \chi_c\left(\coprod_{\lambda\subset\mu}\Omega_\mu^\circ\right)\\ &= \sum_{\lambda\subset\mu}\chi_c(\Omega_\mu^\circ)\\ &= \sum_{\lambda\subset\mu}(-1)^{p(n-p)-|\mu|}\\ &= (-1)^{p(n-p)}\sum_{\lambda\subset\mu}(-1)^{|\mu|}.\end{aligned}$$

So we should count the numbers of Young diagrams; the number of Young diagrams containing λ with at most p rows $n-p$ columns whose number of boxes is even, and the number of ones whose number of boxes is odd.

Definition 1.9.

(i) $e_n(p) = \sharp \left\{ \mu \;\middle|\; \begin{array}{l} \mu \text{ is a Young diagram with at most } p \text{ rows} \\ \text{and } n-p \text{ columns. } |\mu| \text{ is even.} \end{array} \right\}$,

(ii) $o_n(p) = \sharp \left\{ \mu \;\middle|\; \begin{array}{l} \mu \text{ is a Young diagram with at most } p \text{ rows} \\ \text{and } n-p \text{ columns. } |\mu| \text{ is odd.} \end{array} \right\}$.

Proposition 1.10. *We have*

(i) $e_n(p) = \frac{1}{2}\left\{ \binom{n}{p} + \mu_n(p) \right\}$,

(ii) $o_n(p) = \frac{1}{2}\left\{ \binom{n}{p} - \mu_n(p) \right\}$.

Proof. By the definition above, we have

$$\begin{aligned} e_n(p) - o_n(p) &= \mu_n(p), \\ e_n(p) + o_n(p) &= \binom{n}{p}. \end{aligned}$$

Here $\mu_n(p)$ is the Euler-Poincaré index of $F_n(p)$. □

Under the preparation above, we calculate $\chi_c(\Omega^{m,k})$.

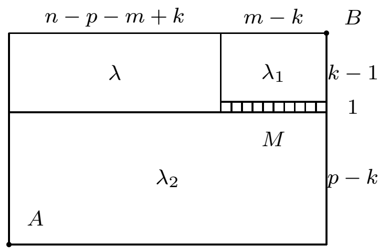
Proposition 1.11. *We have*

$$\begin{aligned} &\chi_c\left(\Omega^{m,k}\right) \\ &= \begin{cases} (-1)^{p(n-p)} \left| \displaystyle\sum_{l=0}^{m-k} (-1)^l \mu_{l+k-1}(l)\mu_{n-k-l}(p-k) \right| & (k \geq 1), \\ (-1)^{p(n-p)}\mu_n(p) & (k = 0). \end{cases} \end{aligned}$$

Proof. We denote $eo_n^j(p) = \frac{1}{2}\left\{ \binom{n}{p} + (-1)^j \mu_n(p) \right\}$.

Then we have

$$
\begin{aligned}
&(-1)^{p(n-p)} \cdot \chi_c\left(\Omega^{m,k}\right) \\
&= \begin{cases}
\begin{vmatrix} \displaystyle\sum_{l=0}^{m-k} eo^{l-1}_{l+k-1}(l) o_{n-k-l}(p-k) + eo^{l}_{l+k-1}(l) e_{n-k-l}(p-k) \\ -eo^{l}_{l+k-1}(l) o_{n-k-l}(p-k) - eo^{l-1}_{l+k-1}(l) e_{n-k-l}(p-k) \end{vmatrix} & (k \geq 1), \\
\mu_n(p) & (k = 0)
\end{cases} \\
&= \begin{cases}
\left| \displaystyle\sum_{l=0}^{m-k} (-1)^l \mu_{l+k-1}(l) \mu_{n-k-l}(p-k) \right| & (k \geq 1), \\
\mu_n(p) & (k = 0)\ .
\end{cases}
\end{aligned}
$$



(figure 2)

Here, λ_1 is the Young diagram with k rows and $m-k$ columns. Further λ_2 is the Young diagram with $p-k$ rows and $n-p$ columns.

For example, we consider the case when $k(m-k)$ is even. We count the number of Young diagrams which have even boxes in λ_1 and odd boxes in λ_2. We divide into the cases when diagrams have $m-k-j$ boxes at M from the left $(0 \leq j \leq m-k)$. If j is even, the number of Young diagrams that we should count is $e_{k-1+j}(k-1) \times o_{n-k-j}(p-k)$. If j is odd, the number of Young diagrams that we should count is $o_{k-1+j}(k-1) \times o_{n-k-j}(p-k)$ (see figure 2). $\square$

At the end of this section, we calculate $\chi_c(\Omega^{m,0}) - \chi_c(\Omega^{m,1})$ in another way, which has an important meaning in the later section.

Proposition 1.12. *We have*

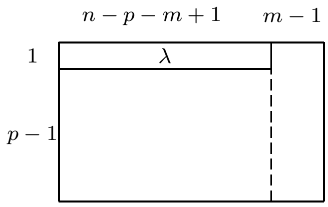
$$\chi_c(\Omega^{m,0}) - \chi_c(\Omega^{m,1}) = (-1)^{p(n-p)} \mu_{n-m}(p).$$

Proof. We denote by $e_p^\lambda(n)$ (resp. $o_p^\lambda(n)$) the number of Young diagrams containing λ with p rows and $n-p$ columns whose number of boxes is even (resp. odd).

Then for the Young diagram $\lambda = (n-p-m+1, 0, \cdots, 0)$, we have

$$\begin{aligned}(-1)^{p(n-p)}\{\chi_c(\Omega^{m,0}) - \chi_c(\Omega^{m,1})\} &= \mu_n(p) - (e_p^\lambda(n) - o_p^\lambda(n)) \\ &= (e_p(n) - e_p^\lambda(n)) - (o_p(n) - o_p^\lambda(n)) \\ &= e_p^{\lambda'}(n) - o_p^{\lambda'}(n) \\ &= \mu_{n-m}(p),\end{aligned}$$

where $\lambda' = (n-p-m, 0, \cdots, 0)$ (see figure 3).



(figure 3)

□

1.3 Constructible functions

We review the notation and results on constructible functions without proofs. For more details, we refer to [10].

Let X be a real analytic manifold.

Definition 1.13. A function $\phi : X \to \mathbb{Z}$ is constructible if:

(i) For all $m \in \mathbb{Z}$, $\phi^{-1}(m)$ is subanalytic,

(ii) the family $\{\phi^{-1}(m)\}_{m\in\mathbb{Z}}$ is locally finite in X.

We denote by $CF(X)$ the abelian group of all the constructible functions on X, and by $\mathscr{CF}_X$ the sheaf $U \mapsto CF(U)$ on X.

It follows from the Hardt triangulation theorem that ϕ is constructible if and only if there exists a locally finite family of compact subanalytic contractible subsets $\{K_i\}_i$ of X such that

$$\phi = \sum_i c_i \mathbf{1}_{K_i}.$$

Here $c_i \in \mathbb{Z}$ and $\mathbf{1}_A$ is the characteristic function of the subset A.

Let $F \in Ob(\mathbf{D}^b_{\mathbb{R}-c}(X))$ (the base ring is a field k with characteristic zero). Then its local Euler-Poincaré index

$$\chi(F)(x) = \sum_j (-1)^j \mathrm{dim} H^j(F)_x$$

is clearly a constructible function. Moreover :

Proposition 1.14.

(i) *Let F, $G \in Ob(\mathbf{D}^b_{\mathbb{R}-c}(X))$. Then we have*

(a) $\chi(F \oplus G) = \chi(F) + \chi(G)$,

(b) $\chi(F \otimes G) = \chi(F) \cdot \chi(G)$.

