Lemma 5.6.

In[1]:=
```
Collect[Expand[(1 + t^2) (e0 + e2 t^2 + e4 t^4) -
   (c0 + c2 t^2)^3 - (e4 - c2^3) (t^2 - h1) (t^2 - h2) (t^2 - h3)], t]
```

Out[1]=

$$-c0^3 + e0 - c2^3\, h1\, h2\, h3 + e4\, h1\, h2\, h3 + \left(-3\, c0^2\, c2 + e0 + e2 + c2^3\, h1\, h2 - e4\, h1\, h2 + c2^3\, h1\, h3 - e4\, h1\, h3 + c2^3\, h2\, h3 - e4\, h2\, h3\right) t^2 + \left(-3\, c0\, c2^2 + e2 + e4 - c2^3\, h1 + e4\, h1 - c2^3\, h2 + e4\, h2 - c2^3\, h3 + e4\, h3\right) t^4$$

In[2]:=
```
Solve[{-c0^3 + e0 - c2^3 h1 h2 h3 + e4 h1 h2 h3 == 0,
  -3 c0^2 c2 + e0 + e2 + c2^3 h1 h2 - e4 h1 h2 + c2^3 h1 h3 - e4 h1 h3 + c2^3 h2 h3 - e4 h2 h3 == 0,
  -3 c0 c2^2 + e2 + e4 - c2^3 h1 + e4 h1 - c2^3 h2 + e4 h2 - c2^3 h3 + e4 h3 == 0}, {e0, e2, e4}]
```

Out[2]=

$$\Big\{\Big\{e0 \to -\left(-c0^3 - c0^3\, h1 - c0^3\, h2 - c0^3\, h1\, h2 - c0^3\, h3 - c0^3\, h1\, h3 - c0^3\, h2\, h3 - 3\, c0^2\, c2\, h1\, h2\, h3 + 3\, c0\, c2^2\, h1\, h2\, h3 - c2^3\, h1\, h2\, h3\right) \Big/ \left((1 + h1)\,(1 + h2)\,(1 + h3)\right),$$

$$e2 \to -\left(c0^3 - 3\, c0^2\, c2 + c0^3\, h1 - 3\, c0^2\, c2\, h1 + c0^3\, h2 - 3\, c0^2\, c2\, h2 - 3\, c0\, c2^2\, h1\, h2 + c2^3\, h1\, h2 + c0^3\, h3 - 3\, c0^2\, c2\, h3 - 3\, c0\, c2^2\, h1\, h3 + c2^3\, h1\, h3 - 3\, c0\, c2^2\, h2\, h3 + c2^3\, h2\, h3 - 3\, c0\, c2^2\, h1\, h2\, h3 + c2^3\, h1\, h2\, h3\right) \Big/ \left((1 + h1)\,(1 + h2)\,(1 + h3)\right),$$

$$e4 \to -\left(-c0^3 + 3\, c0^2\, c2 - 3\, c0\, c2^2 - c2^3\, h1 - c2^3\, h2 - c2^3\, h1\, h2 - c2^3\, h3 - c2^3\, h1\, h3 - c2^3\, h2\, h3 - c2^3\, h1\, h2\, h3\right) \Big/ \left((1 + h1)\,(1 + h2)\,(1 + h3)\right)\Big\}\Big\}$$

In[3]:=
```
e0 := -(-c0^3 - c0^3 h1 - c0^3 h2 - c0^3 h1 h2 - c0^3 h3 - c0^3 h1 h3 - c0^3 h2 h3 -
     3 c0^2 c2 h1 h2 h3 + 3 c0 c2^2 h1 h2 h3 - c2^3 h1 h2 h3) / ((1 + h1) (1 + h2) (1 + h3))
```

In[4]:=
```
e2 := -(c0^3 - 3 c0^2 c2 + c0^3 h1 - 3 c0^2 c2 h1 + c0^3 h2 - 3 c0^2 c2 h2 - 3 c0 c2^2 h1 h2 +
     c2^3 h1 h2 + c0^3 h3 - 3 c0^2 c2 h3 - 3 c0 c2^2 h1 h3 + c2^3 h1 h3 - 3 c0 c2^2 h2 h3 +
     c2^3 h2 h3 - 3 c0 c2^2 h1 h2 h3 + c2^3 h1 h2 h3) / ((1 + h1) (1 + h2) (1 + h3))
```

In[5]:=
```
e4 := -(-c0^3 + 3 c0^2 c2 - 3 c0 c2^2 - c2^3 h1 - c2^3 h2 - c2^3 h1 h2 -
     c2^3 h3 - c2^3 h1 h3 - c2^3 h2 h3 - c2^3 h1 h2 h3) / ((1 + h1) (1 + h2) (1 + h3))
```

