J.Boman氏の最近の２つの関連する結果, distributionの台と解析性，Radon変換と楕円体領域の特殊な関係性についての解説

(On J. Boman's recent 2 related results about the support of a distribution and its analyticity, and a special relationship between Radon transformations and ellipsoidal regions)

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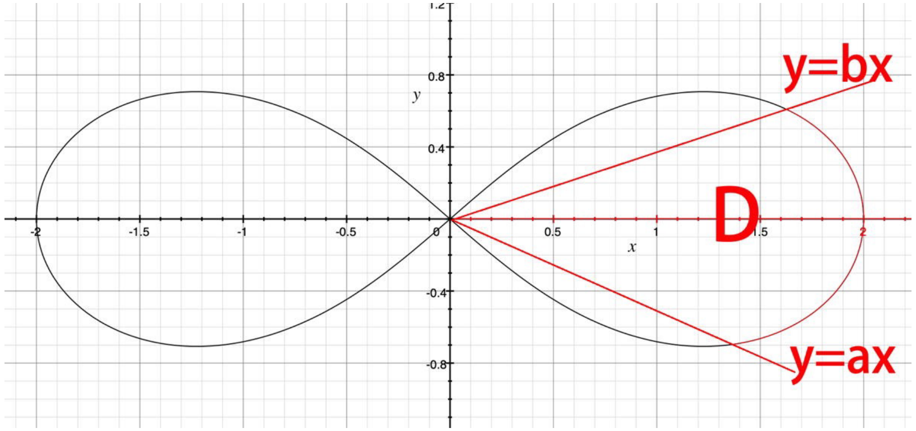
1. Real analyticity of the C^1surface which is the support of a distribution (Jan Boman) : Regularity of a distribution and of the boundary of its support, The Journal of Geometric Analysis vol.32, Article number 300 (2022).

2. The support of a Radon transform and ellipsoidal regions (Jan Boman) : A hypersurface containing the support of a Radon transform must be an ellipsoid. II: The general case; J. Inverse Ill-Posed Probl. 2021; 29(3): 351—367.

3. A new proof of the Arnold conjecture and some related results (I.Newton, V.I.Arnold, A.Koldobsky-A.Merkurjev-V.Yaskin, M.Agranovsky, J.Boman)

The Arnold Conjecture. There is a book "Arnold's Problem" 2nd edition, Springer-Verlag, Berlin, 2004, 656 pages by Vladimir I. Arnold:
Problem 1990-27(=1987-14) *An ovaloid in $\mathbb{R}^n$ (that is, a closed hypersurface bounding a convex body) is said to be algebraically integrable if the volume cut off by a hyperplane from this ovaloid is an algebraic function of the hyperplane. Do there exist algebraically integrable smooth ovaloids different from ellipsoid in $\mathbb{R}^n$ with odd n?* This is generalization of Newton ' s theorem for higher dimensions.

Newton's theorem about ovals (lemma 28 of section VI of book 1 of Newton's Principia) *There is no convex smooth (meaning infinitely differentiable) curve such that the area cut off by a line $ax + by = c$ is an algebraic function of a, b, and c.* As for the assumption, the smoothness of convex curves is necessary, because triangles and Huygens lemniscate $(x^2+y^2)^2 = 2\alpha^2(x^2-y^2)$ are algebraically integrable.