(ii) *Let $F' \to F \to F'' \stackrel{+1}{\to}$ be a distinguished triangle. Then we have*

$$\chi(F) = \chi(F') + \chi(F'').$$

We shall denote by $\mathbf{K}_{\mathbb{R}-c}(X)$ the Grothendieck group of $\mathbf{D}^b_{\mathbb{R}-c}(X)$. This group is obtained as the quotient group of the free abelian group generated by $Ob(\mathbf{D}^b_{\mathbb{R}-c}(X))$ under the equivalence relations; $F = F' + F''$ if there exists a distinguished triangle $F' \to F \to F'' \stackrel{+1}{\to}$. By the proposition above, we have a group homomorphism induced by the local Euler-Poincaré index χ:

$$\chi : \mathbf{K}_{\mathbb{R}-c}(X) \to CF(X).$$

We shall denote by $\mathscr{L}_X$ the sheaf on T^*X of Lagrangian cycles over the ring $\mathbb{Z}$. Then the characteristic cycle CC defines the following homomorphism:

$$CC : \mathbf{K}_{\mathbb{R}-c}(X) \to H^0(T^*X; \mathscr{L}_X).$$

Theorem 1.15. ([10, Theorem 9.7.11]) *The diagram:*

$$\begin{tikzcd}
 & \mathbf{K}_{\mathbb{R}-c}(X) \arrow[dl, "CC"'] \arrow[dr, "\chi"] & \\
H^0(T^*X; \mathscr{L}_X) \arrow[rr, "Eu"'] & & H^0(X; \mathscr{C}\mathscr{F}_X)
\end{tikzcd}$$

is commutative, and the arrows are isomorphisms.

We remark that a representative element of the inverse image of a constructible function $\phi = \sum_{\alpha\in A} m_\alpha \mathbf{1}_{X_\alpha}$ by χ is

$$\bigoplus_{\alpha\in A'} k_{X_\alpha}^{|m_\alpha|}\left[\frac{1-\operatorname{sgn}(m_\alpha)}{2}\right],$$

where $\{X_\alpha\}$ is a subanalytic stratification of X and $A' = \{\alpha \in A | m_\alpha \neq 0\}$.

Next, we review operations on constructible functions [10]. These operations are induced by operations of $\mathbf{K}_{\mathbb{R}-c}(X)$ through the Euler-Poincaré index χ.

Definition 1.16. Let X and Y be two real analytic manifolds, and $f : Y \to X$ be a real analytic map.

(i) The inverse image : Let $\phi \in CF(X)$. We set

$$f^*\phi(y) = \phi(f(y)).$$

If $\phi = \chi(F)$, then clearly $f^*\phi = \chi(f^{-1}F)$.

(ii) The integral : Let $\phi \in CF(X)$ be represented by $\phi = \chi(F) = \sum_i c_i \mathbf{1}_{K_i}$ for a $F \in Ob(\mathbf{D}^b_{\mathbb{R}-c}(X))$, and $\{K_i\}$ is a locally finite family of compact subanalytic contractible subsets. Assume that ϕ has compact support. Then we define

$$\int_X \phi = \sum_i c_i = \chi(R\Gamma(X;F)).$$

(iii) The direct image : Let $\psi \in CF(Y)$ whose support is proper over X. We define

$$\left(\int_f \psi\right)(x) = \int_Y (\psi \cdot \mathbf{1}_{f^{-1}(x)}).$$

If $\psi = \chi(G)$ such that f is proper on $\operatorname{supp}(G)$, then $\int_f \psi = \chi(Rf_!G)$.

Remark 1.17. We remark about the integral of the characteristic function of a locally closed subset A of a manifold X. It is not the usual integral, but a kind of topological integrals. By Theorem 1.15 and the definition, we must calculate the following object

$$\int_X \mathbf{1}_A = \chi(R\Gamma(X; k_A)) = \chi(R\Gamma(X; i_! i^{-1} k_X)) = \chi(R\Gamma_c(A; k_A)) = \chi_c(A).$$

Here k is $\mathbb{R}$ or $\mathbb{C}$, $i : A \to X$ is an inclusion morphism and χ_c is the topological Euler-Poincaré index with compact supports.

Let A_1, A_2 be two locally closed subsets of a manifold X. Then we have distinguished triangles

$$\mathbb{C}_{A_1 \setminus A_2} \to \mathbb{C}_{A_1} \to \mathbb{C}_{A_2} \xrightarrow{+1},$$
$$R\Gamma_c(X; \mathbb{C}_{A_1 \setminus A_2}) \to R\Gamma_c(X; \mathbb{C}_{A_1}) \to R\Gamma_c(X; \mathbb{C}_{A_2}) \xrightarrow{+1} .$$

Therefore we have the additivity of the Euler-Poincaré index with compact supports;

$$\chi_c(A_1) = \chi_c(A_1 \setminus A_2) + \chi_c(A_2).$$

For example, we have

$$\int_{\mathbb{R}} \mathbf{1}_{[0,1]} = 1, \quad \int_{\mathbb{R}} \mathbf{1}_{[0,1)} = 0, \quad \int_{\mathbb{R}} \mathbf{1}_{(0,1)} = -1.$$

Proposition 1.18.

(i) *The following operations are well-defined morphisms of sheaves;*

  (a) $f^* : f^{-1}\mathscr{CF}_X \to \mathscr{CF}_Y$,

  (b) $\int_f : f_! \mathscr{CF}_Y \to \mathscr{CF}_X$.

(ii) *Inverse and direct images have functorial properties. That is, if* $f : Y \to X$ *and* $g : Z \to Y$ *are real analytic maps, then we have;*

  (a) $g^* \circ f^* = (f \circ g)^*$,

  (b) $\int_{f \circ g} = \int_f \circ \int_g$.

(iii) *Consider a Cartesian diagram of morphisms of real analytic manifolds:*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

Then, if $\psi \in CF(Y)$ *such that* f *is proper on* $\mathrm{supp}\psi$*, we have*

$$g^* \int_f \psi = \int_{f'} (h^* \psi).$$

1.4 Radon transforms of constructible functions and Schapira's formula

We review the definition of Radon transforms of constructible functions and Schapira's formula [14].

Let X and Y be two real analytic manifolds and let $S \subset X \times Y$ be a locally closed subanalytic subset of $X \times Y$. Denote by p_1 and p_2 the first and second projection defined on $X \times Y$ and by f and g the restriction of p_1 and p_2 to S respectively:

$$\begin{tikzcd} & X \times Y \arrow[ddl, "p_1"'] \arrow[ddr, "p_2"] & \\ & \cup & \\ & S \arrow[dl, "f"] \arrow[dr, "g"'] & \\ X & & Y. \end{tikzcd}$$

We assume;

$$p_2 \text{ is proper on } \bar{S}, \text{ the closure of } S \text{ in } X \times Y. \tag{1.4.1}$$

Definition 1.19. For a $\phi \in CF(X)$, we define

$$\begin{aligned} \mathcal{R}_S(\phi) &= \int_g f^* \phi \\ &= \int_{p_2} \mathbf{1}_S (p_1^* \phi). \end{aligned}$$

We call $\mathcal{R}_S(\phi)$ the Radon transform of ϕ.

Let $S' \subset Y \times X$ be another locally closed subanalytic subset. We denote again by p_2 and p_1 the first and second projection defined on $Y \times X$, by f' and g' the restriction of p_1 and p_2 to S', and by r the projection $S \underset{Y}{\times} S' \to X \times X$.

Then Schapira posed the hypotheses:

$$p_1 \text{ is proper on } \bar{S}', \text{ the closure of } S' \text{ in } Y \times X, \tag{1.4.2}$$

$$\exists \lambda \neq \mu \in \mathbb{Z} \text{ s.t. } \chi(r^{-1}(x, x')) = \begin{cases} \lambda & (x \neq x'), \\ \mu & (x = x'), \end{cases} \tag{1.4.3}$$

where χ is the topological Euler-Poincaré index.

Under these three hypotheses, we have Schapira's formula.

Theorem 1.20. ([14, Theorem 3.1]) *For any $\phi \in CF(X)$, we have*

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(\phi) = (\mu - \lambda)\phi + \left(\int_X \lambda \phi \right) \mathbf{1}_X.$$

Proof. Denote by h and h' the projections from $S \underset{Y}{\times} S'$ to S and S' respectively.