In[6]:=
```
Collect[Expand[
  a + b z + k1 x^2 + k2 y^2 + k3 z^2 + d z (x^2 + y^2 + z^2) + e (x^2 + y^2 + z^2)^2 /.
   {x → x1, y → y1, z -> z1 + t}], {x1, y1, z1}]
```

Out[6]=

$$a + b\, t + k3\, t^2 + d\, t^3 + e\, t^4 + e\, x1^4 + e\, y1^4 + \left(b + 2\, k3\, t + 3\, d\, t^2 + 4\, e\, t^3\right) z1 + \left(k3 + 3\, d\, t + 6\, e\, t^2\right) z1^2 + (d + 4\, e\, t)\, z1^3 + e\, z1^4 + y1^2 \left(k2 + d\, t + 2\, e\, t^2 + (d + 4\, e\, t)\, z1 + 2\, e\, z1^2\right) + x1^2 \left(k1 + d\, t + 2\, e\, t^2 + 2\, e\, y1^2 + (d + 4\, e\, t)\, z1 + 2\, e\, z1^2\right)$$

In[7]:=
```
Factor[d + 4 e t + 2 s (k3 + 3 d t + 6 e t^2) + 2 s^2 (b + 2 k3 t + 3 d t^2 + 4 e t^3) /.
  {s → -(b + 2 k3 t + 3 d t^2 + 4 e t^3) / (4 (b t + k3 t^2 + d t^3 + e t^4))}]
```

Out[7]=

$$-\left(-b^3 - 2\, b^2\, k3\, t - 5\, b^2\, d\, t^2 - 20\, b^2\, e\, t^3 + 5\, b\, d^2\, t^4 - 20\, b\, e\, k3\, t^4 + 4\, b\, d\, e\, t^5 + 2\, d^2\, k3\, t^5 - 8\, e\, k3^2\, t^5 + d^3\, t^6 + 8\, b\, e^2\, t^6 - 4\, d\, e\, k3\, t^6\right) \Big/ \left(8\, t^2 \left(b + k3\, t + d\, t^2 + e\, t^3\right)^2\right)$$

t - equation is :

In[8]:=
```
Factor[
 -(-b^3 - 2 b^2 k3 t - 5 b^2 d t^2 - 20 b^2 e t^3 + 5 b d^2 t^4 - 20 b e k3 t^4 + 4 b d e t^5 + 2 d^2 k3 t^5 -
     8 e k3^2 t^5 + d^3 t^6 + 8 b e^2 t^6 - 4 d e k3 t^6) /. {b -> -1, k1 -> c0, k2 -> c2,
   k3 -> (e0 - e2 + e4) / (c0 - c2)^2, d -> ((c0 + c2) e2 - 2 c2 e0 - 2 c0 e4) / (c0 - c2)^2,
   e -> (c2^2 e0 - c0 c2 e2 + c0^2 e4) / (c0 - c2)^2}]
```

Out[8]=
```
((-1 - h1 - h2 - h1 h2 - h3 - h1 h3 - h2 h3 - h1 h2 h3 + 2 c0 t +
     2 c2 h1 t + 2 c0 h2 t + 2 c2 h1 h2 t + 2 c0 h3 t + 2 c2 h1 h3 t + 2 c0 h2 h3 t +
     2 c2 h1 h2 h3 t - c0^2 t^2 + c0^2 h1 t^2 - 2 c0 c2 h1 t^2 - c0^2 h2 t^2 - c2^2 h1 h2 t^2 -
     c0^2 h3 t^2 - c2^2 h1 h3 t^2 - 2 c0 c2 h2 h3 t^2 + c2^2 h2 h3 t^2 - c2^2 h1 h2 h3 t^2)
   (1 + h1 + h2 + h1 h2 + h3 + h1 h3 + h2 h3 + h1 h2 h3 - 2 c0 t - 2 c0 h1 t - 2 c2 h2 t -
     2 c2 h1 h2 t - 2 c0 h3 t - 2 c0 h1 h3 t - 2 c2 h2 h3 t - 2 c2 h1 h2 h3 t +
     c0^2 t^2 + c0^2 h1 t^2 - c0^2 h2 t^2 + 2 c0 c2 h2 t^2 + c2^2 h1 h2 t^2 + c0^2 h3 t^2 +
     2 c0 c2 h1 h3 t^2 - c2^2 h1 h3 t^2 + c2^2 h2 h3 t^2 + c2^2 h1 h2 h3 t^2)
   (1 + h1 + h2 + h1 h2 + h3 + h1 h3 + h2 h3 + h1 h2 h3 - 2 c0 t - 2 c0 h1 t - 2 c0 h2 t -
     2 c0 h1 h2 t - 2 c2 h3 t - 2 c2 h1 h3 t - 2 c2 h2 h3 t - 2 c2 h1 h2 h3 t + c0^2 t^2 +
     c0^2 h1 t^2 + c0^2 h2 t^2 + 2 c0 c2 h1 h2 t^2 - c2^2 h1 h2 t^2 - c0^2 h3 t^2 + 2 c0 c2 h3 t^2 +
     c2^2 h1 h3 t^2 + c2^2 h2 h3 t^2 + c2^2 h1 h2 h3 t^2)) / ((1 + h1)^3 (1 + h2)^3 (1 + h3)^3)
```