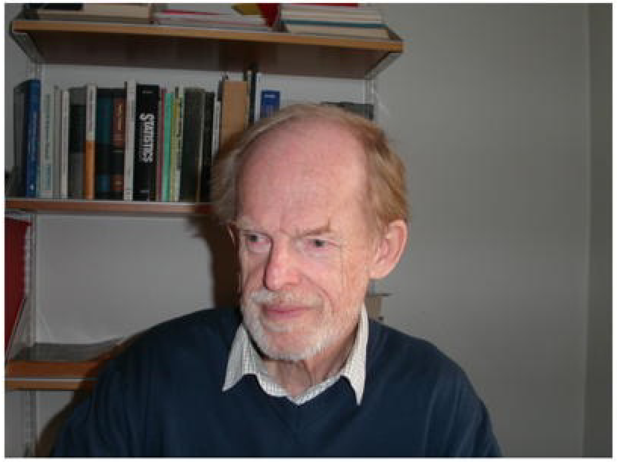
Indeed, for $\alpha > 0, -1 < a < b < 1$, put

$$D := \{(x,y) \in \mathbb{R}^2 \mid (x^2+y^2)^2 \leq 2\alpha^2(x^2-y^2), ax \leq y \leq bx\}.$$

Then,

$$|D| = \alpha^2 \left(\frac{b}{1+b^2} - \frac{a}{1+a^2} \right).$$

Jan Boman



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1. Real analyticity of the C^1surface which is the support of a distribution

Theorem 1.1 U : a neighborhood of $\mathring{x}(\in \mathbb{R}^n)$, $\Psi(x) \in C^1(U)$.
$\Sigma := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \mid y = \Psi(x), x \in U\}$.
If a distribution: $f(x,y) \in \mathscr{D}'(U \times \mathbb{R})$ satisfies the following (i),(ii):

(i) $(\mathring{x}, \Psi(\mathring{x})) \in \mathrm{supp}(f) \subset \Sigma$,

(ii) $\mathrm{WF_A}(f) \cap \{(\mathring{x}, \Psi(\mathring{x}); \xi \cdot dx + 0 \cdot dy) \mid 0 \neq \xi \in \mathbb{R}^n\} = \emptyset$
(That is, $f(x,y)$ depends real analytically on x).

Then, $\Psi(x)$ is analytic at $x = \mathring{x}$.

Remark 1.2. A similar result does not hold in C^∞ category.
$f(x,y) = \varphi(x) e^{-1/x^2} \delta(y - |x|^3)$ $(\varphi(x) \in C_0^\infty(\mathbb{R}), \varphi(x) \equiv 1$ $(|x| < \epsilon))$.

Corollary 1.3. Let $\mathsf{Y}(t)$ be the Heaviside function, and $u(x,y)(\not\equiv 0)$ be an analytic function at $(\overset{\circ}{x}, \Psi(\overset{\circ}{x}))$. Then, Theorem 1.1 also holds for $f(x,y) = u(x,y)\mathsf{Y}(y - \Psi(x))$.

Remark 1.4. Because we can apply the theorem to $g := u_y \cdot \partial_y f - u_y \cdot f = u(x,y)\delta(y - \Psi(x))$. On the other hand, a similar result does not hold for a distribution supported by the half space $\{y \geq \Psi(x)\}$. For example,

$$f(x,y) = Y(y - x^2) + Y(y - x^2 + x^3)$$

is supported by $\{y \geq x^2 - (x)_+^3\}$, and f satisfies the condition (ii) at $(0,0)$, where $(t)_+ = t \ (t \geq 0), = 0 \ (t < 0)$. But, its boundary is only of C^2 class, but not of C^3 class at $(0,0)$.

Proof of Theorem 1.1. By translation, we may assume $\Psi(\overset{\circ}{x}) > 0$. By the assumption on $\mathrm{WF}_A(f)$, we can substitute any fixed value for x in $f(x,y)$. Therefore, we can write $f(x,y)$ as follows:

$$f(x,y)=\sum_{j=0}^{m-1}(-1)^j q_j(x)\delta^{(j)}(y-\Psi(x)),$$

where $m(\geq 1)$ depends on x, but m is locally bounded since f is a distribution on $U\times\mathbb{R}$. Further, again by the WF_A assumption,

$$h_\ell(x):=\int_{-\infty}^{\infty} f(x,y)y^\ell dy=\sum_{j=0}^{m-1}\ell(\ell-1)\cdots(\ell-j+1)q_j(x)\Psi(x)^{\ell-j}$$

is an analytic function of x at $\overset{\circ}{x}$ for $\ell=0,1,2,\ldots$. Put

$$Q(x)={}^t(q_0(x),\ldots,q_{m-1}(x)),\quad H_s={}^t(h_s(x),\ldots,h_{m-1+s}(x)).$$