Consider the following diagram:

$$\begin{tikzcd}
 & & S \underset{Y}{\times} S' \arrow[dl, "h"'] \arrow[d, "r"] \arrow[dr, "h'"] & & \\
 & S \arrow[dl, "f"'] \arrow[dr, "g"] & X \times X \arrow[dll, "q_1"'] \arrow[drr, "q_2"] & S' \arrow[dl, "g'"'] \arrow[dr, "f'"] & \\
X & & Y & & X.
\end{tikzcd}$$

Since the square

$$\begin{tikzcd}
 & S \underset{Y}{\times} S' \arrow[dl, "h"'] \arrow[dr, "h'"] & \\
S \arrow[dr, "g"'] \arrow[rr, phantom, "\square"] & & S' \arrow[dl, "g'"] \\
 & Y &
\end{tikzcd}$$

is of Cartesian, we have

$$\begin{aligned}
\mathcal{R}_{S'} \circ \mathcal{R}_S(\phi) &= \int_{f'} (g'^* \int_g (f^*\phi)) \\
&= \int_{f' \circ h'} ((f \circ h)^* \phi) \\
&= \int_{q_2} \int_r r^* q_1^* \phi \\
&= \int_{q_2} k(x, x') q_1^* \phi.
\end{aligned}$$

Here we have

$$\begin{aligned}
k(x, x') &= \int_r r^* \mathbf{1}_{X \times X} \\
&= \int_r \mathbf{1}_{S \underset{Y}{\times} S'}.
\end{aligned}$$

Hence, it is enough to note that, under the hypothesis,

$$\begin{aligned}
\int_r \mathbf{1}_{S \underset{Y}{\times} S'} &= \mu \mathbf{1}_{\Delta_X} + \lambda \mathbf{1}_{X \setminus \Delta_X} \\
&= (\mu - \lambda) \mathbf{1}_{\Delta_X} + \lambda \mathbf{1}_{X \times X},
\end{aligned}$$

where Δ_X is the diagonal of $X \times X$.

Since $\displaystyle\int_{q_2} \mathbf{1}_{\Delta_X}\ q_1^*\phi = \phi$ and $\displaystyle\int_{q_2} \mathbf{1}_{X \times X}\ q_1^*\phi = \int_X \phi$, we get the result. □

In [14], Schapira applied this formula to correspondences of real flag manifolds.

We consider the following diagram, which is called a correspondence

$$\begin{tikzcd}
 & F_{n+1}(1, q) \arrow[dl, "f"'] \arrow[dr, "g"] & \\
F_{n+1}(1) & & F_{n+1}(q),
\end{tikzcd}$$

where f and g are projections.

We set $\mathcal{R}_{(n+1;1,q)} = \mathcal{R}_S$ and $\mathcal{R}_{(n+1;q,1)} = \mathcal{R}_{S'}$, where $S = F_{n+1}(1, q)$ and $S' = F_{n+1}(q, 1)$. Then this situation satisfies the hypotheses of the previous theorem, because we have

$$r^{-1}(x, x') \simeq \begin{cases} F_{n-1}(q-2) & (x \neq x'), \\ F_n(q-1) & (x = x'). \end{cases}$$

Therefore we can apply the above formula to this case.

Proposition 1.21. ([14, Proposition4.1]) *For any $\phi \in CF(F_{n+1}(1))$, we have*

$$\begin{aligned} &\mathcal{R}_{(n+1;q,1)} \circ \mathcal{R}_{(n+1;1,q)}(\phi) \\ = \ & (\mu_n(q-1) - \mu_{n-1}(q-2))\phi + \mu_{n-1}(q-2) \left(\int_{F_{n+1}(1)} \phi \right) \mathbf{1}_{F_{n+1}(1)}. \end{aligned}$$

In particular, when n is odd and $q = n$, we can obtain an inversion formula for $\mathcal{R}_{(n+1;1,q)}$.

2 Inversion transforms of Radon transforms of constructible functions

We generalize the situation which we apply Schapira's formula in the previous subsection. Namely we consider the following diagrams;

$$\begin{tikzcd} & F_{n+1}(p) \times F_{n+1}(q) \arrow[dddl, "p_1"'] \arrow[dddr, "p_2"] & \\ & \cup & \\ & F_{n+1}(p,q) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ F_{n+1}(p) & & F_{n+1}(q). \end{tikzcd}$$

We consider the following problems in this paper;

(i) an inversion formula for $\mathcal{R}_{(n+1;1,q)}$ in the case when n is even or $q \neq n$,

(ii) an inversion formula for $\mathcal{R}_{(n+1;p,q)}$ in the case when $1 < p$ and $1 < q$ hold.

That is, we consider the reconstruction of ϕ from $\mathcal{R}_S(\phi)$ on more general Grassmann manifolds.

2.1 A minor modification of Schapira's formula

First, we will modify Schapira's formula. We inherit the situation from Section 1.4.

Definition 2.1. For a $\psi \in CF(Y)$, we define

$$\begin{aligned}\mathcal{R}_0(\psi) &= \int_{p_1} \mathbf{1}_{X\times Y}(p_2^*\psi)\\ &= \int_{p_1} (p_2^*\psi)\\ &= \left(\int_Y \psi\right) \mathbf{1}_X.\end{aligned}$$

Proposition 2.2. *Let $\phi \in CF(X)$. Then we have*

$$\mathcal{R}_0 \circ \mathcal{R}_S(\phi) = \int_X (\mu\phi)\, \mathbf{1}_X.$$

Proof. A constructible function ϕ is represented by $\phi = \sum_i c_i \mathbf{1}_{K_i}$, where $\{K_i\}$ is a locally finite family of compact subanalytic contractible subsets. By the linearity of transforms, it is enough to show this formula only for a characteristic function $\mathbf{1}_K$ of a compact subanalytic contractible subset K.

Since the square

$$\begin{tikzcd} & X\times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "a_X"'] & \square & Y \arrow[dl, "a_Y"] \\ & \{pt\} & \end{tikzcd}$$

is of Cartesian, we have

$$\begin{aligned}\mathcal{R}_0 \circ \mathcal{R}_S(\mathbf{1}_K) &= \int_{p_1} p_2^* \int_{p_2} (\mathbf{1}_S \cdot p_1^*\mathbf{1}_K)\\ &= a_X^* \int_{a_Y} \int_{p_2} (\mathbf{1}_S \cdot p_1^*\mathbf{1}_K)\\ &= a_X^* \int_{a_X} \int_{p_1} (\mathbf{1}_S \cdot p_1^*\mathbf{1}_K).\end{aligned}$$

Here for any $\phi \in CF(X)$, we have

$$\begin{aligned}\left(a_X^* \int_{a_X} \phi\right)(x) &= \left(\int_{a_X} \phi\right)(\{pt\})\\ &= \int_X \phi(x')\mathbf{1}_{a_X^{-1}(\{pt\})}(x')\\ &= \left(\int_X \phi\right) \mathbf{1}_X(x),\end{aligned}$$

and

$$\left(\int_{p_1} \mathbf{1}_S \cdot p_1^* \mathbf{1}_K\right)(x) = \int_{X\times Y} \mathbf{1}_S(x',y')\mathbf{1}_{K\times Y}(x',y')\mathbf{1}_{\{x\}\times Y}(x',y')$$
$$= \int_{X\times Y} \mathbf{1}_{((\{x\}\cap K)\times Y)\cap S}(x',y')$$
$$= \mu \mathbf{1}_K.$$

This is because the Euler-Poincaré index of

$$\{x\} \times Y \cap S \simeq \{y \in Y \mid (x,y) \in S\}$$

is μ which is the Euler-Poincaré index of the fibers in the case of $x = x'$ of Schapira's hypotheses.

Therefore we get

$$\mathcal{R}_0 \circ \mathcal{R}_S(\phi) = a_X^* \int_{a_X} \int_{p_1} (\mathbf{1}_S \cdot p_1^* \mathbf{1}_K)$$
$$= a_X^* \int_{a_X} (\chi(Z_x)\mathbf{1}_K)$$
$$= \left(\int_X \mu \mathbf{1}_K\right) \mathbf{1}_X.$$

□

Definition 2.3. For a $\psi \in CF(Y)$, we define

$$\mathcal{R}^{-1}(\psi) = \int_{p_1} (\mu \mathbf{1}_{S'} - \lambda \mathbf{1}_{X\times Y})(p_2^* \psi)$$
$$= \mu \mathcal{R}_{S'}(\psi) - \lambda \mathcal{R}_0(\psi).$$

Proposition 2.4. *Let $\phi \in CF(X)$. Then we have*

$$\mathcal{R}^{-1} \circ \mathcal{R}_S(\phi) = \mu(\mu - \lambda)\phi.$$

In particular, if $\mu(\mu-\lambda)$ is not zero, we can reconstruct the original constructible function ϕ from its Radon transform $\mathcal{R}_S(\phi)$ by dividing the last term by this constant $\mu(\mu-\lambda)$.