The solution t for the third factor is :

In[9]:=
```
Solve[(1 + h1 + h2 + h1 h2 + h3 + h1 h3 + h2 h3 + h1 h2 h3 - 2 c0 t -
     2 c0 h1 t - 2 c0 h2 t - 2 c0 h1 h2 t - 2 c2 h3 t - 2 c2 h1 h3 t - 2 c2 h2 h3 t -
     2 c2 h1 h2 h3 t + c0^2 t^2 + c0^2 h1 t^2 + c0^2 h2 t^2 + 2 c0 c2 h1 h2 t^2 - c2^2 h1 h2 t^2 -
     c0^2 h3 t^2 + 2 c0 c2 h3 t^2 + c2^2 h1 h3 t^2 + c2^2 h2 h3 t^2 + c2^2 h1 h2 h3 t^2) == 0, t]
```

Out[9]=
```
{{t -> (2 c0 + 2 c0 h1 + 2 c0 h2 + 2 c0 h1 h2 + 2 c2 h3 + 2 c2 h1 h3 + 2 c2 h2 h3 + 2 c2 h1 h2 h3 -
       √((-2 c0 - 2 c0 h1 - 2 c0 h2 - 2 c0 h1 h2 - 2 c2 h3 - 2 c2 h1 h3 - 2 c2 h2 h3 -
             2 c2 h1 h2 h3)^2 - 4 (1 + h1 + h2 + h1 h2 + h3 + h1 h3 + h2 h3 + h1 h2 h3)
           (c0^2 + c0^2 h1 + c0^2 h2 + 2 c0 c2 h1 h2 - c2^2 h1 h2 - c0^2 h3 +
             2 c0 c2 h3 + c2^2 h1 h3 + c2^2 h2 h3 + c2^2 h1 h2 h3))) /
     (2 (c0^2 + c0^2 h1 + c0^2 h2 + 2 c0 c2 h1 h2 - c2^2 h1 h2 - c0^2 h3 + 2 c0 c2 h3 +
          c2^2 h1 h3 + c2^2 h2 h3 + c2^2 h1 h2 h3))},
  {t -> (2 c0 + 2 c0 h1 + 2 c0 h2 + 2 c0 h1 h2 + 2 c2 h3 + 2 c2 h1 h3 + 2 c2 h2 h3 + 2 c2 h1 h2 h3 +
       √((-2 c0 - 2 c0 h1 - 2 c0 h2 - 2 c0 h1 h2 - 2 c2 h3 - 2 c2 h1 h3 - 2 c2 h2 h3 -
             2 c2 h1 h2 h3)^2 - 4 (1 + h1 + h2 + h1 h2 + h3 + h1 h3 + h2 h3 + h1 h2 h3)
           (c0^2 + c0^2 h1 + c0^2 h2 + 2 c0 c2 h1 h2 - c2^2 h1 h2 - c0^2 h3 +
             2 c0 c2 h3 + c2^2 h1 h3 + c2^2 h2 h3 + c2^2 h1 h2 h3))) /
     (2 (c0^2 + c0^2 h1 + c0^2 h2 + 2 c0 c2 h1 h2 - c2^2 h1 h2 - c0^2 h3 + 2 c0 c2 h3 +
          c2^2 h1 h3 + c2^2 h2 h3 + c2^2 h1 h2 h3))}}
```

denominator :

In[10]:=
```
Simplify[(2 (c0^2 + c0^2 h1 + c0^2 h2 + 2 c0 c2 h1 h2 -
       c2^2 h1 h2 - c0^2 h3 + 2 c0 c2 h3 + c2^2 h1 h3 + c2^2 h2 h3 + c2^2 h1 h2 h3))]
```

