Concrete expressions of circles on Blum cyclides

Lemma

A parameter expression of a circle

```
x : x0 + (2 r (U1 + V1 t)) / ((U1 + V1 t)^2 + (U2 + V2 t)^2 + (U3 + V3 t)^2)

y : y0 + (2 r (U2 + V2 t)) / ((U1 + V1 t)^2 + (U2 + V2 t)^2 + (U3 + V3 t)^2)

z : z0 + (2 r (U3 + V3 t)) / ((U1 + V1 t)^2 + (U2 + V2 t)^2 + (U3 + V3 t)^2)
```

contained in y - y0 = a (x - x0) + b (z - z0)
with center (x0 + W1, y0 + W2, z0 + W3) and
radius $\sqrt{W1^2 + W2^2 + W3^2}$. We find a, b, W1, W2, W3.

```
In[1]:= Simplify[Cancel[((U1 + V1 t)^2 + (U2 + V2 t)^2 + (U3 + V3 t)^2)
          (((2 r (U1 + V1 t)) / ((U1 + V1 t)^2 + (U2 + V2 t)^2 + (U3 + V3 t)^2) - W1)^2 +
            ((2 r (U2 + V2 t)) / ((U1 + V1 t)^2 + (U2 + V2 t)^2 + (U3 + V3 t)^2) - W2)^2 +
            ((2 r (U3 + V3 t)) / ((U1 + V1 t)^2 + (U2 + V2 t)^2 + (U3 + V3 t)^2) - W3)^2 -
           W1^2 - W2^2 - W3^2)]]

Out[1]= -4 r (-r + U1 W1 + t V1 W1 + U2 W2 + t V2 W2 + U3 W3 + t V3 W3)

In[2]:= Solve[{V2 - a V1 - b V3 == 0, U2 - a U1 - b U3 == 0}, {a, b}]

Out[2]= {{a -> -(-U3 V2 + U2 V3)/(U3 V1 - U1 V3), b -> -(U2 V1 - U1 V2)/(-U3 V1 + U1 V3)}}

In[3]:= Solve[{V1 W1 + V2 W2 + V3 W3 == 0, U1 W1 + U2 W2 + U3 W3 - r == 0,
          W2 - (-(-U3 V2 + U2 V3)/(U3 V1 - U1 V3)) W1 - (-(U2 V1 - U1 V2)/(-U3 V1 + U1 V3)) W3 == 0}, {W1, W2, W3}]

Out[3]= {{W1 -> (r (-U2 V1 V2 + U1 V2^2 - U3 V1 V3 + U1 V3^2)) / (U2^2 V1^2 + U3^2 V1^2 - 2 U1 U2 V1 V2 +
              U1^2 V2^2 + U3^2 V2^2 - 2 U1 U3 V1 V3 - 2 U2 U3 V2 V3 + U1^2 V3^2 + U2^2 V3^2),
          W2 -> (r (U2 V1^2 - U1 V1 V2 - U3 V2 V3 + U2 V3^2)) / (U2^2 V1^2 + U3^2 V1^2 - 2 U1 U2 V1 V2 +
              U1^2 V2^2 + U3^2 V2^2 - 2 U1 U3 V1 V3 - 2 U2 U3 V2 V3 + U1^2 V3^2 + U2^2 V3^2),
          W3 -> (r (U3 V1^2 + U3 V2^2 - U1 V1 V3 - U2 V2 V3)) / (U2^2 V1^2 + U3^2 V1^2 - 2 U1 U2 V1 V2 +
              U1^2 V2^2 + U3^2 V2^2 - 2 U1 U3 V1 V3 - 2 U2 U3 V2 V3 + U1^2 V3^2 + U2^2 V3^2)}}
```

Therefore W1^2 + W2^2 + W3^2 is

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In[4]:= Simplify[
          (-r (U2 V1 V2 - U1 V2^2 + U3 V1 V3 - U1 V3^2) / (U2^2 V1^2 + U3^2 V1^2 - 2 U1 U2 V1 V2 + U1^2 V2^2 +
                U3^2 V2^2 - 2 U1 U3 V1 V3 - 2 U2 U3 V2 V3 + U1^2 V3^2 + U2^2 V3^2))^2 +
           (-r (-U2 V1^2 + U1 V1 V2 + U3 V2 V3 - U2 V3^2) / (U2^2 V1^2 + U3^2 V1^2 - 2 U1 U2 V1 V2 +
                U1^2 V2^2 + U3^2 V2^2 - 2 U1 U3 V1 V3 - 2 U2 U3 V2 V3 + U1^2 V3^2 + U2^2 V3^2))^2 +
           (-r (-U3 V1^2 - U3 V2^2 + U1 V1 V3 + U2 V2 V3) / (U2^2 V1^2 + U3^2 V1^2 - 2 U1 U2 V1 V2 +
                U1^2 V2^2 + U3^2 V2^2 - 2 U1 U3 V1 V3 - 2 U2 U3 V2 V3 + U1^2 V3^2 + U2^2 V3^2))^2]