Proof. By Schapira's formula and Proposition 2.2, we have

$$\begin{aligned}\mathcal{R}^{-1} \circ \mathcal{R}_S(\phi) &= \mu \mathcal{R}_{S'} \circ \mathcal{R}_S(\phi) - \lambda \mathcal{R}_0 \circ \mathcal{R}_S(\phi) \\ &= \mu \left\{ (\mu - \lambda)\phi + \lambda \left(\int_X \phi \right) \mathbf{1}_X \right\} - \lambda\mu \left(\int_X \phi \right) \mathbf{1}_X \\ &= \mu(\mu - \lambda)\phi.\end{aligned}$$

□

We apply this result to the complex or real Grassmann manifolds [1, 5]. We consider the following correspondence,

$$\begin{tikzcd} & F_{n+1}(1,q) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ F_{n+1}(1) & & F_{n+1}(q). \end{tikzcd}$$

Then Schapira's hypotheses are satisfied. We remark that

$$\begin{aligned}\mu &= \mu_n(q-1), \\ \lambda &= \mu_{n-1}(q-2).\end{aligned}$$

First, when we consider the complex case [5], we have

$$\mu_n(p) = \binom{n}{p}. \tag{2.1.1}$$

Therefore since for any $q > 1$ the hypotheses of proposition are satisfied, we can obtain an inversion formula for Radon transform $\mathcal{R}_{(n+1;1,q)}$.

Next, when we consider the real case [11, 12], we have

$$\mu_n(p) = \begin{cases} 0 & (\text{if } p(n-p) \text{ is odd}), \\ \dbinom{E\left(\frac{n}{2}\right)}{E\left(\frac{p}{2}\right)} & (\text{if } p(n-p) \text{ is even}). \end{cases} \tag{2.1.2}$$

Here $E\left(\frac{n}{2}\right)$ denotes the integral part of $\frac{n}{2}$, $\binom{a}{b}$ is the binomial coefficient.

If q is odd in $1 < q < n+1$, the hypotheses are satisfied. Then we can obtain an inversion formula for Radon transform $\mathcal{R}_{(n+1;1,q)}$.

2.2 Inversion formulas on Grassmann manifolds

For $p < q$, we consider the following situation:

$$\begin{tikzcd}[column sep=small]
 & F_{n+1}(p) \times F_{n+1}(q) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
 & \cup & \\
 & F_{n+1}(p,q) \arrow[dl, "f"] \arrow[dr, "g"'] & \\
F_{n+1}(p) & & F_{n+1}(q).
\end{tikzcd}$$

We set $X = F_{n+1}(p)$, $Y = F_{n+1}(q)$ and $S = F_{n+1}(p,q)$.

We remark that Schapira's hypotheses are not satisfied for $1 < p$.
We introduce the new sets to construct to an inversion formula for $\mathcal{R}_{(n+1;p,q)}$.

Definition 2.5.

(i) $S_i = \{(y,x) \in Y \times X \mid \dim(y \cap x) = i\}$ for $i = 0, 1, \cdots p$,

(ii) $Z_j = \{(x_1,x_2) \in X \times X \mid \dim(x_1 \cap x_2) = j\}$ for $j = 0, 1, \cdots p$.

Remark 2.6. We remark that we have

$$X \times X = \coprod_{j=0}^{p} Z_j.$$

Consider the following diagram:

$$\begin{tikzcd}
 & & S \underset{Y}{\times} S_i \arrow[dl, "h"'] \arrow[d, "r"] \arrow[dr, "h'"] & & \\
 & S \arrow[dl, "f"'] \arrow[dr, "g"] & X \times X \arrow[dll, "q_1"] \arrow[drr, "q_2"'] & S_i \arrow[dl, "g'"'] \arrow[dr, "f'"] & \\
X & & Y & & X.
\end{tikzcd}$$

Note that $Z_p = \{(x_1,x_2) \in X \times X \mid x_1 = x_2\}$ and we have $\displaystyle\int_{q_2} \mathbf{1}_{Z_p} q_1^* \phi = \phi$.
By the same argument as Section 1.4 and Section 2.1, we should modify the kernel such that $\displaystyle\int_r$(kernel) is equal to $\mathbf{1}_{Z_p}$.

We calculate $\displaystyle\int_r \mathbf{1}_{S \underset{Y}{\times} S_i}$;

$$\left(\int_r \mathbf{1}_{S \underset{Y}{\times} S_i}\right)(x_1,x_2) = \sum_{j=0}^{p} \left(\int_{S \underset{Y}{\times} S_i} \mathbf{1}_{r^{-1}(x_1,x_2) \cap r^{-1}(Z_j)}\right) \cdot \mathbf{1}_{Z_j}.$$

We calculate the homeomorphism of $r^{-1}(x_1, x_2) \cap S \underset{Y}{\times} S_i$ when we fix $(x_1, x_2) \in Z_j$. We have

$$
\begin{aligned}
& r^{-1}(x_1, x_2) \cap S \underset{Y}{\times} S_i \\
= & \left\{ y \in F_{n+1}(q) \,\middle|\, \begin{array}{l} x_1 \subset y \\ \dim(x_2 \cap y) = i \end{array} \right\} \quad (\dim(x_1 \cap x_2) = j).
\end{aligned}
$$

Here we consider conditions in the quotient space E/x_1. Then we have

$$
\begin{aligned}
& r^{-1}(x_1, x_2) \cap S \underset{Y}{\times} S_i \\
\simeq & \left\{ \begin{array}{ll} \emptyset & (i < j), \\ \{ y \in F_{n+1-p}(q-p) \,|\, \dim(x \cap y) = i - j \ \} & (i \geq j) \end{array} \right. \\
& (\dim x = p - j) \\
= & \left\{ \begin{array}{ll} \emptyset & (i < j), \\ \Omega_{i,j} \setminus \Omega_{i+1,j} & (i \geq j), \end{array} \right.
\end{aligned}
$$

where we denote by

$$
\Omega_{i,j} = \{ y \in F_{n+1-p}(q-p) \,|\, \dim(x \cap y) \geq i - j \} \quad (\dim x = p - j).
$$

This set is the Schubert variety.

Therefore we should calculate the Euler-Poincaré index $\chi_c(\Omega_{i,j} \setminus \Omega_{i+1,j})$ with compact supports. Moreover we have the additivity

$$
\chi_c(\Omega_{i,j} \setminus \Omega_{i+1,j}) = \chi_c(\Omega_{i,j}) - \chi_c(\Omega_{i+1,j}).
$$

So it is enough to calculate the Euler-Poincaré index $\chi_c(\Omega_{i,j})$ with compact supports. In Section 1.2, we calculated these values.

First we consider complex Grassmann manifolds i.e. $E = \mathbb{C}^{n+1}$.
Therefore we get

$$
\begin{aligned}
& \chi_c(r^{-1}(x_1, x_2) \cap r^{-1}(Z_j) \cap S \underset{Y}{\times} S_i) \\
= & \left\{ \begin{array}{ll} 0 & (i < j), \\ \dbinom{p-j}{i-j} \dbinom{n+1-2p+j}{q-p-i+j} & (i \geq j) \end{array} \right. \\
= & : c_{ij}.
\end{aligned}
$$

Next we consider real Grassmann manifolds i.e. $E = \mathbb{R}^{n+1}$.