Out[10]=
```
2 (c0^2 (1 + h1 + h2 - h3) + 2 c0 c2 (h1 h2 + h3) + c2^2 (h2 h3 + h1 (h2 (-1 + h3) + h3)))
```

square root part :

```
In[11]:= Simplify[
  ((-2 c0 - 2 c0 h1 - 2 c0 h2 - 2 c0 h1 h2 - 2 c2 h3 - 2 c2 h1 h3 - 2 c2 h2 h3 - 2 c2 h1 h2 h3)^2 -
    4 (1 + h1 + h2 + h1 h2 + h3 + h1 h3 + h2 h3 + h1 h2 h3) (c0^2 + c0^2 h1 + c0^2 h2 + 2 c0 c2 h1 h2 -
       c2^2 h1 h2 - c0^2 h3 + 2 c0 c2 h3 + c2^2 h1 h3 + c2^2 h2 h3 + c2^2 h1 h2 h3))]

Out[11]= 4 (c0 - c2)^2 (1 + h1) (1 + h2) (h1 - h3) (h2 - h3)
```

other part :

```
In[12]:= Simplify[
  2 c0 + 2 c0 h1 + 2 c0 h2 + 2 c0 h1 h2 + 2 c2 h3 + 2 c2 h1 h3 + 2 c2 h2 h3 + 2 c2 h1 h2 h3]

Out[12]= 2 (1 + h1) (1 + h2) (c0 + c2 h3)
```

Therefore, setting w : c0 / c2, k : $\sqrt{(1 + h1) (1 + h2) (h1 - h3) (h2 - h3)}$,
t : ((w + h3) (1 + h1) (1 + h2) + (w - 1) k) /
(c2 ((1 + h1 + h2 - h3) w^2 + 2 (h1 h2 + h3) w + h1 h3 + h2 h3 - h1 h2 + h1 h2 h3)),

First we prove s : - (b + 2 k3 t + 3 d t^2 + 4 e t^3) / (4 (b t + k3 t^2 + d t^3 + e t^4))

is equal to
s : - c2 $\left(-h1\,h2 + h1\,h3 + h2\,h3 + h1\,h2\,h3 + (2\,h1\,h2 + 2\,h3)\,w + (1 + h1 + h2 - h3)\,w^2\right)$ k / {2 (1 + h1) (1 + h2) (h1 - h3) (h2 - h3) (-1 + w) }

We consider the difference (times the denominator) :

```
In[13]:= ds := - (b + 2 k3 t + 3 d t^2 + 4 e t^3) - (4 (b t + k3 t^2 + d t^3 + e t^4))
   (- c2 (-h1 h2 + h1 h3 + h2 h3 + h1 h2 h3 + (2 h1 h2 + 2 h3) w + (1 + h1 + h2 - h3) w^2)
     k / {2 (1 + h1) (1 + h2) (h1 - h3) (h2 - h3) (-1 + w) })

In[14]:= Simplify[Factor[(ds /. {{b → -1, k1 → c0, k2 → c2, k3 → (e0 - e2 + e4) / (c0 - c2)^2,
        d → ((c0 + c2) e2 - 2 c2 e0 - 2 c0 e4) / (c0 - c2)^2,
        e → (c2^2 e0 - c0 c2 e2 + c0^2 e4) / (c0 - c2)^2}}) /.
     {c0 → w c2, t → ((w + h3) (1 + h1) (1 + h2) + (w - 1) k) /
        (c2 ((1 + h1 + h2 - h3) w^2 + 2 (h1 h2 + h3) w + h1 h3 + h2 h3 - h1 h2 + h1 h2 h3))}] /.
   {k → √ (1 + h1) (1 + h2) (h1 - h3) (h2 - h3)}]

Out[14]= {{0}}
```

Therefore we get ds = 0.

Here we introduce u, v as follows :

```
In[15]:= u := h1 + 2 h1^2 + h1^3 + h2 - 2 h1 h2 - h1^2 h2 + 2 h1^3 h2 + 2 h2^2 - h1 h2^2 - 4 h1^2 h2^2 + h2^3 +
   2 h1 h2^3 - 2 h3 + h1 h3 + 2 h1^2 h3 - h1^3 h3 + h2 h3 + h1^2 h2 h3 + 2 h2^2 h3 + h1 h2^2 h3 -
   h2^3 h3 - 4 h3^2 - h1 h3^2 + 2 h1^2 h3^2 - h2 h3^2 - 2 h1 h2 h3^2 + 2 h2^2 h3^2 - 2 h3^3 - h1 h3^3 - h2 h3^3

In[16]:= v := 2 - 2 h1^2 + 4 h1 h2 - 2 h2^2 + 4 h3 + 2 h3^2
```

Next we calculate 1 / e''' = 1 / a' = 1 / (b t + k3 t^2 + d t^3 + e t^4)

We prove 1 / e''' = -c2 $\left((u + v\,k)\left(w + \left(\frac{h1\,h2 + h3 + k}{1 + h1 + h2 - h3}\right)\right)\text{^}4\right) \Big/$
$\left((w - 1)\text{^}3\,(h1 - h2)^2\,(h1 - h3)\,(h2 - h3)\,(1 + h3)\right)$