We may assume $q_{m-1}(x)\not\equiv 0$ at $\overset{\circ}{x}$. Then we have a series of linear equations:

$$M_s Q = H_s \ \ (C^\omega\text{-vector}), \quad (\forall s = 0, 1, 2, \dots),$$

where

$$M_s = \left(c_{s+k,j} \Psi^{s+k-j}\right)_{k,j=0,1,\dots,m-1}, \quad c_{\ell,j} := \ell!/(\ell - j)!.$$

Lemma 1.5. For $\forall s = 0, 1, 2, \dots$

$$\Psi(x)^{ms} q_{m-1}(x)^m$$

is analytic at $x = \overset{\circ}{x}$.

Proof of Lemma 1.5. Put

$$N = \left((k+1)\delta_{k+1,j}\right)_{k,j=0,1,\dots,m-1}.$$

Then, we have

$$M_{s+1} = M_s(\Psi I + N) = M_0(\Psi I + N)^{s+1},$$

$$N^\ell = \Big((k+1)\cdots(k+\ell)\delta_{k+\ell,j}\Big)_{k,j=0,1,\ldots,m-1}.$$

Note that

$$\det(M_sQ,\cdots,M_{s+m-1}Q) = \det(H_s,\cdots,H_{s+m-1}) \in C^\omega \quad (s=0,1,2,\ldots).$$

The left hand side is equal to

$$\begin{aligned}
&\det\Big(M_s\cdot\big(Q,(\Psi I+N)Q,\cdots,(\Psi I+N)^{m-1}Q\big)\Big)\\
&=\det M_s\cdot\det\Big(Q,(\Psi I+N)Q,\cdots,(\Psi I+N)^{m-1}Q\Big)\\
&=\det M_0\ \det(\Psi I+N)^s\cdot\det\Big(Q,\cdots,(\Psi I+N)^{m-1}Q\Big)\\
&=\det M_0\cdot\Psi^{ms}\cdot\det(Q,NQ,N^2Q,\cdots,N^{m-1}Q)\\
&=1!2!\cdots(m-1)!\cdot\Psi^{ms}\cdot(\pm1)\left(\prod_{p=1}^{m-1}p^p\right)\cdot q_{m-1}^m
\end{aligned}$$

$$= (\pm 1)(m-1)!^m \Psi^{ms} q_{m-1}^m.$$

This completes the proof of Lemma 1.5. By this lemma, $B(x) := q_{m-1}(x)^m \not\equiv 0, A(x) := \Psi(x)^m q_{m-1}(x)^m$ are analytic at $\overset{\circ}{x}$ (put $s=0$, or $s=1$). Therefore $\Psi(x)^m = A(x)/B(x)$, and

$$B(x)(A(x)/B(x))^s$$

is analytic at $\overset{\circ}{x}$ for any $s = 0, 1, 2, \ldots$. Since the ring R of all analytic functions at $\overset{\circ}{x}$ is a UFD (unique factorization domain), $A(x)/B(x)$ must belong to R (use uniqueness of prime factorization!). Hence $\Psi(x)^m$ is analytic at $\overset{\circ}{x}$. Remembering $\Psi(\overset{\circ}{x}) > 0$, we have the analyticity of $\Psi(x)$ at $\overset{\circ}{x}$. Further by using the equations:

$$M_0 Q = H_0, \quad \det M_0 = 1!2! \cdots (m-1)! \neq 0,$$

we get the analyticity of $Q = {}^t(q_0(x), \ldots, q_{m-1}(x))$ at $\overset{\circ}{x}$.