Then we get

$$
\begin{aligned}
&(-1)^{(q-p)(n+1-q)}\chi_c(r^{-1}(x_1,x_2)\cap r^{-1}(Z_j)\cap S\underset{Y}{\times}S_i)\\
=&\left\{\begin{array}{cr}
0, & (i<j)\\
\left|\displaystyle\sum_{l=0}^{p-i}(-1)^l\mu_{l+i-j-1}(l)\mu_{n+1-p-i+j-l}(q-p-i+j)\right| & \\
\quad -\left|\displaystyle\sum_{l=0}^{p-i-1}(-1)^l\mu_{l+i-j}(l)\mu_{n-p-i+j-l}(q-p-i+j-1)\right|, & (i>j)\\
\mu_{n+1-2p+j}(q-p) & (i=j)
\end{array}\right.\\
=&:(-1)^{(q-p)(n+1-q)}c_{ij}.
\end{aligned}
$$

We can unify these two cases.

Note that c_{ij} is independent of the choice of (x_1,x_2) in Z_j. Therefore we have

$$\left(\int_r \mathbf{1}_{S\underset{Y}{\times}S_i}\right)(x_1,x_2)=\sum_{j=0}^{p}c_{ij}\mathbf{1}_{Z_j}.$$

Here, we define the $(p+1)$-type matrix $C^{p,q}=(c_{ij})_{0\le i,j\le p}$. This is the lower triangular matrix, so the absolute value of this determinant $\det C^{p,q}$ is equal to

$$|\mathrm{detC^{p,q}}|=\prod_{j=0}^{p}\mu_{n+1-2p+j}(q-p) \tag{2.2.1}$$

in the both situations. In particular it is $\mathbb{Z}$-valued.

Under the preparation above, we define the kernel function of an inversion formula for $\mathcal{R}_{(n+1;p,q)}$.

We get the equation

$$C^{p,q}\begin{pmatrix}\mathbf{1}_{Z_0}\\ \mathbf{1}_{Z_1}\\ \vdots\\ \mathbf{1}_{Z_p}\end{pmatrix}=\begin{pmatrix}\int_r \mathbf{1}_{S\underset{Y}{\times}S_0}\\ \int_r \mathbf{1}_{S\underset{Y}{\times}S_1}\\ \vdots\\ \int_r \mathbf{1}_{S\underset{Y}{\times}S_p}\end{pmatrix}.$$

By Cramer's formula, we can solve this equation with respect to $\mathbf{1}_{Z_p}$;

$$\det C^{p,q} \cdot \mathbf{1}_{Z_p} = \det \begin{pmatrix} c_{00} & 0 & \cdots & 0 & \int_r \mathbf{1}_{S \underset{Y}{\times} S_0} \\ c_{10} & c_{11} & \ddots & \vdots & \int_r \mathbf{1}_{S \underset{Y}{\times} S_1} \\ \vdots & \vdots & \ddots & 0 & \vdots \\ c_{p-1,0} & c_{p-1,1} & \cdots & c_{p-1,p-1} & \int_r \mathbf{1}_{S \underset{Y}{\times} S_{p-1}} \\ c_{p,0} & c_{p,1} & \cdots & c_{p,p-1} & \int_r \mathbf{1}_{S \underset{Y}{\times} S_p} \end{pmatrix}.$$

Definition 2.7. We define the kernel function of an inversion formula for the Radon transform as follows;

$$K_{p,q} = \det \begin{pmatrix} c_{00} & 0 & \cdots & 0 & \mathbf{1}_{S_0} \\ c_{10} & c_{11} & \ddots & \vdots & \mathbf{1}_{S_1} \\ \vdots & \vdots & \ddots & 0 & \vdots \\ c_{p-1,0} & c_{p-1,1} & \cdots & c_{p-1,p-1} & \mathbf{1}_{S_{p-1}} \\ c_{p,0} & c_{p,1} & \cdots & c_{p,p-1} & \mathbf{1}_{S_p} \end{pmatrix}.$$

Then we can define $\mathcal{R}^{-1}(\psi)$ for a $\psi \in CF(F_{n+1}(q))$ by

$$\mathcal{R}^{-1}(\psi) = \int_{p_1} K_{p,q} \cdot (p_2^* \psi).$$

Therefore we have the main result.

Theorem 2.8. *For any $\phi \in CF(F_{n+1}(p))$, we have*

$$\mathcal{R}^{-1} \circ \mathcal{R}_{(n+1;p,q)}(\phi) = \det C^{p,q} \cdot \phi.$$

In particular, if $\det C^{p,q}$ is not equal to zero, we can reconstruct the original constructible function ϕ from its Radon transform $\mathcal{R}_{(n+1;p,q)}(\phi)$ by dividing the last term by this constant $\det C^{p,q}$.

Proof. We can calculate in the same way as Schapira's formula. So we have

$$\begin{aligned}
&\mathcal{R}^{-1} \circ \mathcal{R}_{(n+1;p,q)}(\phi) \\
&= \int_{p_1} K_{p,q} \cdot \left(p_2^* \int_{p_2} 1_S \cdot p_1^* \phi \right) \\
&= \int_{q_2} \left\{ \det \begin{pmatrix} c_{00} & 0 & \cdots & 0 & \int_r \mathbf{1}_{S \underset{Y}{\times} S_0} \\ c_{10} & c_{11} & \ddots & \vdots & \int_r \mathbf{1}_{S \underset{Y}{\times} S_1} \\ \vdots & \vdots & \ddots & 0 & \vdots \\ c_{p-1,0} & c_{p-1,1} & \cdots & c_{p-1,p-1} & \int_r \mathbf{1}_{S \underset{Y}{\times} S_{p-1}} \\ c_{p,0} & c_{p,1} & \cdots & c_{p,p-1} & \int_r \mathbf{1}_{S \underset{Y}{\times} S_p} \end{pmatrix} \right\} q_1^* \phi \\
&= \int_{q_2} \det C^{p,q} \mathbf{1}_{Z_p} q_1^* \phi \\
&= \det C^{p,q} \cdot \phi \ . \\
&\quad (\text{Remark that } \int_{q_2} \mathbf{1}_{Z_p} q_1^* \phi = \phi.)
\end{aligned}$$

□

First, we consider the complex case. If $p+q \leq n+1$ is satisfied, we have $\det C^{p,q} \neq 0$ from (2.1.1) and (2.2.1). Then we obtain an inversion formula for $\mathcal{R}_{(n+1;p,q)}$.

Next, we consider the real case. If it is satisfied that $p+q \leq n+1$ and $q-p$ is even, we have $\det C^{p,q} \neq 0$ from (2.1.2) and (2.2.1). Then we obtain an inversion formula for $\mathcal{R}_{(n+1;p,q)}$.

In the case of $p = q$, the inversion formula is clear because $\mathcal{R}_{(n+1;p,q)}(\phi) = \phi$ holds.

Moreover, in the case of $p > q$, by the duality, we can get;

$$\begin{cases} (n+1-p) < (n+1-q), \\ F_{n+1}(n+1-p) \simeq F_{n+1}(p), \\ F_{n+1}(n+1-q) \simeq F_{n+1}(q). \end{cases}$$

So, we have only to consider the result in the case of $p < q$. That is, in the complex case, for $p+q \geq n+1$ we obtain an inversion formula for $\mathcal{R}_{(n+1l;p,q)}$. In the real case, when $p+q \geq n+1$ and $p-q$ is even, we obtain an inversion formula for $\mathcal{R}_{(n+1;p,q)}$.

Remark 2.9. In Section 2.1, we obtain an inversion formula at $p = 1$. In the terminologies of Section 2.2, we note that

$$\mathbf{1}_{F_{n+1}(1)\times F_{n+1}(q)} = \mathbf{1}_{S_0} + \mathbf{1}_{S_1}.$$

Therefore the formula in Section 2.1 is the same one in Section 2.2 at $p = 1$.

3 The inverse transform of the Radon transform

In the previous section, we constructed inversion formulas of Radon transforms. For general p and q, these formulas are not always inverse transforms of Radon transforms. In this section, we show that these inversion formulas are inverse transforms of Radon transforms when $p+q = n+1$ holds. Inversion formulas constructed in the previous section give left inverse transforms, speaking more concretely, so we show that these formulas give right inverse transforms.