To do so, we consider the difference :

In[17]:=
```
die3 := 1 - (b t + k3 t^2 + d t^3 + e t^4) (-c2 ((u + v k) (w + ((h1 h2 + h3 + k)/(1 + h1 + h2 - h3)))^4) /
    ((w - 1)^3 (h1 - h2)^2 (h1 - h3) (h2 - h3) (1 + h3)))
```

In[18]:=
```
Simplify[((die3 /. {{b → -1, k1 → c0, k2 → c2, k3 → (e0 - e2 + e4) / (c0 - c2)^2,
        d → ((c0 + c2) e2 - 2 c2 e0 - 2 c0 e4) / (c0 - c2)^2,
        e → (c2^2 e0 - c0 c2 e2 + c0^2 e4) / (c0 - c2)^2}}) /.
    {c0 → w c2, t → ((w + h3) (1 + h1) (1 + h2) + (w - 1) k) /
        (c2 ((1 + h1 + h2 - h3) w^2 + 2 (h1 h2 + h3) w + h1 h3 + h2 h3 - h1 h2 + h1 h2 h3))}) /.
  {k → √ (1 + h1) (1 + h2) (h1 - h3) (h2 - h3)}]
```

Out[18]= {0}

In[19]:=
```
Simplify[u + v √ (1 + h1) (1 + h2) (h1 - h3) (h2 - h3) /. {h3 → r, h2 → q + r, h1 → p + q + r}]
```

Out[19]=
$$p^3 (1 + 2 q + r) + p (1 + r)^2 (1 + 2 q + r) + 2 p^2 \left(1 + q + q^2 + 2 r + q r + r^2 - \sqrt{(q (p + q) (1 + q + r) (1 + p + q + r))}\right) + 2 (1 + r)^2 \left(q + q^2 + q r + \sqrt{(q (p + q) (1 + q + r) (1 + p + q + r))}\right)$$

In[20]:=
```
Expand[(p (1 + 2 q + r) + 2 (1 + q + q^2 + 2 r + q r + r^2))^2 - 4 q (p + q) (1 + q + r) (1 + p + q + r)]
```

Out[20]=
$$4 + 4 p + p^2 + 8 q + 8 p q + 8 q^2 + 16 r + 12 p r + 2 p^2 r + 24 q r + 16 p q r + 16 q^2 r + 24 r^2 + 12 p r^2 + p^2 r^2 + 24 q r^2 + 8 p q r^2 + 8 q^2 r^2 + 16 r^3 + 4 p r^3 + 8 q r^3 + 4 r^4$$

As for k3''' = (k3 + 3 d t + 6 e t^2) + 3 (b + 2 k3 t + 3 d t^2 + 4 e t^3) s + 6 (b t + k3 t^2 + d t^3 + e t^4) s^2

we prove k3''' = (c2 (h1 + h2 + 2 h1 h2 - 2 h3 - h1 h3 - h2 h3) (-1 + w)) / (2 (1 + h1) (1 + h2) (1 + h3))

Consider the difference

In[21]:=
```
dk3 := (k3 + 3 d t + 6 e t^2) +
   3 (b + 2 k3 t + 3 d t^2 + 4 e t^3) s + 6 (b t + k3 t^2 + d t^3 + e t^4) s^2 -
   (c2 (h1 + h2 + 2 h1 h2 - 2 h3 - h1 h3 - h2 h3) (-1 + w)) / (2 (1 + h1) (1 + h2) (1 + h3))
```

In[22]:=
```
Simplify[((dk3 /. {{b → -1, k1 → c0, k2 → c2, k3 → (e0 - e2 + e4) / (c0 - c2)^2,
        d → ((c0 + c2) e2 - 2 c2 e0 - 2 c0 e4) / (c0 - c2)^2,
        e → (c2^2 e0 - c0 c2 e2 + c0^2 e4) / (c0 - c2)^2}}) /.
    {c0 → w c2, t → ((w + h3) (1 + h1) (1 + h2) + (w - 1) k) /
        (c2 ((1 + h1 + h2 - h3) w^2 + 2 (h1 h2 + h3) w + h1 h3 + h2 h3 - h1 h2 + h1 h2 h3)),
      s -> - c2 (-h1 h2 + h1 h3 + h2 h3 + h1 h2 h3 + (2 h1 h2 + 2 h3) w + (1 + h1 + h2 - h3) w^2)
        k / {2 (1 + h1) (1 + h2) (h1 - h3) (h2 - h3) (-1 + w) }}) /.
  {k → √ (1 + h1) (1 + h2) (h1 - h3) (h2 - h3)}]
```