2. The support of a Radon transform and ellipsoidal regions

$D \subset \mathbb{R}^n$: a bounded convex open set such that $0 \in D$.
$f(x) \in \mathscr{D}'(\mathbb{R}^n)$ such that $\operatorname{supp} f \subset \overline{D}$ (the closure of D). Then the Radon transform of f:

$$Rf(\omega, p) := \int_{x\cdot\omega=p} f(x)dS = \int_{\mathbb{R}^n} \delta(x\cdot\omega-p)\, f(x)dx, \quad (\omega, p) \in S^{n-1}\times\mathbb{R}$$

Here dS is the $(n-1)$-dim measure on hyperplanes. Define

$$\rho(\omega) := \sup\{x \cdot \omega \mid x \in D\}.$$

Then

$$\operatorname{supp}\big(Rf(\omega, p)\big) \subset \{(\omega, p) \in S^{n-1} \times \mathbb{R} \mid -\rho(-\omega) \le p \le \rho(\omega)\}.$$

Theorem 2.1. Suppose that $f \not\equiv 0$, and that

$$\text{supp}\left(Rf(\omega,p)\right) \subset \Sigma_D := \{(\omega,p) \in S^{n-1}\times\mathbb{R} \mid p = \rho(\omega), \text{or } -\rho(-\omega)\}.$$

Further suppose that ∂D is a strictly convex C^2 boundary. Then, D is an ellipsoidal region. That is, after some translation and some rotation of coordinates, we have

$$D = \left\{x \in \mathbb{R}^n \mid \sum_{j=1}^{n} \frac{x_j^2}{\beta_j} - 1 < 0\right\}$$

with some $\beta_1, \ldots, \beta_n > 0$.

Remark 2.2. He only assumed that D is a bounded open convex domain in his paper in 2021. But there is a proof gap: $q_{m-1}(\omega) \not\equiv 0 \overset{?}{\Rightarrow} q_{m-1}(\omega)q_{m-1}(-\omega) \not\equiv 0$ for some function $q_{m-1}(\omega)$ on S^{n-1}. Concerning this, he assumend $D = -D$ (symmetric condition) in the former paper in 2020. Together with the result of Theorem 1.1, we can conclude Theorem 2.1 under the strictly convex C^2 boundary condition because $\Sigma_D = \{p = \pm\rho(\pm\omega)\}$ becomes a C^1 surface in $S^{n-1} \times \mathbb{R}$. Hence, $\rho(\omega)$ and $q_{m-1}(\omega)$ becomes real analytic, and so $q_{m-1}(\omega) \not\equiv 0 \Rightarrow q_{m-1}(\omega)q_{m-1}(-\omega) \not\equiv 0$.

Example 2.3. Set $D = \{x \in \mathbb{R}^2 \mid |x| = \sqrt{x_1^2 + x_2^2} < 1\}$, and

$$f(x) := \frac{1}{\pi}\Big((1 - |x|^2)_+^{-1/2} + \Delta_x (1 - |x|^2)_+^{1/2}\Big),$$

then, $Rf(\omega, p) = \delta(p-1) + \delta(p+1)$. This is because

$$Rf(\omega, p) = \chi_{\{|p|<1\}}(\omega) + \partial_p^2 \left(\frac{1}{2}(1-p^2)_+\right) = \delta(p-1) + \delta(p+1).$$

Remark 2.4. (Application of Theorem 1.1 to Radon transforms)