For $p < q$ and $p + q = n + 1$ we consider the following diagram:

$$\begin{tikzcd}
 & F_{n+1}(p) \times F_{n+1}(q) \arrow[dddl, "p_1"'] \arrow[dddr, "p_2"] & \\
 & \cup & \\
 & F_{n+1}(p, q) \arrow[dl, "f"] \arrow[dr, "g"'] & \\
F_{n+1}(p) & & F_{n+1}(q).
\end{tikzcd}$$

We set $X = F_{n+1}(p)$, $Y = F_{n+1}(q)$ and $S = F_{n+1}(p, q)$.

We introduce the following sets similarly as in the previous section.

Definition 3.1.

(i) $S_i = \{(y, x) \in Y \times X \mid \dim(y \cap x) = i \}$ for $i = 0, 1, \cdots p$,

(ii) $Z'_j = \{(y_1, y_2) \in Y \times Y \mid \dim(y_1 \cap y_2) = j + (q - p) \}$ for $j = 0, 1, \cdots p$.

Remark 3.2. Let l be the dimension of $y_1 \cap y_2$. Because $\mathrm{span}[y_1, y_2]$ is a subspace of the total space $E = k^{n+1}$ ($k = \mathbb{R}$ or $\mathbb{C}$), we have an inequality

$$2q - l \leq n + 1 = p + q.$$

Therefore we have $q - p \leq l \leq q$.

Moreover we remark that

$$Y \times Y = \coprod_{j=0}^{p} Z'_j.$$

We calculate $\mathcal{R}_S \circ \mathcal{R}^{-1}$ similarly in previous sections.
Consider the following diagram:

$$\begin{tikzcd}
 & & S_i \underset{X}{\times} S \arrow[dl, "h'"'] \arrow[d, "r'"] \arrow[dr, "h"] & & \\
 & S_i \arrow[dl, "g'"'] \arrow[dr, "f'"] & Y \times Y \arrow[dll, "q'_1"'] \arrow[drr, "q'_2"] & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\
Y & & X & & Y.
\end{tikzcd}$$

In the same way as Schapira's formula, we calculate $\displaystyle\int_{r'} \mathbf{1}_{S_i \underset{X}{\times} S}$;

Proposition 3.3. *We have*

$$\left(\int_{r'} \mathbf{1}_{S_i \underset{X}{\times} S} \right) = \sum_{j=0}^{p} c_{ij} \cdot \mathbf{1}_{Z'_j},$$

where $C^{p,q} = (c_{ij})_{0 \le i,j \le p}$ is the coefficient matrix defined in the previous section.

Proof.

$$\left(\int_{r'} \mathbf{1}_{S_i \underset{X}{\times} S} \right)(y_1, y_2) = \sum_{j=0}^{p} \left(\int_{S_i \underset{X}{\times} S} \mathbf{1}_{r'^{-1}(y_1,y_2) \cap r'^{-1}(Z'_j)} \right) \cdot \mathbf{1}_{Z'_j}.$$

First, we calculate the homeomorphism of $r'^{-1}(y_1, y_2) \cap S_i \underset{X}{\times} S$ for $(y_1, y_2) \in Z'_j$.

We have

$$\begin{aligned}
& r'^{-1}(y_1, y_2) \cap S_i \underset{X}{\times} S \\
= \ & \left\{ x \in F_{n+1}(p) \;\middle|\; \begin{array}{c} x \subset y_1 \\ \dim(x \cap y_2) = i \end{array} \right\} \quad (\dim(y_1 \cap y_2) = j + q - p) \\
= \ & \{ x \in F_q(p) \mid \dim(x \cap y_1 \cap y_2) = i \} \\
= \ & \Omega'_{i,j} \setminus \Omega'_{i+1,j},
\end{aligned}$$

where we denote by

$$\Omega'_{i,j} = \{x \in F_q(p) \mid \dim(x \cap y) \geq i\} \quad (\dim y = j + q - p).$$

This set is the Schubert variety.

Therefore we should calculate the Euler-Poincaré index $\chi_c(\Omega'_{i,j} \setminus \Omega'_{i+1,j})$ with compact supports. Moreover we have the additivity

$$\chi_c(\Omega'_{i,j} \setminus \Omega'_{i+1,j}) = \chi_c(\Omega'_{i,j}) - \chi_c(\Omega'_{i+1,j}).$$

So it is enough to calculate the Euler-Poincaré index $\chi_c(\Omega'_{i,j})$ with compact supports.

Here we consider Young diagrams corresponding to $\Omega_{i,j}$ in the previous section and that of $\Omega'_{i,j}$. We remember the definition of $\Omega_{i,j}$;

$$\Omega_{i,j} = \{y \in F_q(q-p) \,|\dim(x \cap y) = i - j\} \quad (\dim x = p - j).$$

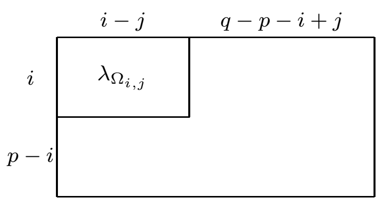
So the Young diagram $\lambda_{\Omega_{i,j}} = (a_1, a_2, \cdots, a_{q-p})$ with at most $q-p$ rows p columns corresponding to $\Omega_{i,j}$ is

$$a_l = \begin{cases} i & (1 \leq l \leq i - j), \\ 0 & (i - j + 1 \leq l \leq q - p). \end{cases}$$

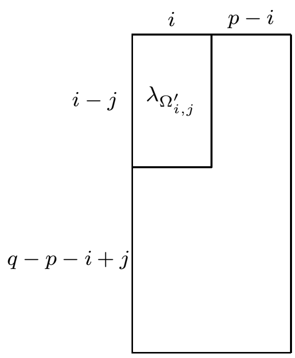
On the other hand, the Young diagram $\lambda_{\Omega'_{i,j}} = (b_1, b_2, \cdots, b_p)$ with at most p rows $q-p$ columns corresponding to $\Omega'_{i,j}$ is

$$b_l = \begin{cases} i - j & (1 \leq l \leq i), \\ 0 & (i + 1 \leq l \leq p). \end{cases}$$

These Young diagrams have the following shapes;



$(\lambda_{\Omega_{i,j}})$



$(\lambda_{\Omega'_{i,j}})$

(figure 4)

This implies that we have

$$\chi_c(\Omega_{i,j}) = \chi_c(\Omega'_{i,j}).$$

Therefore we have

$$\chi_c(r'^{-1}(y_1, y_2) \cap r'^{-1}(Z'_j) \cap S_i \underset{X}{\times} S) = c_{ij}.$$

$\square$

Theorem 3.4. *Let $p + q = n + 1$ hold. If all the conditions are satisfied for the construction of the inversion formula of the Radon transform in the previous section, this inversion formula becomes the inverse transform of the Radon transform. Namely, we get the following statement in each case.*

(i) *In the complex case, the Radon transform is the non-trivial isomorphism from $CF(F_{n+1}(p))$ to $CF(F_{n+1}(q))$ up to constant.*

(ii) *In the real case, if $q - p$ is even the Radon transform is the non-trivial isomorphism from $CF(F_{n+1}(p))$ to $CF(F_{n+1}(q))$ up to constant.*

Moreover, through the Euler-Poincaré index χ, the Radon transform gives the non-trivial isomorphism between Grothendieck groups.

Proof. It is enough to show that $\mathcal{R}^{-1}$ is a right inverse of $\mathcal{R}_{(n+1;p,q)}$. By the same argument as the previous section, we should show this statement only when $p < q$ holds.