Out[22]= {{0}}

As for k2''' := k2 + d t + 2 e t^2 + (b + 2 k3 t + 3 d t^2 + 4 e t^3) s + 2 (b t + k3 t^2 + d t^3 + e t^4) s^2

We prove $k2''' = -\frac{c2\ (2 + h1 + h2)\ (-1 + w)}{2\ (1 + h1)\ (1 + h2)}$

Consider the difference

In[23]:=
```
dk2 := k2 + d t + 2 e t^2 + (b + 2 k3 t + 3 d t^2 + 4 e t^3) s +
   2 (b t + k3 t^2 + d t^3 + e t^4) s^2 - (-(c2 (2 + h1 + h2) (-1 + w))/(2 (1 + h1) (1 + h2)))
```

In[24]:=
```
Simplify[((dk2 /. {{b → -1, k1 → c0, k2 → c2, k3 → (e0 - e2 + e4) / (c0 - c2)^2,
        d → ((c0 + c2) e2 - 2 c2 e0 - 2 c0 e4) / (c0 - c2)^2,
        e → (c2^2 e0 - c0 c2 e2 + c0^2 e4) / (c0 - c2)^2}}) /.
    {c0 → w c2, t → ((w + h3) (1 + h1) (1 + h2) + (w - 1) k) /
       (c2 ((1 + h1 + h2 - h3) w^2 + 2 (h1 h2 + h3) w + h1 h3 + h2 h3 - h1 h2 + h1 h2 h3)),
     s -> - c2 (-h1 h2 + h1 h3 + h2 h3 + h1 h2 h3 + (2 h1 h2 + 2 h3) w + (1 + h1 + h2 - h3) w^2)
       k / {2 (1 + h1) (1 + h2) (h1 - h3) (h2 - h3) (-1 + w) }}) /.
  {k → √ (1 + h1) (1 + h2) (h1 - h3) (h2 - h3)}]
```

Out[24]= {{0}}

As for k1''' :=
k1 + d t + 2 e t^2 + (b + 2 k3 t + 3 d t^2 + 4 e t^3) s + 2 (b t + k3 t^2 + d t^3 + e t^4) s^2

We prove k1''' = $\frac{c2\ (h1 + h2 + 2\ h1\ h2)\ (-1 + w)}{2\ (1 + h1)\ (1 + h2)}$

Consider the difference

In[25]:=
```
dk1 := k1 + d t + 2 e t^2 + (b + 2 k3 t + 3 d t^2 + 4 e t^3) s +
  2 (b t + k3 t^2 + d t^3 + e t^4) s^2 - (c2 (h1 + h2 + 2 h1 h2) (-1 + w) / (2 (1 + h1) (1 + h2)))
```

In[26]:=
```
Simplify[((dk1 /. {{b → -1, k1 → c0, k2 → c2, k3 → (e0 - e2 + e4) / (c0 - c2)^2,
        d → ((c0 + c2) e2 - 2 c2 e0 - 2 c0 e4) / (c0 - c2)^2,
        e → (c2^2 e0 - c0 c2 e2 + c0^2 e4) / (c0 - c2)^2}}) /.
    {c0 → w c2, t → ((w + h3) (1 + h1) (1 + h2) + (w - 1) k) /
       (c2 ((1 + h1 + h2 - h3) w^2 + 2 (h1 h2 + h3) w + h1 h3 + h2 h3 - h1 h2 + h1 h2 h3)),
     s -> - c2 (-h1 h2 + h1 h3 + h2 h3 + h1 h2 h3 + (2 h1 h2 + 2 h3) w + (1 + h1 + h2 - h3) w^2)
       k / {2 (1 + h1) (1 + h2) (h1 - h3) (h2 - h3) (-1 + w) }}) /.
  {k → √ (1 + h1) (1 + h2) (h1 - h3) (h2 - h3)}]
```

Out[26]= {{0}}

As for a''' := e + (d + 4 e t) s + (k3 + 3 d t + 6 e t^2) s^2 +
(b + 2 k3 t + 3 d t^2 + 4 e t^3) s^3 + (b t + k3 t^2 + d t^3 + e t^4) s^4

We prove a''' = -c2^3 (u + v k)
$\left(w + \left(\frac{h1\ h2 + h3 + k}{1 + h1 + h2 - h3}\right)\right)$^4 / $\left(16\ (1 + h1)^2\ (1 + h2)^2\ (h1 - h3)\ (h2 - h3)\ (1 + h3)\ (-1 + w)\right)$