Out[4]= (r^2 (V1^2 + V2^2 + V3^2)) /
         (U3^2 (V1^2 + V2^2) - 2 U1 U3 V1 V3 - 2 U2 V2 (U1 V1 + U3 V3) + U2^2 (V1^2 + V3^2) + U1^2 (V2^2 + V3^2))
```

Proposition (circles on a Blum surface)

Let (x0, y0, z0) be any point on a Blum cyclide {(x^2 + y^2 + z^2)^2 − 2 a x^2 − 2 b y^2 − 2 c z^2 + d^2 = 0}. Here we suppose a > c > d > 0, −b > d.
We consider the inversion at (x0, y0, z0) as follows (x1, y1, z1 are new variables) :

In[5]:=
```
Simplify[Cancel[
  (x1^2 + y1^2 + z1^2) ^2 ((x^2 + y^2 + z^2) ^2 - 2 a x^2 - 2 b y^2 - 2 c z^2 + d^2) -
   16 ((A x0 x1 + B y0 y1 + C0 z0 z1) (x1^2 + y1^2 + z1^2) +
      A x1^2 + B y1^2 + C0 z1^2 + (x0 x1 + y0 y1 + z0 z1 + 1) ^2) /.
  {x → x0 + 2 x1 / (x1^2 + y1^2 + z1^2), y → y0 + 2 y1 / (x1^2 + y1^2 + z1^2),
   z → z0 + 2 z1 / (x1^2 + y1^2 + z1^2), A → (x0^2 + y0^2 + z0^2 - a) / 2,
   B → (x0^2 + y0^2 + z0^2 - b) / 2, C0 → (x0^2 + y0^2 + z0^2 - c) / 2,
   d → √(2 a x0^2 + 2 b y0^2 + 2 c z0^2 - (x0^2 + y0^2 + z0^2) ^2)}]]
```

Out[5]= 0

Therefore Blum equation
(x^2 + y^2 + z^2)^2 − 2 a x^2 − 2 b y^2 − 2 c z^2 + d^2 = 0
reduces to
(A x0 x1 + B y0 y1 + C0 z0 z1) (x1^2 + y1^2 + z1^2) + A x1^2 + B y1^2 + C0 z1^2 + (x0 x1 + y0 y1 + z0 z1 + 1)^2 = 0 under the inversion at (x0, y0, z0).

We impose an aditional condition
A x0 x1 + B y0 y1 + C0 z0 z1 + k = 0 with a parameter k, and we eliminate z1 :

In[6]:=
```
Collect[C0^2 z0^2
   (-k (x1^2 + y1^2 + z1^2) + A x1^2 + B y1^2 + C0 z1^2 + (x0 x1 + y0 y1 + z0 z1 + 1) ^2) /.
  {z1 → -(A x0 x1 + B y0 y1 + k) / (C0 z0)}, {x1, y1}, Simplify]
```

Out[6]=
$$\begin{aligned}&C0^2\, z0^2 + C0\, k\,(k - 2\, z0^2) + k^2\,(-k + z0^2) + 2\,(C0 - k)\, y0\, y1\,(C0\, z0^2 + B\,(k - z0^2)) +\\&x1^2\,(A\, C0\,(C0 - 2\, x0^2)\, z0^2 + C0^2\,(-k + x0^2)\, z0^2 + A^2\, x0^2\,(C0 - k + z0^2)) +\\&y1^2\,(B\, C0\,(C0 - 2\, y0^2)\, z0^2 + C0^2\,(-k + y0^2)\, z0^2 + B^2\, y0^2\,(C0 - k + z0^2)) +\\&x1\,(2\,(C0 - k)\, x0\,(C0\, z0^2 + A\,(k - z0^2)) +\\&\quad 2\, x0\, y0\, y1\,(C0\,(-B + C0)\, z0^2 + A\,(-C0\, z0^2 + B\,(C0 - k + z0^2))))\end{aligned}$$

The coefficient of x1^2 : A2

In[7]:=
```
Simplify[A C0 (C0 - 2 x0²) z0² + C0² (-k + x0²) z0² + A² x0² (C0 - k + z0²) -
  (C0^2 z0^2 (A - k) + A^2 x0^2 (C0 - k) + z0^2 x0^2 (C0 - A) ^2)]
```

Out[7]= 0

A2 : C0^2 z0^2 (A − k) + A^2 x0^2 (C0 − k) + z0^2 x0^2 (C0 − A)^2

The coefficient of x1 y1 : B2

In[8]:=
```
Simplify[2 x0 y0 (C0 (-B + C0) z0² + A (-C0 z0² + B (C0 - k + z0²))) -
  (2 A B (C0 - k) x0 y0 + 2 (C0 - A) (C0 - B) x0 y0 z0^2)]
```

Out[8]= 0

B2 : 2 A B (C0 − k) x0 y0 + 2 (C0 − A) (C0 − B) x0 y0 z0^2

The coefficient of y1^2 : C2

In[9]:= Simplify[B C0 (C0 - 2 y0²) z0² + C0² (-k + y0²) z0² + B² y0² (C0 - k + z0²) -
(C0^2 z0^2 (B - k) + B^2 y0^2 (C0 - k) + z0^2 y0^2 (C0 - B)^2)]

Out[9]= 0

C2 : C0^2 z0^2 (B - k) + B^2 y0^2 (C0 - k) + z0^2 y0^2 (C0 - B)^2

The coefficient of x1 : D2

In[10]:= Simplify[
2 (C0 - k) x0 (C0 z0² + A (k - z0²)) - (2 (C0 - k) k A x0 + 2 (C0 - A) (C0 - k) x0 z0^2)]

Out[10]= 0

D2 : 2 (C0 - k) k A x0 + 2 (C0 - A) (C0 - k) x0 z0^2

The coefficient of y1 : E2

In[11]:= Simplify[
2 (C0 - k) y0 (C0 z0² + B (k - z0²)) - (2 (C0 - k) k B y0 + 2 (C0 - B) (C0 - k) y0 z0^2)]

Out[11]= 0

E2 : 2 (C0 - k) k B y0 + 2 (C0 - B) (C0 - k) y0 z0^2

The constant term : F2

In[12]:= Simplify[C0² z0² + C0 k (k - 2 z0²) + k² (-k + z0²) - ((C0 - k) k^2 + (C0 - k)^2 z0^2)]

Out[12]= 0

F2 : (C0 - k) k^2 + (C0 - k)^2 z0^2

Therefore for the splitting of A2 x1^2 + B2 x1 y1 + C2 y1^2 + D2 x1 + E2 y1 + F2 into two first order polynomials, we consider the discriminant in x1 of A2 x1^2 + B2 x1 y1 + C2 y1^2 + D2 x1 + E2 y1 + F2 = A2 x1^2 + (B2 y1 + D2) x1 + C2 y1^2 + E2 y1 + F2

In[13]:= Collect[(B2 y1 + D2)^2 - 4 A2 (C2 y1^2 + E2 y1 + F2), y1]

Out[13]= $D2^2 - 4\,A2\,F2 + (2\,B2\,D2 - 4\,A2\,E2)\,y1 + \left(B2^2 - 4\,A2\,C2\right) y1^2$

Hence for the splitting, k must satisfies the vanishing of the discriminant of the above quadratic polynomial in y1, and the positivity of $B2^2 - 4\,A2\,C2$.

As for the vanishing of the discriminant in y1 :

In[14]:= Simplify[(B2 D2 - 2 A2 E2)^2 - (B2^2 - 4 A2 C2) (D2^2 - 4 A2 F2)]

Out[14]= $4\,A2\left(-B2\,D2\,E2 + A2\,E2^2 + B2^2\,F2 + C2\left(D2^2 - 4\,A2\,F2\right)\right)$

Hence we impose $-B2\,D2\,E2 + A2\,E2^2 + B2^2\,F2 + C2\left(D2^2 - 4\,A2\,F2\right) = 0$ on k :

In[15]:= Simplify[-B2 D2 E2 + A2 E2² + B2² F2 + C2 (D2² - 4 A2 F2) /.
{A2 -> C0^2 z0^2 (A - k) + A^2 x0^2 (C0 - k) + z0^2 x0^2 (C0 - A)^2,
B2 -> 2 A B (C0 - k) x0 y0 + 2 (C0 - A) (C0 - B) x0 y0 z0^2,
C2 -> C0^2 z0^2 (B - k) + B^2 y0^2 (C0 - k) + z0^2 y0^2 (C0 - B)^2,
D2 -> 2 (C0 - k) k A x0 + 2 (C0 - A) (C0 - k) x0 z0^2, E2 ->
2 (C0 - k) k B y0 + 2 (C0 - B) (C0 - k) y0 z0^2, F2 -> (C0 - k) k^2 + (C0 - k)^2 z0^2}]

Out[15]= $4\,C0^4\,(A - k)\,(C0 - k)\,(-B + k)\,z0^4\left(k^2 + A\,x0^2 + B\,y0^2 + C0\,z0^2 - k\left(x0^2 + y0^2 + z0^2\right)\right)$

The last factor is equal to

$$\left(k+\frac{1}{2}\left(-d-x0^2-y0^2-z0^2\right)\right)\left(k+\frac{1}{2}\left(d-x0^2-y0^2-z0^2\right)\right)$$

because

In[16]:=
```
Simplify[(x0^2 + y0^2 + z0^2)^2 - d^2 - 4 (A x0^2 + B y0^2 + C0 z0^2) /.
  {A → (x0^2 + y0^2 + z0^2 - a) / 2,
   B → (x0^2 + y0^2 + z0^2 - b) / 2, C0 → (x0^2 + y0^2 + z0^2 - c) / 2,
   d → √(2 a x0^2 + 2 b y0^2 + 2 c z0^2 - (x0^2 + y0^2 + z0^2)^2)}]
```

Out[16]= 0

Thus k should be either one of A, B, C0, (x0^2 + y0^2 + z0^2 - d) / 2, (x0^2 + y0^2 + z0^2 + d) / 2

Further concerning the positivity of B2^2 - 4 A2 C2, for each k we have the following :

Case k = A = (x0^2 + y0^2 + z0^2 - a) / 2

In[17]:=
```
Simplify[
 (B2^2 - 4 A2 C2 /. {A2 -> C0^2 z0^2 (A - k) + A^2 x0^2 (C0 - k) + z0^2 x0^2 (C0 - A)^2,
     B2 -> 2 A B (C0 - k) x0 y0 + 2 (C0 - A) (C0 - B) x0 y0 z0^2,
     C2 -> C0^2 z0^2 (B - k) + B^2 y0^2 (C0 - k) + z0^2 y0^2 (C0 - B)^2}) /. {k → A}]
```

Out[17]= $-4\,(A-B)\,(A-C0)\,C0^2\,x0^2\,z0^2\left(A^2+B\,y0^2+C0\,z0^2-A\left(y0^2+z0^2\right)\right)$

In[18]:=
```
Simplify[A^2 + B y0^2 + C0 z0^2 - A (y0^2 + z0^2) /. {A → (x0^2 + y0^2 + z0^2 - a) / 2,
   B -> (x0^2 + y0^2 + z0^2 - b) / 2, C0 -> (x0^2 + y0^2 + z0^2 - c) / 2}]
```

Out[18]= $\frac{1}{4}\left(a^2-2\,a\,x0^2+x0^4-2\,b\,y0^2+y0^4-2\,c\,z0^2+2\,y0^2\,z0^2+z0^4+2\,x0^2\left(y0^2+z0^2\right)\right)$

This is equal to ((x0^2 + y0^2 + z0^2)^2 - 2 a x0^2 - 2 b y0^2 - 2 c z0^2 + a^2) / 4 = (a^2 - d^2) / 4

Hence the last factor is (a^2 - d^2) / 4. Therefore

(a - b) (c - a) (a + d) (a - d) > 0

Case k = B = (x0^2 + y0^2 + z0^2 - b) / 2

In[19]:=
```
Simplify[
 (B2^2 - 4 A2 C2 /. {A2 -> C0^2 z0^2 (A - k) + A^2 x0^2 (C0 - k) + z0^2 x0^2 (C0 - A)^2,
     B2 -> 2 A B (C0 - k) x0 y0 + 2 (C0 - A) (C0 - B) x0 y0 z0^2,
     C2 -> C0^2 z0^2 (B - k) + B^2 y0^2 (C0 - k) + z0^2 y0^2 (C0 - B)^2}) /. {k → B}]
```

Out[19]= $4\,(A-B)\,(B-C0)\,C0^2\,y0^2\,z0^2\left(B^2+A\,x0^2+C0\,z0^2-B\left(x0^2+z0^2\right)\right)$

In[20]:=
```
Simplify[B^2 + A x0^2 + C0 z0^2 - B (x0^2 + z0^2) /. {A → (x0^2 + y0^2 + z0^2 - a) / 2,
   B -> (x0^2 + y0^2 + z0^2 - b) / 2, C0 -> (x0^2 + y0^2 + z0^2 - c) / 2}]
```

Out[20]= $\frac{1}{4}\left(b^2-2\,a\,x0^2+x0^4-2\,b\,y0^2+2\,x0^2\,y0^2+y0^4-2\,c\,z0^2+2\,x0^2\,z0^2+2\,y0^2\,z0^2+z0^4\right)$

This is equal to (b^2 - d^2) / 4. Therefore
(a - b) (b - c) (b + d) (b - d) > 0

Case k = C0 = (x0^2 + y0^2 + z0^2 - c) / 2

In[21]:=
```
Simplify[
 (B2^2 - 4 A2 C2 /. {A2 -> C0^2 z0^2 (A - k) + A^2 x0^2 (C0 - k) + z0^2 x0^2 (C0 - A) ^2,
     B2 -> 2 A B (C0 - k) x0 y0 + 2 (C0 - A) (C0 - B) x0 y0 z0^2,
     C2 -> C0^2 z0^2 (B - k) + B^2 y0^2 (C0 - k) + z0^2 y0^2 (C0 - B) ^2}) /. {k → C0}]
```

Out[21]= $4\,(A - C0)\,C0^2\,(-B + C0)\,\left(C0^2 + A\,x0^2 + B\,y0^2 - C0\,\left(x0^2 + y0^2\right)\right)\,z0^4$

In[22]:=
```
Simplify[C0² + A x0² + B y0² - C0 (x0² + y0²) /. {A → (x0^2 + y0^2 + z0^2 - a) / 2,
   B -> (x0^2 + y0^2 + z0^2 - b) / 2, C0 -> (x0^2 + y0^2 + z0^2 - c) / 2}]
```

Out[22]= $\frac{1}{4}\left(c^2 - 2\,a\,x0^2 + x0^4 - 2\,b\,y0^2 + 2\,x0^2\,y0^2 + y0^4 - 2\,c\,z0^2 + 2\,x0^2\,z0^2 + 2\,y0^2\,z0^2 + z0^4\right)$

This is equal to (c^2 - d^2) / 4. Therefore
(a - c) (b - c) (c + d) (d - c) > 0

Case k = (x0^2 + y0^2 + z0^2 - d) / 2

In[23]:=
```
Simplify[
 ((B2^2 - 4 A2 C2) - (C0^2 z0^2 (d - a) (d - b) (d - c) (x0^2 + y0^2 + z0^2 + d) ^2 / 8) /.
     {A2 -> C0^2 z0^2 (A - k) + A^2 x0^2 (C0 - k) + z0^2 x0^2 (C0 - A) ^2,
      B2 -> 2 A B (C0 - k) x0 y0 + 2 (C0 - A) (C0 - B) x0 y0 z0^2,
      C2 -> C0^2 z0^2 (B - k) + B^2 y0^2 (C0 - k) + z0^2 y0^2 (C0 - B) ^2}) /.
   {k → (x0^2 + y0^2 + z0^2 - d) / 2, A → (x0^2 + y0^2 + z0^2 - a) / 2,
    B -> (x0^2 + y0^2 + z0^2 - b) / 2, C0 -> (x0^2 + y0^2 + z0^2 - c) / 2} /.
  d → √(2 a x0^2 + 2 b y0^2 + 2 c z0^2 - (x0^2 + y0^2 + z0^2) ^2)]
```

Out[23]= 0

Therefore
(d - a) (d - b) (d - c) > 0

Case k = (x0^2 + y0^2 + z0^2 + d) / 2

In[24]:=
```
Simplify[
 ((B2^2 - 4 A2 C2) - (-C0^2 z0^2 (d + a) (d + b) (d + c) (x0^2 + y0^2 + z0^2 - d) ^2 / 8) /.
     {A2 -> C0^2 z0^2 (A - k) + A^2 x0^2 (C0 - k) + z0^2 x0^2 (C0 - A) ^2,
      B2 -> 2 A B (C0 - k) x0 y0 + 2 (C0 - A) (C0 - B) x0 y0 z0^2,
      C2 -> C0^2 z0^2 (B - k) + B^2 y0^2 (C0 - k) + z0^2 y0^2 (C0 - B) ^2}) /.
   {k → (x0^2 + y0^2 + z0^2 + d) / 2, A → (x0^2 + y0^2 + z0^2 - a) / 2,
    B -> (x0^2 + y0^2 + z0^2 - b) / 2, C0 -> (x0^2 + y0^2 + z0^2 - c) / 2} /.
  d → √(2 a x0^2 + 2 b y0^2 + 2 c z0^2 - (x0^2 + y0^2 + z0^2) ^2)]
```

Out[24]= 0

Therefore
- (d + a) (d + b) (d + c) > 0

Hence, only k = (x0^2 + y0^2 + z0^2 - c) / 2, (x0^2 + y0^2 + z0^2 - d) / 2,
(x0^2 + y0^2 + z0^2 + d) / 2 give the positive signatures.

Case 1. k = C0 = (x0^2 + y0^2 + z0^2 - c) / 2

Solve the system of equations
with respect to y1, z1 :
-k (x1^2 + y1^2 + z1^2) + A x1^2 + B y1^2 + C0 z1^2 + (x0 x1 + y0 y1 + z0 z1 + 1) ^2 == 0,
A x0 x1 + B y0 y1 + C0 z0 z1 + k == 0.

Setting v1 = $\sqrt{(a - c)\,(c - b)\,\left(c^2 - d^2\right)}$ **, we show that the following is the solution :**