For any $\psi \in CF(F_{n+1}(q))$, we have

$$\begin{aligned}
&\mathcal{R}_S \circ \mathcal{R}^{-1}(\psi) \\
&= \int_{p_2} 1_S \cdot \left(p_1^* \int_{p_1} K_{p,q} \cdot p_2^* \psi \right) \\
&= \int_{q'_2} \left\{ \det \begin{pmatrix} c_{00} & 0 & \cdots & 0 & \int_r \mathbf{1}_{S_0 \underset{X}{\times} S} \\ c_{10} & c_{11} & \ddots & \vdots & \int_r \mathbf{1}_{S_1 \underset{X}{\times} S} \\ \vdots & \vdots & \ddots & 0 & \vdots \\ c_{p-1,0} & c_{p-1,1} & \cdots & c_{p-1,p-1} & \int_r \mathbf{1}_{S_{p-1} \underset{X}{\times} S} \\ c_{p,0} & c_{p,1} & \cdots & c_{p,p-1} & \int_r \mathbf{1}_{S_p \underset{X}{\times} S} \end{pmatrix} \right\} q_1'^* \psi.
\end{aligned}$$

We set

$$z = \det \begin{pmatrix} c_{00} & 0 & \cdots & 0 & \int_r \mathbf{1}_{S_0 \underset{X}{\times} S} \\ c_{10} & c_{11} & \ddots & \vdots & \int_r \mathbf{1}_{S_1 \underset{X}{\times} S} \\ \vdots & \vdots & \ddots & 0 & \vdots \\ c_{p-1,0} & c_{p-1,1} & \cdots & c_{p-1,p-1} & \int_r \mathbf{1}_{S_{p-1} \underset{X}{\times} S} \\ c_{p,0} & c_{p,1} & \cdots & c_{p,p-1} & \int_r \mathbf{1}_{S_p \underset{X}{\times} S} \end{pmatrix}.$$

By Cramer's formula, there exist constructible functions x_0, x_1, $\cdots$, x_{p-1} such that we have

$$C^{p,q} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{p-1} \\ z \end{pmatrix} = (\det C^{p,q}) \begin{pmatrix} \int_r \mathbf{1}_{S_0 \underset{X}{\times} S} \\ \int_r \mathbf{1}_{S_1 \underset{X}{\times} S} \\ \vdots \\ \int_r \mathbf{1}_{S_{p-1} \underset{X}{\times} S} \\ \int_r \mathbf{1}_{S_p \underset{X}{\times} S} \end{pmatrix} = (\det C^{p,q}) C^{p,q} \begin{pmatrix} \mathbf{1}_{Z'_0} \\ \mathbf{1}_{Z'_1} \\ \vdots \\ \mathbf{1}_{Z'_{p-1}} \\ \mathbf{1}_{Z'_p} \end{pmatrix}.$$

By multiplying the matrix $(\det C^{p,q})(C^{p,q})^{-1}$ whose coefficients are $\mathbb{Z}$-valued, we obtain

$$\det C^{p,q} z = (\det C^{p,q})^2 \mathbf{1}_{Z'_p}.$$

So we have $z = \det C^{p,q} \cdot \mathbf{1}_{Z'_p}$.

Therefore for any $\psi \in CF(F_{n+1}(q))$ we have

$$\begin{aligned} \mathcal{R}_{(n+1;p,q)} \circ \mathcal{R}^{-1}(\psi) &= \int_{q'_2} \det C^{p,q} \mathbf{1}_{Z'_p} q'^*_1 \psi \\ &= \det C^{p,q} \cdot \psi, \end{aligned}$$

where we remark that $\int_{q'_2} \mathbf{1}_{Z'_p} q'^*_1 \psi = \psi$. □

4 Application : The indices of Radon transforms for $\mathcal{D}$-modules

In this section we apply our result to the calculation of indices of $\mathcal{D}$-modules.

We review definition of Radon-Penrose transforms of $\mathcal{D}$-modules. For more details on basic properties of $\mathcal{D}$-modules and Radon-Penrose transforms for $\mathcal{D}$-modules, we refer to [3, 9].

Consider a correspondence of complex analytic manifolds:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y. \end{tikzcd}$$

Definition 4.1. For $F \in Ob(\mathbf{D}^b(X))$, we set:

$$\Phi_S(F) = Rg_! f^{-1}(F)[\dim S - \dim X].$$

Definition 4.2. For $\mathcal{M} \in Ob(\mathbf{D}^b(\mathcal{D}_X))$, we set:

$$\underline{\Phi}_S \mathcal{M} = \int_g Df^* \mathcal{M}.$$

We call this the Radon-Penrose transform for $\mathcal{D}$-module.

Proposition 4.3. ([3, Proposition 2.6]) *Let $\mathcal{M} \in Ob(\mathbf{D}^b_{\text{good}}(\mathcal{D}_X))$. Assume that f is non-characteristic for $\mathcal{M}$, and that g is proper on $f^{-1}(\operatorname{supp}\mathcal{M})$. Then we have*

$$\Phi_S R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S \mathcal{M}, \mathcal{O}_Y).$$

Here we consider the index of the Radon-Penrose transforms of $\mathcal{D}$-modules.

Let $\mathcal{M} \in Ob(\mathbf{D}^b_{\text{rh}}(\mathcal{D}_X))$ be a complex of regular holonomic $\mathcal{D}$-modules. Then we remark that

$$Sol(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$$

is $\mathbb{C}$-constructible ([8, 10]). Moreover we define

$$\chi(\mathcal{M}) = \chi(Sol(\mathcal{M})).$$

Proposition 4.4. *Let $\mathcal{M} \in Ob(\mathbf{D}^b_{\text{rh}}(\mathcal{D}_X))$. Assume f is non-characteristic for $\mathcal{M}$, and that g is proper on $f^{-1}(\operatorname{supp}\mathcal{M})$. Then we have*

$$\chi(\underline{\Phi}_S(\mathcal{M})) = (-1)^{\dim S - \dim X} \mathcal{R}_S(\chi(\mathcal{M})).$$

Proof. Under the hypotheses $R\mathcal{H}om$ and Radon transforms are compatible and so are χ and operations.

$$\begin{aligned}
\chi(\underline{\Phi}_S(\mathcal{M})) &= \chi(R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S(\mathcal{M}), \mathcal{O}_Y)) \\
&= \chi(\Phi_S R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \\
&= \chi(Rg_! f^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)[\dim S - \dim X]) \\
&= \int_g f^* \chi(Sol(\mathcal{M})[\dim S - \dim X]) \\
&= (-1)^{\dim S - \dim X} \mathcal{R}_S(\chi(Sol(\mathcal{M}))) \\
&= (-1)^{\dim S - \dim X} \mathcal{R}_S(\chi(\mathcal{M})).
\end{aligned}$$

□

We consider complex Grassmann manifolds. Set $X = F_{n+1}(p)$, $Y = F_{n+1}(q)$, $S = F_{n+1}(p,q)$ for $p+q \leq n+1$, $p < q$ or $p+q \geq n+1$, $p > q$. Then since we have an inversion formula for $\mathcal{R}_S$, we can calculate the index of $\chi(Sol(\mathcal{M}))$ from the index of its Radon transform $\chi(\underline{\Phi_S}(\mathcal{M}))$.

5 The image of the Radon transform of the characteristic function on a Schubert cell

In this section, we characterize the image of the Radon transform of the characteristic function on a Schubert cell. We consider the following correspondence;

$$\begin{array}{ccc} & S = F_{n+1}(p,q) & \\ {}^{f}\swarrow & & \searrow^{g} \\ X = F_{n+1}(p) & & F_{n+1}(q) = Y. \end{array}$$

In this section, we consider the complement of Young diagram λ^c when we consider the Schubert cell. We denote the whole of increasing sequences like this by $\Lambda_{p,n-p}$. We denote the Schubert cell corresponding to $\lambda^c = \alpha \in \Lambda_{p,n-p}$ by Ω°_α in this section.

Definition 5.1.

(i) For an $\alpha = (a_1, a_2, \cdots, a_p) \in \Lambda_{p,n-p}$, we define a new sequence $\hat{\alpha} \in \Lambda_{p,n}$;

$$\hat{\alpha} = (a_1 + 1, a_2 + 2, \cdots, a_p + p).$$

(ii) Let $\alpha = (a_1, a_2, \cdots, a_p) \in \Lambda_{p,n}$ and $\beta = (b_1, b_2, \cdots, b_q) \in \Lambda_{q,m}$ for $p < q$. Then we define $\alpha \subset \beta$ if and only if for each i $(1 \leq i \leq p)$ there exists j $(1 \leq j \leq q)$ such that $a_i = b_j$. We denote this correspondence of numbers by $\sigma_{\alpha,\beta}$, that is $\sigma_{\alpha,\beta}(i) = j$.