Consider the difference

In[27]:=
```
da := e + (d + 4 e t) s + (k3 + 3 d t + 6 e t^2) s^2 + (b + 2 k3 t + 3 d t^2 + 4 e t^3) s^3 +
  (b t + k3 t^2 + d t^3 + e t^4) s^4 - (-c2^3 (u + v k) (w + ((h1 h2 + h3 + k) / (1 + h1 + h2 - h3)))^4 /
     (16 (1 + h1)^2 (1 + h2)^2 (h1 - h3) (h2 - h3) (1 + h3) (-1 + w)))
```

In[28]:=
```
Simplify[((da /. {{b → -1, k1 → c0, k2 → c2, k3 → (e0 - e2 + e4) / (c0 - c2)^2,
      d → ((c0 + c2) e2 - 2 c2 e0 - 2 c0 e4) / (c0 - c2)^2,
      e → (c2^2 e0 - c0 c2 e2 + c0^2 e4) / (c0 - c2)^2}}) /.
   {c0 → w c2, t → ((w + h3) (1 + h1) (1 + h2) + (w - 1) k) /
      (c2 ((1 + h1 + h2 - h3) w^2 + 2 (h1 h2 + h3) w + h1 h3 + h2 h3 - h1 h2 + h1 h2 h3)),
    s -> - c2 (-h1 h2 + h1 h3 + h2 h3 + h1 h2 h3 + (2 h1 h2 + 2 h3) w + (1 + h1 + h2 - h3) w^2)
      k / {2 (1 + h1) (1 + h2) (h1 - h3) (h2 - h3) (-1 + w) }}) /.
  {k → √ (1 + h1) (1 + h2) (h1 - h3) (h2 - h3)}]
```

Out[28]= $\{\{0\}\}$

Therefore A := -k1''' / (2 e''') is (under u → U, v → V)

In[29]:=
```
Simplify[(-(c2 (h1 + h2 + 2 h1 h2) (-1 + w))/(2 (1 + h1) (1 + h2))) (-c2 ((U + V k) (w + ((h1 h2 + h3 + k)/(1 + h1 + h2 - h3)))^4) /
      ((w - 1)^3 (h1 - h2)^2 (h1 - h3) (h2 - h3) (1 + h3))) / 2]
```

Out[29]= $$\left(c2^2\,(h1+h2+2\,h1\,h2)\,(U+k\,V)\left(\frac{h1\,h2+h3+k}{1+h1+h2-h3}+w\right)^4\right)\Big/\left(4\,(1+h1)\,(h1-h2)^2\,(1+h2)\,(h1-h3)\,(h2-h3)\,(1+h3)\,(-1+w)^2\right)$$

In[30]:=
```
A0 := (c2^2 (h1 + h2 + 2 h1 h2) (U + k V) ((h1 h2 + h3 + k)/(1 + h1 + h2 - h3) + w)^4) /
   (4 (1 + h1) (h1 - h2)^2 (1 + h2) (h1 - h3) (h2 - h3) (1 + h3) (-1 + w)^2)
```

B := -k2''' / (2 e''') is

In[31]:=
```
Simplify[-(-(c2 (2 + h1 + h2) (-1 + w))/(2 (1 + h1) (1 + h2))) (-c2 ((U + V k) (w + ((h1 h2 + h3 + k)/(1 + h1 + h2 - h3)))^4) /
      ((w - 1)^3 (h1 - h2)^2 (h1 - h3) (h2 - h3) (1 + h3))) / 2]
```

Out[31]= $$-\left(c2^2\,(2+h1+h2)\,(U+k\,V)\left(\frac{h1\,h2+h3+k}{1+h1+h2-h3}+w\right)^4\right)\Big/\left(4\,(1+h1)\,(h1-h2)^2\,(1+h2)\,(h1-h3)\,(h2-h3)\,(1+h3)\,(-1+w)^2\right)$$

In[32]:=
```
B0 := -(c2^2 (2 + h1 + h2) (U + k V) ((h1 h2 + h3 + k)/(1 + h1 + h2 - h3) + w)^4) /
   (4 (1 + h1) (h1 - h2)^2 (1 + h2) (h1 - h3) (h2 - h3) (1 + h3) (-1 + w)^2)
```

C := -k3''' / (2 e''') is

In[33]:=
```
Simplify[
  -((c2 (h1 + h2 + 2 h1 h2 - 2 h3 - h1 h3 - h2 h3) (-1 + w)) / (2 (1 + h1) (1 + h2) (1 + h3)))
   (-c2 ((U + V k) (w + ((h1 h2 + h3 + k)/(1 + h1 + h2 - h3)))^4) /
      ((w - 1)^3 (h1 - h2)^2 (h1 - h3) (h2 - h3) (1 + h3))) / 2]
```