```
y1 -> (2 (a - c) (-b + c) x0 y0 - v1 (-c + x0^2 + y0^2 + z0^2) )
   x1 / ((c - b) (c^2 - d^2 + 2 (a - c) x0^2)), z1 -> ((c - b) (d^2 - c^2 + 2 (c - a) x0^2) +
     ((b - c) x0 (( a - c) (x0^2 + y0^2 - z0^2) + a c - d^2) + v1 y0 (x0^2 + y0^2 + z0^2 - b))
      x1 ) / ((c - b) z0 (c^2 - d^2 + 2 (a - c) x0^2))
```

```
In[25]:= Simplify[ (A x0 x1 + B y0 y1 + C0 z0 z1 + k /.
           {A → (x0^2 + y0^2 + z0^2 - a) / 2, B -> (x0^2 + y0^2 + z0^2 - b) / 2,
            C0 -> (x0^2 + y0^2 + z0^2 - c) / 2, k -> (x0^2 + y0^2 + z0^2 - c) / 2}) /.
         {y1 -> (2 (a - c) (-b + c) x0 y0 - v1 (-c + x0^2 + y0^2 + z0^2) )
            x1 / ((c - b) (c^2 - d^2 + 2 (a - c) x0^2)), z1 -> ((c - b) (d^2 - c^2 + 2 (c - a) x0^2) +
              ((b - c) x0 (( a - c) (x0^2 + y0^2 - z0^2) + a c - d^2) +
                 v1 y0 (x0^2 + y0^2 + z0^2 - b)) x1 ) / ((c - b) z0 (c^2 - d^2 + 2 (a - c) x0^2))} /.
        {d → √(2 a x0^2 + 2 b y0^2 + 2 c z0^2 - (x0^2 + y0^2 + z0^2) ^2)}]

Out[25]= 0
```

```
In[26]:= Simplify[
          (-k (x1^2 + y1^2 + z1^2) + A x1^2 + B y1^2 + C0 z1^2 + (x0 x1 + y0 y1 + z0 z1 + 1) ^2 /.
             {A → (x0^2 + y0^2 + z0^2 - a) / 2, B -> (x0^2 + y0^2 + z0^2 - b) / 2,
              C0 -> (x0^2 + y0^2 + z0^2 - c) / 2, k -> (x0^2 + y0^2 + z0^2 - c) / 2}) /.
           {y1 -> (2 (a - c) (-b + c) x0 y0 - v1 (-c + x0^2 + y0^2 + z0^2) )
              x1 / ((c - b) (c^2 - d^2 + 2 (a - c) x0^2)),
            z1 -> ((c - b) (d^2 - c^2 + 2 (c - a) x0^2) +   ((b - c) x0 (( a - c)
                       (x0^2 + y0^2 - z0^2) + a c - d^2) + v1 y0 (x0^2 + y0^2 + z0^2 - b)) x1 ) /
              ((c - b) z0 (c^2 - d^2 + 2 (a - c) x0^2))} /. {v1 -> √((a - c) (c - b) (c^2 - d^2))} /.
          {d → √(2 a x0^2 + 2 b y0^2 + 2 c z0^2 - (x0^2 + y0^2 + z0^2) ^2)}]

Out[26]= 0
```

The original coordinate (x, y, z) of the point (x1, y1, z1) is

x :

```
In[27]:= Simplify[x0 + 2 x1 / (x1^2 + y1^2 + z1^2) /.
         {y1 → (2 (a - c) (-b + c) x0 y0 - v1 (-c + x0^2 + y0^2 + z0^2) )
            x1 / ((c - b) (c^2 - d^2 + 2 (a - c) x0^2)), z1 → ((c - b) (d^2 - c^2 + 2 (c - a) x0^2) +
              ((b - c) x0 (( a - c) (x0^2 + y0^2 - z0^2) + a c - d^2) +
                 v1 y0 (x0^2 + y0^2 + z0^2 - b)) x1 ) / ((c - b) z0 (c^2 - d^2 + 2 (a - c) x0^2))}]