Definition 5.2. Let $\alpha = (a_1, a_2, \cdots, a_p) \in \Lambda_{p,n-p}$, $\beta = (b_1, b_2, \cdots, b_q) \in \Lambda_{q,n-q}$ such that $\alpha \subset \beta$. Then we define

$$c_{\alpha,\beta} = \sum_{k=1}^{p} \sigma_{\alpha,\beta}(k) - k.$$

We characterize the image of the Radon transform of $\mathbf{1}_{\Omega^\circ_\alpha}$.

Theorem 5.3. *Let $\alpha \in \Lambda_{p,n-p}$.*

(i) *In the complex case, we have*

$$\mathcal{R}_S(\mathbf{1}_{\Omega^\circ_\alpha}) = \begin{cases} \displaystyle\sum_{\hat{\alpha}\subset\hat{\beta}} \mathbf{1}_{\Omega^\circ_\beta} & \text{for } p \le q, \\ \displaystyle\sum_{\hat{\alpha}\supset\hat{\beta}} \mathbf{1}_{\Omega^\circ_\beta} & \text{for } p \ge q, \end{cases}$$

where $\hat{\beta}$ ranges through sequences in $\Lambda_{q,n}$ containing (or contained by) $\hat{\alpha}$.

(ii) *In the real case, we have*

$$\mathcal{R}_S(\mathbf{1}_{\Omega^\circ_\alpha}) = \begin{cases} \displaystyle\sum_{\hat{\alpha}\subset\hat{\beta}} (-1)^{c_{\hat{\alpha},\hat{\beta}}} \mathbf{1}_{\Omega^\circ_\beta} & \text{for } p \le q, \\ \displaystyle\sum_{\hat{\alpha}\supset\hat{\beta}} (-1)^{c_{\hat{\beta},\hat{\alpha}}} \mathbf{1}_{\Omega^\circ_\beta} & \text{for } p \ge q, \end{cases}$$

where $\hat{\beta}$ ranges through sequences in $\Lambda_{q,n}$ containing (or contained by) $\hat{\alpha}$.

Proof. It is enough to show when $p < q$.

We calculate

$$\mathcal{R}_S(\mathbf{1}_{\Omega^\circ_\alpha})(y) = \int_X \mathbf{1}_{g^{-1}(y)\cap S\cap f^{-1}(\Omega^\circ_\alpha)}.$$

Here we have

$$\begin{aligned} & g^{-1}(y) \cap S \cap f^{-1}(\Omega^\circ_\alpha) \\ \simeq \ & \left\{ x \in F_{n+1}(p) \;\middle|\; \begin{array}{l} x \subset y, \\ \ \dim(x \cap V_{a_i+i}) = i, \\ \ \dim(x \cap V_{a_i+i-1}) = i-1 \end{array} \quad (i = 1, 2, \cdots, p) \right\}. \end{aligned}$$

Here we fix an $x \in \Omega_\alpha$. To satisfy $x \subset y$, y must have the same gaps of dimensions of intersection with the complete flag of E as ones of x. Therefore $\hat{\beta}$ must satisfy $\hat{\beta} \supset \hat{\alpha}$.

Let $x \in \Omega_\alpha$, $y \in \Omega_\beta$ where $\hat{\alpha} \subset \hat{\beta}$. Then we can modify the basis of the whole space E to contain the basis of y without modifying the original complete flag. y has the complete flag which is a subflag of the complete flag of E. That is, we define the complete flag of y;

$$\begin{array}{ccccccccc} V_{b_1+1} & \subset & V_{b_2+2} & \subset & \cdots & \subset & V_{b_q+q} & & \\ \| & & \| & & \cdots & & \| & & \\ V'_1 & \subset & V'_2 & \subset & \cdots & \subset & V'_q & = y. \end{array}$$

By considering in y, then the fiber above is a Schubert cell of $F_q(p)$;

$$\begin{aligned} & g^{-1}(y) \cap S \cap f^{-1}(\Omega^\circ_\alpha) \\ \simeq & \left\{ x \in F_q(p) \;\middle|\; \begin{array}{l} \dim(x \cap V_{a_i+i}) = i, \\ \dim(x \cap V_{a_i+i-1}) = i-1 \end{array} \quad (i = 1, 2, \cdots, p) \right\} \\ \simeq & \left\{ x \in F_q(p) \;\middle|\; \begin{array}{l} \dim(x \cap V'_{\sigma_{\hat{\alpha},\hat{\beta}}(i)}) = i, \\ \dim(x \cap V'_{\sigma_{\hat{\alpha},\hat{\beta}}(i)-1}) = i-1 \end{array} \quad (i = 1, 2, \cdots, p) \right\}. \end{aligned}$$

Therefore we should calculate the Euler-Poincaré index with compact supports of this Schubert cell. In the complex case, it is equal to 1. In the real case, it is equal to $(-1)^{c_{\hat{\alpha},\hat{\beta}}}$. So we obtain results. $\square$

We can represent this formula with using Young diagrams when $p = 1$, $q = n$.

Definition 5.4. For a Young diagram $\lambda = (k)$ with at most a row and n columns, we define its dual with at most n rows and a column;

$$\lambda^* = (1, 1, \cdots, 1, \overset{n-k}{1}, 0, 0, \cdots, 0).$$

Definition 5.5. Let λ be a Young diagram with at most a row and n columns. For a Young diagram with at most n rows and a columns μ, we define

$$\tau_\lambda(\mu) = \begin{cases} n - |\lambda| & (\text{for } n - |\lambda| < |\mu|), \\ n - |\lambda| - 1 & (\text{for } n - |\lambda| \geq |\mu|). \end{cases}$$

Proposition 5.6. *Let λ be any Young diagram with a row and n columns.*

(i) *In the complex case, we have*

$$\mathcal{R}_S(\mathbf{1}_{\Omega^\circ_\lambda}) = \sum_{\mu \neq \lambda^*} \mathbf{1}_{\Omega^\circ_\mu},$$

where μ ranges through Young diagrams with at most n rows and a column which are not equal to λ^*.

When we denote Young diagrams by $\lambda_k = (k)$ $(0 \leq k \leq n)$, we can represent this formula by a matrix;

$$\mathcal{R}_S \begin{pmatrix} \mathbf{1}_{\Omega_{\lambda_0}} \\ \mathbf{1}_{\Omega_{\lambda_1}} \\ \vdots \\ \mathbf{1}_{\Omega_{\lambda_n}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_{\Omega_{\lambda_0^*}} \\ \mathbf{1}_{\Omega_{\lambda_1^*}} \\ \vdots \\ \mathbf{1}_{\Omega_{\lambda_n^*}} \end{pmatrix}.$$

(ii) *In the real case, we have*

$$\mathcal{R}_S(\mathbf{1}_{\Omega_\lambda^\circ}) = \sum_{\mu \neq \lambda^*} (-1)^{\tau_\lambda(\mu)} \mathbf{1}_{\Omega_\mu^\circ},$$

where μ ranges through Young diagrams with at most n rows and a column which are not equal to λ^*.

When we denote Young diagrams by $\lambda_k = (k)$ $(0 \leq k \leq n)$, we can represent this formula by a matrix;

$$\mathcal{R}_S \begin{pmatrix} \mathbf{1}_{\Omega_{\lambda_0}} \\ \mathbf{1}_{\Omega_{\lambda_1}} \\ \vdots \\ \mathbf{1}_{\Omega_{\lambda_n}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & (-1)^{n-1} & (-1)^{n-1} & \cdots & (-1)^{n-1} \\ (-1)^{n-1} & 0 & (-1)^{n-2} & \cdots & (-1)^{n-2} \\ (-1)^{n-2} & (-1)^{n-2} & 0 & & (-1)^{n-3} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_{\Omega_{\lambda_0^*}} \\ \mathbf{1}_{\Omega_{\lambda_1^*}} \\ \vdots \\ \mathbf{1}_{\Omega_{\lambda_n^*}} \end{pmatrix}.$$

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Yutaka MATSUI
Graduate School of Mathematical Sciences, the University of Tokyo
3-8-1, Komaba, Meguro-ku, Tokyo, 153-8914, Japan
E-mail: you317@ms.u-tokyo.ac.jp