Out[33]= $$\left(c2^2\,(h1+h2+2\,h1\,h2-2\,h3-h1\,h3-h2\,h3)\,(U+k\,V)\left(\frac{h1\,h2+h3+k}{1+h1+h2-h3}+w\right)^4\right)\Big/\left(4\,(1+h1)\,(h1-h2)^2\,(1+h2)\,(h1-h3)\,(h2-h3)\,(1+h3)^2\,(-1+w)^2\right)$$

In[34]:= $C0 := \left(c2^2\ (h1 + h2 + 2\ h1\ h2 - 2\ h3 - h1\ h3 - h2\ h3)\ (U + k\ V)\ \left(\frac{h1\ h2 + h3 + k}{1 + h1 + h2 - h3} + w\right)^4\right) \Big/ \left(4\ (1 + h1)\ (h1 - h2)^2\ (1 + h2)\ (h1 - h3)\ (h2 - h3)\ (1 + h3)^2\ (-1 + w)^2\right)$

D^2 := a''' / e''' is

In[35]:= $\text{Simplify}\left[\left(-c2\text{^}3\ (U + V\ k)\ \left(w + \left(\frac{h1\ h2 + h3 + k}{1 + h1 + h2 - h3}\right)\right)\text{^}4 \Big/ \left(16\ (1 + h1)^2\ (1 + h2)^2\ (h1 - h3)\ (h2 - h3)\ (1 + h3)\ (-1 + w)\right)\right)\ \left(-c2\ \left((U + V\ k)\ \left(w + \left(\frac{h1\ h2 + h3 + k}{1 + h1 + h2 - h3}\right)\right)\right)\text{^}4\right) \Big/ \left((w - 1)\text{^}3\ (h1 - h2)^2\ (h1 - h3)\ (h2 - h3)\ (1 + h3)\right)\right]$

Out[35]= $\left(c2^4\ (U + k\ V)^2\ \left(\frac{h1\ h2 + h3 + k}{1 + h1 + h2 - h3} + w\right)^8\right) \Big/ \left(16\ (1 + h1)^2\ (h1 - h2)^2\ (1 + h2)^2\ (h1 - h3)^2\ (h2 - h3)^2\ (1 + h3)^2\ (-1 + w)^4\right)$

Therefore D is

$\left(c2^2\ (U + k\ V)\ \left(\frac{h1\ h2 + h3 + k}{1 + h1 + h2 - h3} + w\right)^4\right) \Big/ \left(4\ (1 + h1)\ (h1 - h2)\ (1 + h2)\ (h1 - h3)\ (h2 - h3)\ (1 + h3)\ (-1 + w)^2\right)$

In[36]:= $D0 := \left(c2^2\ (U + k\ V)\ \left(\frac{h1\ h2 + h3 + k}{1 + h1 + h2 - h3} + w\right)^4\right) \Big/ \left(4\ (1 + h1)\ (h1 - h2)\ (1 + h2)\ (h1 - h3)\ (h2 - h3)\ (1 + h3)\ (-1 + w)^2\right)$

In[37]:= **Simplify[B0 / A0]**

Out[37]= $-\frac{2 + h1 + h2}{h1 + h2 + 2\ h1\ h2}$

In[38]:= **Simplify[C0 / A0]**

Out[38]= $\frac{h1 + h2 + 2\ h1\ h2 - 2\ h3 - h1\ h3 - h2\ h3}{(h1 + h2 + 2\ h1\ h2)\ (1 + h3)}$

In[39]:= **Simplify[D0 / A0]**

Out[39]= $\frac{h1 - h2}{h1 + h2 + 2\ h1\ h2}$

Since A0 > 0, we have B0 < 0, D0 > 0. Further

In[40]:= **Simplify[(A0 - C0) / A0]**

Out[40]= $\frac{2\ (1 + h1)\ (1 + h2)\ h3}{(h1 + h2 + 2\ h1\ h2)\ (1 + h3)}$

In[41]:= **Simplify[(C0 - D0) / A0]**

Out[41]= $\frac{2\ (1 + h1)\ (h2 - h3)}{(h1 + h2 + 2\ h1\ h2)\ (1 + h3)}$

In[42]:= **Simplify[(-B0 - D0) / A0]**

Out[42]= $\frac{2\,(1 + h2)}{h1 + h2 + 2\,h1\,h2}$

Hence we have A0 > C0 > D0 > 0, -B0 > D0.

Further

In[43]:= **Simplify[(A0 - C0) / (C0 - B0)]**

Out[43]= h3

In[44]:= **Simplify[(A0 - D0) / (D0 - B0)]**

Out[44]= h2

In[45]:= **Simplify[(A0 + D0) / (-B0 - D0)]**

Out[45]= h1