Out[27]= x0 + (2 x1) / (x1^2 + (x1^2 (-2 (a - c) (-b + c) x0 y0 + v1 (-c + x0^2 + y0^2 + z0^2))^2) /
            ((b - c)^2 (c^2 - d^2 + 2 (a - c) x0^2)^2) +
           ((-b + c) (-c^2 + d^2 + 2 (-a + c) x0^2) + x1 (v1 y0 (-b + x0^2 + y0^2 + z0^2) +
                 (b - c) x0 (a c - d^2 + (a - c) (x0^2 + y0^2 - z0^2))))^2 /
            ((b - c)^2 (c^2 - d^2 + 2 (a - c) x0^2)^2 z0^2))
```

In[28]:=
```
Simplify[
  (2 x1) / (x1^2 + (x1^2 (-2 (a - c) (-b + c) x0 y0 + v1 (-c + x0^2 + y0^2 + z0^2))^2) / ((b - c)^2
        (c^2 - d^2 + 2 (a - c) x0^2)^2) + ((-b + c) (-c^2 + d^2 + 2 (-a + c) x0^2) + x1 (v1 y0 (-b +
              x0^2 + y0^2 + z0^2) + (b - c) x0 (a c - d^2 + (a - c) (x0^2 + y0^2 - z0^2))))^2 /
       ((b - c)^2 (c^2 - d^2 + 2 (a - c) x0^2)^2 z0^2)) - 2 B1^2 z0^2 x1 /
     (B1^2 z0^2 (x1)^2 + B2^2 z0^2 (x1)^2 + (B1 + B3 x1)^2) /.
   {B1 -> (b - c) (c^2 - d^2 + 2 (a - c) x0^2),
    B2 -> 2 (a - c) (-b + c) x0 y0 - v1 (-c + x0^2 + y0^2 + z0^2),
    B3 -> v1 y0 (-b + x0^2 + y0^2 + z0^2) + (b - c) x0 (a c - d^2 + (a - c) (x0^2 + y0^2 - z0^2))}]
```

Out[28]= 0

y :

In[29]:=
```
Simplify[y0 + 2 y1 / (x1^2 + y1^2 + z1^2) /.
  {y1 → (2 (a - c) (-b + c) x0 y0 - v1 (-c + x0^2 + y0^2 + z0^2) )
      x1 / ((c - b) (c^2 - d^2 + 2 (a - c) x0^2)), z1 → ((c - b) (d^2 - c^2 + 2 (c - a) x0^2) +
        ((b - c) x0 ((a - c) (x0^2 + y0^2 - z0^2) + a c - d^2) +
           v1 y0 (x0^2 + y0^2 + z0^2 - b)) x1 ) / ((c - b) z0 (c^2 - d^2 + 2 (a - c) x0^2))}]
```

Out[29]=
```
y0 + (2 x1 (2 (a - c) (-b + c) x0 y0 - v1 (-c + x0^2 + y0^2 + z0^2))) /
  ((-b + c) (c^2 - d^2 + 2 (a - c) x0^2)
     (x1^2 + (x1^2 (-2 (a - c) (-b + c) x0 y0 + v1 (-c + x0^2 + y0^2 + z0^2))^2) /
         ((b - c)^2 (c^2 - d^2 + 2 (a - c) x0^2)^2) +
        ((-b + c) (-c^2 + d^2 + 2 (-a + c) x0^2) + x1 (v1 y0 (-b + x0^2 + y0^2 + z0^2) +
               (b - c) x0 (a c - d^2 + (a - c) (x0^2 + y0^2 - z0^2))))^2 /
         ((b - c)^2 (c^2 - d^2 + 2 (a - c) x0^2)^2 z0^2)))
```

In[30]:=
```
Simplify[(2 x1 (2 (a - c) (-b + c) x0 y0 - v1 (-c + x0^2 + y0^2 + z0^2))) /
    ((-b + c) (c^2 - d^2 + 2 (a - c) x0^2)
      (x1^2 + (x1^2 (-2 (a - c) (-b + c) x0 y0 + v1 (-c + x0^2 + y0^2 + z0^2))^2) /
         ((b - c)^2 (c^2 - d^2 + 2 (a - c) x0^2)^2) +
        ((-b + c) (-c^2 + d^2 + 2 (-a + c) x0^2) + x1 (v1 y0 (-b + x0^2 + y0^2 + z0^2) +
               (b - c) x0 (a c - d^2 + (a - c) (x0^2 + y0^2 - z0^2))))^2 /
         ((b - c)^2 (c^2 - d^2 + 2 (a - c) x0^2)^2 z0^2))) - (-2 B1 B2 z0^2
     x1 / (B1^2 z0^2 (x1)^2 + B2^2 z0^2 (x1)^2 + (B1 + B3 x1)^2)) /.
  {B1 -> (b - c) (c^2 - d^2 + 2 (a - c) x0^2),
   B2 ->
    2 (a - c) (-b + c) x0 y0 - v1 (-c + x0^2 + y0^2 + z0^2),
   B3 -> v1 y0 (-b + x0^2 + y0^2 + z0^2) + (b - c) x0 (a c - d^2 + (a - c) (x0^2 + y0^2 - z0^2))}]
```

Out[30]= 0

z :

In[31]:=
```
Simplify[z0 + 2 z1 / (x1^2 + y1^2 + z1^2) /.
  {y1 → (2 (a - c) (-b + c) x0 y0 - v1 (-c + x0^2 + y0^2 + z0^2) )
     x1 / ((c - b) (c^2 - d^2 + 2 (a - c) x0^2)), z1 → ((c - b) (d^2 - c^2 + 2 (c - a) x0^2) +
       ((b - c) x0 (( a - c) (x0^2 + y0^2 - z0^2) + a c - d^2) +
          v1 y0 (x0^2 + y0^2 + z0^2 - b)) x1 ) / ((c - b) z0 (c^2 - d^2 + 2 (a - c) x0^2))}]
```

Out[31]=
```
z0 + (2 ((-b + c) (-c^2 + d^2 + 2 (-a + c) x0^2) +
         x1 (v1 y0 (-b + x0^2 + y0^2 + z0^2) + (b - c) x0 (a c - d^2 + (a - c) (x0^2 + y0^2 - z0^2))))) /
   ((-b + c) (c^2 - d^2 + 2 (a - c) x0^2) z0
     (x1^2 + (x1^2 (-2 (a - c) (-b + c) x0 y0 + v1 (-c + x0^2 + y0^2 + z0^2))^2) /
        ((b - c)^2 (c^2 - d^2 + 2 (a - c) x0^2)^2) +
       ((-b + c) (-c^2 + d^2 + 2 (-a + c) x0^2) + x1 (v1 y0 (-b + x0^2 + y0^2 + z0^2) +
              (b - c) x0 (a c - d^2 + (a - c) (x0^2 + y0^2 - z0^2))))^2 /
        ((b - c)^2 (c^2 - d^2 + 2 (a - c) x0^2)^2 z0^2)))
```

In[32]:=
```
Simplify[
 (2 ((-b + c) (-c^2 + d^2 + 2 (-a + c) x0^2) + x1 (v1 y0 (-b + x0^2 + y0^2 + z0^2) + (b - c) x0
          (a c - d^2 + (a - c) (x0^2 + y0^2 - z0^2))))) / ((-b + c) (c^2 - d^2 + 2 (a - c) x0^2)
      z0 (x1^2 + (x1^2 (-2 (a - c) (-b + c) x0 y0 + v1 (-c + x0^2 + y0^2 + z0^2))^2) /
          ((b - c)^2 (c^2 - d^2 + 2 (a - c) x0^2)^2) +
         ((-b + c) (-c^2 + d^2 + 2 (-a + c) x0^2) + x1 (v1 y0 (-b + x0^2 + y0^2 + z0^2) +
               (b - c) x0 (a c - d^2 + (a - c) (x0^2 + y0^2 - z0^2))))^2 /
          ((b - c)^2 (c^2 - d^2 + 2 (a - c) x0^2)^2 z0^2))) - (-2 B1 z0
     (B1 + B3 x1) / (B1^2 z0^2 (x1)^2 + B2^2 z0^2 (x1)^2 + (B1 + B3 x1)^2)) /.
  {B1 -> (b - c) (c^2 - d^2 + 2 (a - c) x0^2),
   B2 ->
    2 (a - c) (-b + c) x0 y0 - v1 (-c + x0^2 + y0^2 + z0^2),
   B3 -> v1 y0 (-b + x0^2 + y0^2 + z0^2) + (b - c) x0 (a c - d^2 + (a - c) (x0^2 + y0^2 - z0^2))}]
```

Out[32]= 0

Therefore

x : $x0 + 2\,B1^2\,z0^2\,x1 / \left(B1^2\,z0^2\,(x1)^2 + B2^2\,z0^2\,(x1)^2 + (B1 + B3\,x1)^2\right)$

y : $y0 - 2\,B1\,B2\,z0^2\,x1 / \left(B1^2\,z0^2\,(x1)^2 + B2^2\,z0^2\,(x1)^2 + (B1 + B3\,x1)^2\right)$

z : $z0 - 2\,B1\,z0\,(B1 + B3\,x1) / \left(B1^2\,z0^2\,(x1)^2 + B2^2\,z0^2\,(x1)^2 + (B1 + B3\,x1)^2\right)$

where

B1 -> $(b - c)\left(c^2 - d^2 + 2\,(a - c)\,x0^2\right)$

B2 -> $2\,(a - c)\,(-b + c)\,x0\,y0 - v1\left(-c + x0^2 + y0^2 + z0^2\right)$

B3 -> $v1\,y0\left(-b + x0^2 + y0^2 + z0^2\right) + (b - c)\,x0\left(a\,c - d^2 + (a - c)\left(x0^2 + y0^2 - z0^2\right)\right)$

Hence by setting
$r \to z0\,B1,\ U1 \to 0,\ U2 \to 0,\ U3 \to -B1,\ V1 \to z0\,B1,$
$V2 \to -z0\,B2,\ V3 \to -B3,$ we can apply Lemma.

Thus t is

In[33]:=
```
Simplify[
 -(-U3 V2 + U2 V3)/(U3 V1 - U1 V3) /. {U1 -> 0, U2 -> 0, U3 -> -B1, V1 -> z0 B1, V2 -> -z0 B2, V3 -> -B3}]
```

Out[33]= $-\frac{B2}{B1}$

s is

In[34]:=
```
Simplify[
 -(U2 V1 - U1 V2)/(-U3 V1 + U1 V3) /. {U1 -> 0, U2 -> 0, U3 -> -B1, V1 -> z0 B1, V2 -> -z0 B2, V3 -> -B3}]
```

Out[34]= 0

Consequently

$$T = t = -\frac{B2}{B1} = -\left(2\,(a-c)\,(-b+c)\,x0\,y0 - v1\left(-c + x0^2 + y0^2 + z0^2\right)\right) \Big/ \left((b-c)\left(c^2 - d^2 + 2\,(a-c)\,x0^2\right)\right)$$

Here z0 is

In[35]:= Solve[{(x0^2 + y0^2 + z0^2)^2 - 2 a x0^2 - 2 b y0^2 - 2 c z0^2 + d^2 == 0}, z0]

Out[35]=
$$\left\{\left\{z0 \to -\sqrt{\left(c - x0^2 - y0^2 - \sqrt{\left(c^2 - d^2 + 2\,a\,x0^2 - 2\,c\,x0^2 + 2\,b\,y0^2 - 2\,c\,y0^2\right)}\right)}\right\},\right.$$
$$\left\{z0 \to \sqrt{\left(c - x0^2 - y0^2 - \sqrt{\left(c^2 - d^2 + 2\,a\,x0^2 - 2\,c\,x0^2 + 2\,b\,y0^2 - 2\,c\,y0^2\right)}\right)}\right\},$$
$$\left\{z0 \to -\sqrt{\left(c - x0^2 - y0^2 + \sqrt{\left(c^2 - d^2 + 2\,a\,x0^2 - 2\,c\,x0^2 + 2\,b\,y0^2 - 2\,c\,y0^2\right)}\right)}\right\},$$
$$\left.\left\{z0 \to \sqrt{\left(c - x0^2 - y0^2 + \sqrt{\left(c^2 - d^2 + 2\,a\,x0^2 - 2\,c\,x0^2 + 2\,b\,y0^2 - 2\,c\,y0^2\right)}\right)}\right\}\right\}$$

Hence $-c + x0^2 + y0^2 + z0^2$ is

$$\sqrt{c^2 - d^2 + 2\,(a-c)\,x0^2 + 2\,(b-c)\,y0^2}\,,\ -\sqrt{c^2 - d^2 + 2\,(a-c)\,x0^2 + 2\,(b-c)\,y0^2}$$

The center of the circle is

x :

In[36]:= Simplify[
x0 + (r (-U2 V1 V2 + U1 V2^2 - U3 V1 V3 + U1 V3^2)) / (U2^2 V1^2 + U3^2 V1^2 - 2 U1 U2 V1 V2 + U1^2 V2^2 + U3^2 V2^2 - 2 U1 U3 V1 V3 - 2 U2 U3 V2 V3 + U1^2 V3^2 + U2^2 V3^2) /.
{r → z0 B1, U1 → 0, U2 → 0, U3 → -B1, V1 → z0 B1, V2 → -z0 B2, V3 → -B3}]

Out[36]=
$$\frac{-B1\,B3 + B1^2\,x0 + B2^2\,x0}{B1^2 + B2^2}$$

y :

In[37]:= Simplify[
y0 + (r (U2 V1^2 - U1 V1 V2 - U3 V2 V3 + U2 V3^2)) / (U2^2 V1^2 + U3^2 V1^2 - 2 U1 U2 V1 V2 + U1^2 V2^2 + U3^2 V2^2 - 2 U1 U3 V1 V3 - 2 U2 U3 V2 V3 + U1^2 V3^2 + U2^2 V3^2) /.
{r → z0 B1, U1 → 0, U2 → 0, U3 → -B1, V1 → z0 B1, V2 → -z0 B2, V3 → -B3}]

Out[37]=
$$\frac{B2\,B3 + B1^2\,y0 + B2^2\,y0}{B1^2 + B2^2}$$

z :

In[38]:= Simplify[
z0 + (r (U3 V1^2 + U3 V2^2 - U1 V1 V3 - U2 V2 V3)) / (U2^2 V1^2 + U3^2 V1^2 - 2 U1 U2 V1 V2 + U1^2 V2^2 + U3^2 V2^2 - 2 U1 U3 V1 V3 - 2 U2 U3 V2 V3 + U1^2 V3^2 + U2^2 V3^2) /.
{r → z0 B1, U1 → 0, U2 → 0, U3 → -B1, V1 → z0 B1, V2 → -z0 B2, V3 → -B3}]

Out[38]= 0

(The radius)^2 :

In[39]:=
```
Simplify[(r^2 (V1^2 + V2^2 + V3^2)) / (U3^2 (V1^2 + V2^2) -
      2 U1 U3 V1 V3 - 2 U2 V2 (U1 V1 + U3 V3) + U2^2 (V1^2 + V3^2) + U1^2 (V2^2 + V3^2)) /.
   {r → z0 B1, U1 → 0, U2 → 0, U3 → -B1, V1 → z0 B1, V2 → -z0 B2, V3 → -B3}]
```

Out[39]= $\frac{B3^2 + (B1^2 + B2^2)\, z0^2}{B1^2 + B2^2}$