Assume that D is locally expressed by

$$D = \{x_n > \varphi(x')\}, \quad \left(\varphi_{x_i x_j}(x')\right)_{i,j=1,\dots,n-1} \gg 0,$$

then $x \cdot \omega = x' \cdot \omega' + \varphi(x')\omega_n$ takes the maximum on ∂D at $x' = (x_1, \dots, x_{n-1}) = x'(\omega)$ such that

$$\varphi_{x_i}(x') = -\omega_i/\omega_n \quad (i = 1, \dots, n-1).$$

Hence, by the inverse mapping theorem,

$$\rho(\omega) = x'(\omega) \cdot \omega' + \varphi(x'(\omega)) \cdot \omega_n$$

is a C^1 function of ω. Further, in $\{\omega_n > 0\}$, we can take $y' = \omega'/\omega_n$ as coordinates of S^{n-1}. Since $x_n + x' \cdot y' - p\sqrt{1 + |y'|^2} = 0$,

$$Rf(y', p) = \sqrt{1 + |y'|^2} \int_{\mathbb{R}^{n-1}} f(x', p\sqrt{1 + |y'|^2} - y' \cdot x')dx'.$$

Therefore, the cotangential direction of the analytic wavefront set of the integrand is

$$\eta' dx' + \eta_n d(p\sqrt{1+|y'|^2} - y' \cdot x')$$
$$= (\eta' - \eta_n y')dx' + \eta_n \sqrt{1+|y'|^2}\,dp + \eta_n(**)dy'.$$

Hence,

$$\mathrm{WF}_A(Rf) \subset \left\{(y', p; \xi' dy' \pm \sqrt{1+|y'|^2} dp) \mid y', p, \xi'\right\},$$

and so $Rf(y', p)$ is depending real analytically on y'. Thus Theorem 1.1 can be applied to $Rf(y', p)$ and $\Sigma_D = \{p = \rho(y')\}$.

Proof of Theorem 2.1.: Note that $Rf(-\omega, -p) = Rf(\omega, p)$ by the definition of a Radon transform. Hence we can assume the following form of $Rf(\omega, p)$:

$$Rf(\omega,p) = \sum_{j=0}^{m-1}\Big(q_j(\omega)\delta^{(j)}(p-\rho(\omega)) + (-1)^j q_j(-\omega)\delta^{(j)}(p+\rho(-\omega))\Big).$$

Since $Rf(\omega, p)$ depends real analytically on ω, the argument similar to the proof of Theorem 1.1 is available. Consider the moments of $Rf(\omega, p)$ with respect to p:

$$h_\ell(\omega) := \int_{\mathbb{R}} p^\ell Rf(\omega,p)dp = \int_{\mathbb{R}^n} f(x)(x\cdot\omega)^\ell dx \ \ (\ell = 0, 1, \ldots),$$

where $h_\ell \in \mathcal{P}_\ell := \{\text{homogeneous polynomials of } \omega \in \mathbb{R}^n \text{ with degree } \ell\}$. Therefore, putting $c_{\ell,j} = \ell!/(\ell - j)!$, we have

$$\sum_{j=0}^{m-1}(-1)^j c_{\ell,j}\Big(\rho(\omega)^{\ell-j}q_j(\omega) + (-1)^\ell \rho(-\omega)^{\ell-j}q_j(-\omega)\Big) = h_\ell(\omega) \in \mathcal{P}_\ell.$$

Using the notation $\tilde{q}_j(\omega) = q_j(-\omega), \tilde{\rho}(\omega) = \rho(-\omega)$, we set

$$
\begin{aligned}
Q &:= {}^t(q_0, \dots, q_{m-1}, \tilde{q}_0, \dots, \tilde{q}_{m-1}),\\
H_s &:= {}^t(h_s, \dots, h_{s+m-1}, h_{s+m}, \dots, h_{s+2m-1}),\\
M_s &:= (M'_s,\ M''_s),\\
M'_s &:= \left((-1)^j c_{s+k,j}\rho^{s+k-j}\right)_{k=0,\dots,2m-1, j=0,1,\dots,m-1},\\
M''_s &:= \left((-1)^{j-m+s+k} c_{s+k,j-m}\tilde{\rho}^{\,s+k-j+m}\right)_{k=0,\dots,2m-1, j=m,\dots,2m-1}.
\end{aligned}
$$

Hence we get $M_s Q = H_s \ (s = 0, 1, 2, \dots)$. We prepare 3 lemmas.