```
Case k = (x0^2 + y0^2 + z0^2 - d) / 2

Solve the system of equations
 with respect to y1, z1 :
 -k (x1^2 + y1^2 + z1^2) + A x1^2 + B y1^2 + C0 z1^2 + (x0 x1 + y0 y1 + z0 z1 + 1)^2 == 0,
A x0 x1 + B y0 y1 + C0 z0 z1 + k == 0.
Setting v2 = √(2 (-b + d) (c - d) (a - d))

we show that the following is the solution :

y1 -> (2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) +
      2 x0 y0 ((a b - d^2) (c - d) + (a - b) (c - d) x0^2 -
         (a - b) (c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2) x1 -
      v2 z0 (x0^2 + y0^2 + z0^2 - c) ((d + x0^2 + y0^2 + z0^2) x1 + 2 x0)) /
   (2 (y0^2 (d - c) (b^2 - d^2 + 2 x0^2 (a - b)) + z0^2 (b - d) (d^2 - c^2 + 2 (c - a) x0^2))),
z1 → (-2 z0 ((b - d) (d^2 - c d - (2 a - c - d) x0^2 + (d - c) (y0^2 + z0^2)) -
         ((a c - d^2) (b - d) + (b - d) (a - c) (x0^2 - z0^2) +
            (b c - 2 b d + d c - 2 a c + a b + a d) y0^2) x0 x1) +
      v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + (d + x0^2 + y0^2 + z0^2) x1)) /
   (2 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2))
```

In[40]:=
```
Simplify[(A x0 x1 + B y0 y1 + C0 z0 z1 + k /.
      {A → (x0^2 + y0^2 + z0^2 - a) / 2, B -> (x0^2 + y0^2 + z0^2 - b) / 2,
       C0 -> (x0^2 + y0^2 + z0^2 - c) / 2, k -> (x0^2 + y0^2 + z0^2 - d) / 2}) /.
    {y1 -> (2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) +
          2 x0 y0 ((a b - d^2) (c - d) + (a - b) (c - d) x0^2 -
             (a - b) (c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2) x1 -
          v2 z0 (x0^2 + y0^2 + z0^2 - c) ((d + x0^2 + y0^2 + z0^2) x1 + 2 x0)) /
       (2 (y0^2 (d - c) (b^2 - d^2 + 2 x0^2 (a - b)) + z0^2 (b - d) (d^2 - c^2 + 2 (c - a) x0^2))),
     z1 → (-2 z0 ((b - d) (d^2 - c d - (2 a - c - d) x0^2 + (d - c) (y0^2 + z0^2)) -
             ((a c - d^2) (b - d) + (b - d) (a - c) (x0^2 - z0^2) +
                (b c - 2 b d + d c - 2 a c + a b + a d) y0^2) x0 x1) +
          v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + (d + x0^2 + y0^2 + z0^2) x1)) /
       (2 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2))} /.
   {d → √(2 a x0^2 + 2 b y0^2 + 2 c z0^2 - (x0^2 + y0^2 + z0^2)^2)}]
```

Out[40]= 0

In[41]:=
```
Simplify[
 (-k (x1^2 + y1^2 + z1^2) + A x1^2 + B y1^2 + C0 z1^2 + (x0 x1 + y0 y1 + z0 z1 + 1) ^2 /.
      {A → (x0^2 + y0^2 + z0^2 - a) / 2, B -> (x0^2 + y0^2 + z0^2 - b) / 2,
       C0 -> (x0^2 + y0^2 + z0^2 - c) / 2, k -> (x0^2 + y0^2 + z0^2 - d) / 2}) /.
    {y1 -> (2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) +
         2 x0 y0 ((a b - d^2) (c - d ) + (a - b) ( c - d) x0^2 -
            (a - b) ( c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2) x1 -
         v2 z0 (x0^2 + y0^2 + z0^2 - c) ((d + x0^2 + y0^2 + z0^2) x1 + 2 x0)) /
       (2 (y0^2 (d - c ) (b^2 - d^2 + 2 x0^2 (a - b)) + z0^2 (b - d) (d^2 - c^2 + 2 (c - a) x0^2 ))),
     z1 → (-2 z0 ((b - d) (d^2 - c d - (2 a - c - d) x0^2 + (d - c) (y0^2 + z0^2)) -
            ((a c - d^2) (b - d) + (b - d) (a - c) (x0^2 - z0^2) +
               ( b c - 2 b d + d c - 2 a c + a b + a d) y0^2) x0 x1) +
         v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + (d + x0^2 + y0^2 + z0^2) x1)) /
       (2 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2)
            z0^2))} /. {v2 → √(2 (-b + d) (c - d) (a - d))} /.
  {d → √(2 a x0^2 + 2 b y0^2 + 2 c z0^2 - (x0^2 + y0^2 + z0^2) ^2)}]
```

Out[41]= 0

We introduce C1, C2, C3, C4, C5 as

```
C1 -> (-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2,
C2 -> -v2 x0 z0 (-c + x0^2 + y0^2 + z0^2) +
   (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)),
C3 -> -v2 z0 (-c + x0^2 + y0^2 + z0^2) (d + x0^2 + y0^2 + z0^2) +
   2 x0 y0 ((c - d) (a b - d^2) + (a - b) (c - d) x0^2 - (a - b) (c - d) y0^2 +
       ((a + b) (c + d) - 2 (a b + c d)) z0^2), C4 -> v2 x0 y0 (-b + x0^2 + y0^2 + z0^2) +
   (-b + d) z0 (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2)),
C5 -> v2 y0 (-b + x0^2 + y0^2 + z0^2) (d + x0^2 + y0^2 + z0^2) + 2 x0 z0
    ((b - d) (a c - d^2) + (b (c - 2 d) + c d + a (b - 2 c + d)) y0^2 + (a - c) (b - d) (x0^2 - z0^2))
```

Then, the original coordinate (x, y, z) of the point (x1, y1, z1) is

x :

```
In[42]:= Simplify[x0 + 2 x1 / (x1^2 + y1^2 + z1^2) /.
   {y1 -> (2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) +
        2 x0 y0 ((a b - d^2) (c - d ) + (a - b) ( c - d) x0^2 -
           (a - b) ( c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2) x1 -
        v2 z0 (x0^2 + y0^2 + z0^2 - c) ((d + x0^2 + y0^2 + z0^2) x1 + 2 x0)) /
      (2 (y0^2 (d - c ) (b^2 - d^2 + 2 x0^2 (a - b)) + z0^2 (b - d) (d^2 - c^2 + 2 (c - a) x0^2 ))),
    z1 → (-2 z0 ((b - d) (d^2 - c d - (2 a - c - d) x0^2 + (d - c) (y0^2 + z0^2)) -
           ((a c - d^2) (b - d) + (b - d) (a - c) (x0^2 - z0^2) +
              ( b c - 2 b d + d c - 2 a c + a b + a d) y0^2) x0 x1) +
        v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + (d + x0^2 + y0^2 + z0^2) x1)) /
      (2 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2))}]
```

```
Out[42]= x0 + (2 x1) /
   (x1^2 + (2 x0 x1 y0 ((c - d) (a b - d^2) + (a - b) (c - d) x0^2 - (a - b) (c - d) y0^2 + ((a + b)
                 (c + d) - 2 (a b + c d)) z0^2) +
            2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) -
            v2 z0 (-c + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)))^2 /
       (4 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2)^2) +
      (v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)) -
            2 z0 (-x0 x1 ((b - d) (a c - d^2) + (b c - 2 b d + c d + a (b - 2 c + d))
                    y0^2 + (a - c) (b - d) (x0^2 - z0^2)) +
                 (b - d) (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2))))^2 /
       (4 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2)^2))
```

In[43]:=
```
Simplify[
 (2 x1) / (x1^2 + (2 x0 x1 y0 ((c - d) (a b - d^2) + (a - b) (c - d) x0^2 - (a - b) (c - d) y0^2 +
          ((a + b) (c + d) - 2 (a b + c d)) z0^2) +
        2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) -
        v2 z0 (-c + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)))^2 /
      (4 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2)^2) +
     (v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)) -
        2 z0 (-x0 x1 ((b - d) (a c - d^2) + (b c - 2 b d + c d + a (b - 2 c + d)) y0^2 +
              (a - c) (b - d) (x0^2 - z0^2)) +
           (b - d) (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2))))^2 /
      (4 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 +
              2 (-a + c) x0^2) z0^2)^2)) -
   ((8 C1^2 x1) / (4 C1^2 (x1)^2 + (2 C2 + C3 x1)^2 + (2 C4 + C5 x1)^2)) /.
  {C1 -> (-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 +
     (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2,
   C2 -> -v2 x0 z0 (-c + x0^2 + y0^2 + z0^2) +
     (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)),
   C3 -> -v2 z0 (-c + x0^2 + y0^2 + z0^2) (d + x0^2 + y0^2 + z0^2) +
    2 x0 y0 ((c - d) (a b - d^2) + (a - b) (c - d) x0^2 -
        (a - b) (c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2),
   C4 -> v2 x0 y0 (-b + x0^2 + y0^2 + z0^2) + (-b + d) z0
      (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2)),
   C5 -> v2 y0 (-b + x0^2 + y0^2 + z0^2) (d + x0^2 + y0^2 + z0^2) +
    2 x0 z0 ((b - d) (a c - d^2) + (b (c - 2 d) + c d + a (b - 2 c + d)) y0^2 +
        (a - c) (b - d) (x0^2 - z0^2))}]
```

Out[43]= 0

Therefore

x : x0 + (8 C1^2 x1) / (4 C1^2 (x1)^2 + (2 C2 + C3 x1)^2 + (2 C4 + C5 x1)^2)

y :

```
In[44]:= Simplify[y0 + 2 y1 / (x1^2 + y1^2 + z1^2) /.
   {y1 -> (2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) +
        2 x0 y0 ((a b - d^2) (c - d ) + (a - b) ( c - d) x0^2 -
           (a - b) ( c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2) x1 -
        v2 z0 (x0^2 + y0^2 + z0^2 - c) ((d + x0^2 + y0^2 + z0^2) x1 + 2 x0)) /
      (2 (y0^2 (d - c ) (b^2 - d^2 + 2 x0^2 (a - b)) + z0^2 (b - d) (d^2 - c^2 + 2 (c - a) x0^2 ))),
    z1 → (-2 z0 ((b - d) (d^2 - c d - (2 a - c - d) x0^2 + (d - c) (y0^2 + z0^2)) -
           ((a c - d^2) (b - d) + (b - d) (a - c) (x0^2 - z0^2) +
              ( b c - 2 b d + d c - 2 a c + a b + a d) y0^2) x0 x1) +
        v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + (d + x0^2 + y0^2 + z0^2) x1)) /
      (2 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2))}]

Out[44]= y0 + (2 x0 x1 y0 ((c - d) (a b - d^2) + (a - b) (c - d) x0^2 -
           (a - b) (c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2) +
        2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) -
        v2 z0 (-c + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2))) /
     (((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2)
       (x1^2 + (2 x0 x1 y0 ((c - d) (a b - d^2) + (a - b) (c - d) x0^2 -
                 (a - b) (c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2) +
              2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) -
              v2 z0 (-c + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)))^2 /
          (4 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2)^2) +
         (v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)) -
              2 z0 (-x0 x1 ((b - d) (a c - d^2) + (b (c - 2 d) + c d + a (b - 2 c + d)) y0^2 +
                       (a - c) (b - d) (x0^2 - z0^2)) +
                    (b - d) (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2))))^2 / (4
            ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2)^2)))
```

```
In[45]:= Simplify[(2 x0 x1 y0 ((c - d) (a b - d^2) + (a - b) (c - d) x0^2 -
            (a - b) (c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2) +
          2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) -
          v2 z0 (-c + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2))) /
        (((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2)
          (x1^2 + (2 x0 x1 y0 ((c - d) (a b - d^2) + (a - b) (c - d) x0^2 -
                  (a - b) (c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2) +
                2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) -
                v2 z0 (-c + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)))^2 / (4 ((-c + d)
                   (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2)^2) +
            (v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)) -
                2 z0 (-x0 x1 ((b - d) (a c - d^2) + (b (c - 2 d) + c d + a (b - 2 c + d)) y0^2 +
                      (a - c) (b - d) (x0^2 - z0^2)) + (b - d)
                    (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2))))^2 /
              (4 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 +
                      2 (-a + c) x0^2) z0^2)^2))) -
      (4 C1 (2 C2 + C3 x1) / (4 C1^2 (x1)^2 + (2 C2 + C3 x1)^2 + (2 C4 + C5 x1)^2)) /.
     {C1 ->
       (-c + d)
         (b^2 - d^2 + 2 (a - b) x0^2) y0^2 +
        (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2,
      C2 -> -v2 x0 z0 (-c + x0^2 + y0^2 + z0^2) +
        (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)),
      C3 -> -v2 z0 (-c + x0^2 + y0^2 + z0^2) (d + x0^2 + y0^2 + z0^2) +
        2 x0 y0 ((c - d) (a b - d^2) + (a - b) (c - d) x0^2 -
           (a - b) (c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2),
      C4 -> v2 x0 y0 (-b + x0^2 + y0^2 + z0^2) + (-b + d) z0
          (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2)),
      C5 -> v2 y0 (-b + x0^2 + y0^2 + z0^2) (d + x0^2 + y0^2 + z0^2) +
        2 x0 z0 ((b - d) (a c - d^2) + (b (c - 2 d) + c d + a (b - 2 c + d)) y0^2 +
           (a - c) (b - d) (x0^2 - z0^2))}]

Out[45]= 0
```

Therefore

y : y0 + 4 C1 (2 C2 + C3 x1) / (4 C1^2 (x1)^2 + (2 C2 + C3 x1)^2 + (2 C4 + C5 x1)^2)

z :

In[46]:=
```
Simplify[z0 + 2 z1 / (x1^2 + y1^2 + z1^2) /.
  {y1 -> (2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) +
        2 x0 y0 ((a b - d^2) (c - d ) + (a - b) ( c - d) x0^2 -
           (a - b) ( c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2) x1 -
        v2 z0 (x0^2 + y0^2 + z0^2 - c) ((d + x0^2 + y0^2 + z0^2) x1 + 2 x0)) /
      (2 (y0^2 (d - c ) (b^2 - d^2 + 2 x0^2 (a - b)) + z0^2 (b - d) (d^2 - c^2 + 2 (c - a) x0^2 ))),
   z1 -> (-2 z0 ((b - d) (d^2 - c d - (2 a - c - d) x0^2 + (d - c) (y0^2 + z0^2)) -
          ((a c - d^2) (b - d) + (b - d) (a - c) (x0^2 - z0^2) +
             ( b c - 2 b d + d c - 2 a c + a b + a d) y0^2) x0 x1) +
        v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + (d + x0^2 + y0^2 + z0^2) x1)) /
      (2 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2))}]
```

Out[46]=
```
z0 + (v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)) -
      2 z0 (-x0 x1 ((b - d) (a c - d^2) +
              (b (c - 2 d) + c d + a (b - 2 c + d)) y0^2 + (a - c) (b - d) (x0^2 - z0^2)) +
          (b - d) (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2)))) /
   (((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2)
     (x1^2 + (2 x0 x1 y0 ((c - d) (a b - d^2) + (a - b) (c - d) x0^2 -
                 (a - b) (c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2) +
              2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) -
              v2 z0 (-c + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)))^2 /
         (4 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2)^2) +
        (v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)) -
            2 z0 (-x0 x1 ((b - d) (a c - d^2) + (b (c - 2 d) + c d + a (b - 2 c + d)) y0^2 +
                    (a - c) (b - d) (x0^2 - z0^2)) +
                (b - d) (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2))))^2 / (4
          ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2)^2)))
```

```
In[47]:= Simplify[
  (v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)) - 2 z0 (-x0 x1 ((b - d) (a c -
            d^2) + (b (c - 2 d) + c d + a (b - 2 c + d)) y0^2 + (a - c) (b - d) (x0^2 - z0^2)) +
          (b - d) (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2)))) /
     (((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2)
       (x1^2 + (2 x0 x1 y0 ((c - d) (a b - d^2) + (a - b) (c - d) x0^2 -
                (a - b) (c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2) +
             2 (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)) -
             v2 z0 (-c + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)))^2 / (4 ((-c + d)
                (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2)^2) +
          (v2 y0 (-b + x0^2 + y0^2 + z0^2) (2 x0 + x1 (d + x0^2 + y0^2 + z0^2)) -
              2 z0 (-x0 x1 ((b - d) (a c - d^2) + (b (c - 2 d) + c d + a (b - 2 c + d)) y0^2 +
                   (a - c) (b - d) (x0^2 - z0^2)) + (b - d)
                 (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2))))^2 /
           (4 ((-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 +
                   2 (-a + c) x0^2) z0^2)^2))) -
    (4 C1 (2 C4 + C5 x1) / (4 C1^2 (x1)^2 + (2 C2 + C3 x1)^2 + (2 C4 + C5 x1)^2)) /.
  {C1 ->
     (-c + d)
       (b^2 - d^2 + 2 (a - b) x0^2) y0^2 +
      (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2,
    C2 -> -v2 x0 z0 (-c + x0^2 + y0^2 + z0^2) +
      (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)),
    C3 -> -v2 z0 (-c + x0^2 + y0^2 + z0^2) (d + x0^2 + y0^2 + z0^2) +
      2 x0 y0 ((c - d) (a b - d^2) + (a - b) (c - d) x0^2 -
         (a - b) (c - d) y0^2 + ((a + b) (c + d) - 2 (a b + c d)) z0^2),
    C4 -> v2 x0 y0 (-b + x0^2 + y0^2 + z0^2) + (-b + d) z0
       (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2)),
    C5 -> v2 y0 (-b + x0^2 + y0^2 + z0^2) (d + x0^2 + y0^2 + z0^2) +
      2 x0 z0 ((b - d) (a c - d^2) + (b (c - 2 d) + c d + a (b - 2 c + d)) y0^2 +
         (a - c) (b - d) (x0^2 - z0^2))}]

Out[47]= 0
```

Therefore
z : z0 + 4 C1 (2 C4 + C5 x1) / (4 C1^2 (x1)^2 + (2 C2 + C3 x1)^2 + (2 C4 + C5 x1)^2)

Hence by setting
r → 2 C1, U1 → 0, U2 → 2 C2, U3 → 2 C4,
V1 → 2 C1, V2 → C3, V3 → C5, we can apply Lemma.

Thus t is

In[48]:=
$$\text{Simplify}\left[-\frac{-U3\,V2+U2\,V3}{U3\,V1-U1\,V3}\ /.\ \{U1\to 0,\ U2\to 2\,C2,\ U3\to 2\,C4,\ V1\to 2\,C1,\ V2\to C3,\ V3\to C5\}\right]$$

Out[48]=
$$\frac{C3\,C4-C2\,C5}{2\,C1\,C4}$$

In[49]:=
```
Simplify[(C3 C4 - C2 C5) - 2 C1 (2 (b - c) (a - d) x0 y0 z0 +
      v2 (-b y0^2 - c z0^2 + (x0^2 + y0^2 + z0^2) (y0^2 + z0^2))) /.
   {C1 -> (-c + d) (b^2 - d^2 + 2 (a - b) x0^2) y0^2 + (b - d) (-c^2 + d^2 + 2 (-a + c) x0^2) z0^2,
    C2 -> -v2 x0 z0 (-c + x0^2 + y0^2 + z0^2) +
      (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)),
    C3 -> -v2 z0 (-c + x0^2 + y0^2 + z0^2) (d + x0^2 + y0^2 + z0^2) +
      2 x0 y0 ((c - d) (a b - d^2) + (a - b) (c - d) x0^2 - (a - b) (c - d) y0^2 +
          ((a + b) (c + d) - 2 (a b + c d)) z0^2), C4 -> v2 x0 y0 (-b + x0^2 + y0^2 + z0^2) +
      (-b + d) z0 (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2)),
    C5 -> v2 y0 (-b + x0^2 + y0^2 + z0^2) (d + x0^2 + y0^2 + z0^2) + 2 x0 z0 ((b - d) (a c - d^2) +
          (b (c - 2 d) + c d + a (b - 2 c + d)) y0^2 + (a - c) (b - d) (x0^2 - z0^2))} /.
  d → √(2 a x0^2 + 2 b y0^2 + 2 c z0^2 - (x0^2 + y0^2 + z0^2)^2)]
```

Out[49]= 0

Therefore

$$t:\frac{C3\,C4-C2\,C5}{2\,C1\,C4}=(2\,(b-c)\,(a-d)\,x0\,y0\,z0+$$
$$v2\,(-b\,y0^{\wedge}2-c\,z0^{\wedge}2+(x0^{\wedge}2+y0^{\wedge}2+z0^{\wedge}2)\,(y0^{\wedge}2+z0^{\wedge}2)))\,/\,C4$$

s is

In[50]:=
$$\text{Simplify}\left[-\frac{U2\,V1-U1\,V2}{-U3\,V1+U1\,V3}\ /.\ \{U1\to 0,\ U2\to 2\,C2,\ U3\to 2\,C4,\ V1\to 2\,C1,\ V2\to C3,\ V3\to C5\}\right]$$

Out[50]=
$$\frac{C2}{C4}$$

The partial derivatives of z by x,
y are zx → x0 (a - x0^2 - y0^2 - z0^2) / (z0 (x0^2 + y0^2 + z0^2 - c)),
zy → y0 (b - x0^2 - y0^2 - z0^2) / (z0 (x0^2 + y0^2 + z0^2 - c)).

Then T := (t + s zx) / (1 - s zy)

In[51]:=
```
Solve[(x0^2 + y0^2 + z0^2)^2 - 2 a x0^2 - 2 b y0^2 - 2 c z0^2 + d^2 == 0, z0]
```

Out[51]=
$$\left\{\left\{z0\to-\sqrt{\left(c-x0^2-y0^2-\sqrt{\left(c^2-d^2+2\,a\,x0^2-2\,c\,x0^2+2\,b\,y0^2-2\,c\,y0^2\right)}\right)}\right\},\right.$$
$$\left\{z0\to\sqrt{\left(c-x0^2-y0^2-\sqrt{\left(c^2-d^2+2\,a\,x0^2-2\,c\,x0^2+2\,b\,y0^2-2\,c\,y0^2\right)}\right)}\right\},$$
$$\left\{z0\to-\sqrt{\left(c-x0^2-y0^2+\sqrt{\left(c^2-d^2+2\,a\,x0^2-2\,c\,x0^2+2\,b\,y0^2-2\,c\,y0^2\right)}\right)}\right\},$$
$$\left.\left\{z0\to\sqrt{\left(c-x0^2-y0^2+\sqrt{\left(c^2-d^2+2\,a\,x0^2-2\,c\,x0^2+2\,b\,y0^2-2\,c\,y0^2\right)}\right)}\right\}\right\}$$

When $z0\to\sqrt{\left(c-x0^2-y0^2+\sqrt{\left(c^2-d^2+2\,(a-c)\,x0^2+2\,(b-c)\,y0^2\right)}\right)}$,
we show that T = P / Q with

In[52]:=
```
Q := - (b - d) (c - d)^2 (2 c + d) (c + d) + (b - d) (- c + d) (6 a c - 7 c^2 + 4 (a - c ) d + d^2) x0^2 -
   (b - c) (b - d) (c - d) (3 c + d) y0^2 - 2 (a - c) (b - d) (2 a - 3 c + d) x0^4 +
   2 (b - c) (- 2 a b + a c + 2 b c + (a - 3 c) d + d^2) x0^2 y0^2 -
   √ (c^2 - d^2 + 2 ( a - c) x0^2 + 2 (b - c) y0^2)
    ((b - d) (c - d) (2 c - d) (c + d) + (b - d) (4 a c - 5 c^2 - 2 (a - c) d + d^2) x0^2 +
      (b - c) (b - d) (c - d) y0^2 - 2 (a - c) (b - d) x0^4 - 2 (b - c) (a - d) x0^2 y0^2)
```

In[53]:=
```
P := x0 y0 (- (c^2 - d^2) (3 a b - 2 a c - 2 b c - (a + b - 4 c ) d - d^2) +
      2 (a - c) (- 2 a b + a c + 2 b c + (a - 3 c) d + d^2) x0^2 + 2 (b - c)
       (- 2 a b + 2 a c + b c + (b - 3 c) d + d^2) y0^2 + √ (c^2 - d^2 + 2 ( a - c) x0^2 + 2 (b - c) y0^2)
       (- 3 a b c + 2 a c^2 + 2 b c^2 + (a b + a c + b c - 4 c^2 ) d -
         (a + b - c ) d^2 + d^3 + 2 (a - c) (b - d) x0^2 + 2 (b - c) (a - d) y0^2)) -
   v2 √ (c - x0^2 - y0^2 + √ (c^2 - d^2 + 2 ( a - c) x0^2 + 2 (b - c) y0^2))
    (c (c^2 - d^2 + 2 (a - c) x0^2 + 2 (b - c) y0^2) +
      (x0^2 + y0^2 + z0^2 - c) (c^2 - d^2 + (a - c) x0^2 + (b - c) y0^2))
```

In[54]:=
```
Simplify[(t + s zx) / (1 - s zy) - P / Q /. {t -> (2 (b - c) (a - d) x0 y0 z0 +
          v2 (- b y0^2 - c z0^2 + (x0^2 + y0^2 + z0^2) (y0^2 + z0^2))) / C4,
     s → C2 / C4, zx → x0 (a - x0^2 - y0^2 - z0^2) / (z0 (x0^2 + y0^2 + z0^2 - c)),
     zy → y0 (b - x0^2 - y0^2 - z0^2) / (z0 (x0^2 + y0^2 + z0^2 - c))} /.
   {C2 -> - v2 x0 z0 (- c + x0^2 + y0^2 + z0^2) +
       (c - d) y0 (b d - d^2 + 2 (a - b) x0^2 + (b - d) (x0^2 + y0^2 + z0^2)),
    C4 -> v2 x0 y0 (- b + x0^2 + y0^2 + z0^2) + (- b + d) z0
       (d^2 - 2 a x0^2 - c (d - x0^2 + y0^2 + z0^2) + d (x0^2 + y0^2 + z0^2))} /.
  {z0 → √ (c - x0^2 - y0^2 + √ (c^2 - d^2 + 2 ( a - c) x0^2 + 2 (b - c) y0^2)),
   v2 → √(2 (b - d) (c - d) (d - a))}]
```

Out[54]= 0

The center of the circle is

x :

In[55]:=
```
Simplify[
 x0 + (r (- U2 V1 V2 + U1 V2^2 - U3 V1 V3 + U1 V3^2)) / (U2^2 V1^2 + U3^2 V1^2 - 2 U1 U2 V1 V2 +
        U1^2 V2^2 + U3^2 V2^2 - 2 U1 U3 V1 V3 - 2 U2 U3 V2 V3 + U1^2 V3^2 + U2^2 V3^2) /.
  {r → 2 C1, U1 → 0, U2 → 2 C2, U3 → 2 C4, V1 → 2 C1, V2 → C3, V3 → C5}]
```

Out[55]=

$$-\frac{2\,C1^2\,(C2\,C3 + C4\,C5)}{4\,C1^2\,\left(C2^2 + C4^2\right) + (C3\,C4 - C2\,C5)^2} + x0$$

y :

In[56]:=
```
Simplify[
 y0 + (r (U2 V1^2 - U1 V1 V2 - U3 V2 V3 + U2 V3^2)) / (U2^2 V1^2 + U3^2 V1^2 - 2 U1 U2 V1 V2 + U1^2 V2^2 +
      U3^2 V2^2 - 2 U1 U3 V1 V3 - 2 U2 U3 V2 V3 + U1^2 V3^2 + U2^2 V3^2) /.
  {r → 2 C1, U1 → 0, U2 → 2 C2, U3 → 2 C4, V1 → 2 C1, V2 → C3, V3 → C5}]
```

Out[56]=
$$\frac{C1 \left(4\, C1^2\, C2 + C5\, (-C3\, C4 + C2\, C5)\right)}{4\, C1^2 \left(C2^2 + C4^2\right) + (C3\, C4 - C2\, C5)^2} + y0$$

In[57]:=
```
Simplify[
 z0 + (r (U3 V1^2 + U3 V2^2 - U1 V1 V3 - U2 V2 V3)) / (U2^2 V1^2 + U3^2 V1^2 - 2 U1 U2 V1 V2 + U1^2 V2^2 +
      U3^2 V2^2 - 2 U1 U3 V1 V3 - 2 U2 U3 V2 V3 + U1^2 V3^2 + U2^2 V3^2) /.
  {r → 2 C1, U1 → 0, U2 → 2 C2, U3 → 2 C4, V1 → 2 C1, V2 → C3, V3 → C5}]
```

Out[57]=
$$\frac{C1 \left(4\, C1^2\, C4 + C3\, (C3\, C4 - C2\, C5)\right)}{4\, C1^2 \left(C2^2 + C4^2\right) + (C3\, C4 - C2\, C5)^2} + z0$$

(**The radius**) ^2 **is**

In[58]:=
```
Simplify[(r^2 (V1^2 + V2^2 + V3^2)) / (U3^2 (V1^2 + V2^2) -
    2 U1 U3 V1 V3 - 2 U2 V2 (U1 V1 + U3 V3) + U2^2 (V1^2 + V3^2) + U1^2 (V2^2 + V3^2)) /.
  {r → 2 C1, U1 → 0, U2 → 2 C2, U3 → 2 C4, V1 → 2 C1, V2 → C3, V3 → C5}]
```

Out[58]=
$$\frac{C1^2 \left(4\, C1^2 + C3^2 + C5^2\right)}{4\, C1^2 \left(C2^2 + C4^2\right) + (C3\, C4 - C2\, C5)^2}$$