Lemma 2.5. $(\rho(\omega)\tilde{\rho}(\omega))^m$ is a polynomial in ω.

Proof. Setting

$$N' = \left((k+1)\delta_{k+1,j}\right)_{k,j=0,1,\dots,m-1}, N'' = \left((k-m+1)\delta_{k+1,j}\right)_{k,j=m,\dots,2m-1}$$

we have

$$M'_{s+1} = \rho M'_s(\rho I_m - N'), \qquad M''_{s+1} = M''_s(-\tilde{\rho} I_m + N'').$$

Therefore,

$$M_{s+1} = M_s \begin{pmatrix} \rho I_m - N' & 0 \\ 0 & -\tilde{\rho} I_m + N'' \end{pmatrix} = M_0 \begin{pmatrix} \rho I_m - N' & 0 \\ 0 & -\tilde{\rho} I_m + N'' \end{pmatrix}^{s+1}.$$

So we consider the following determinant as before:

$$\det \left(M_s Q, \ldots, M_{s+2m-1} Q \right) = \det(H_s, \ldots, H_{s+2m-1}) \in \mathcal{P}_*,$$

with $* = s + (s+1) + \cdots + (s+2m-1) = m(2s+2m-1)$. Then the left side is equal to

$$\det M_s \cdot \det(Q, KQ, \ldots, K^{2m-1} Q),$$

where

$$K = \begin{pmatrix} \rho I_m - N' & 0 \\ 0 & -\tilde{\rho} I_m + N'' \end{pmatrix}, \qquad M_s = M_0 \, K^s.$$

We have

$$\det M_s = \det M_0 \cdot (\det K)^s = \det M_0 \cdot (-\rho \tilde{\rho})^{ms}.$$

As for $\det M_0$, we can find the value:

$$\det M_0 = C_m \cdot (\rho + \tilde{\rho})^{m^2},$$

where C_m is a non-zero constant. Since $\rho(\omega), \tilde{\rho}(\omega) = \rho(-\omega)$ are positive, $\det M_0 \neq 0$.

Further, as for

$$A := \det(Q, KQ, \dots, K^{2m-1}Q),$$

putting $Q = {}^t(Q', Q'')$, $Q' := {}^t(q_0, \dots, q_{m-1})$, $Q'' := {}^t(\tilde{q}_0, \dots, \tilde{q}_{m-1})$, we obtain

$$K^\ell Q = \begin{pmatrix} (\rho I_m - N')^\ell Q' \\ (-\tilde{\rho} I_m + N'')^\ell Q'' \end{pmatrix}.$$

So,

$$A = \det \begin{pmatrix} Q' & (\rho I_m - N')Q' \cdots & (\rho I_m - N')^{2m-1} Q' \\ Q'' & (-\tilde{\rho} I_m + N'')Q'' \cdots & (-\tilde{\rho} I_m + N'')^{2m-1} Q'' \end{pmatrix},$$

and we can find its value

$$A = C'_m (q_{m-1}\tilde{q}_{m-1})^m (\rho + \tilde{\rho})^{m^2},$$

where C'_m is a non-zero constant. Hence we have

$$(q_{m-1}\tilde{q}_{m-1})^m (\rho + \tilde{\rho})^{2m^2} \cdot (\rho\tilde{\rho})^{ms} \in \mathcal{P}_{m(2s+2m-1)} \quad (s = 0, 1, \ldots).$$

So, if $q_{m-1}(\omega)\tilde{q}_{m-1}(\omega) = q_{m-1}(\omega)q_{m-1}(-\omega) \not\equiv 0$, the argument similar to the proof of Theorem 1.1 works because the polynomial ring is a UFD. Hence $(\rho(\omega)\tilde{\rho}(\omega))^m$ is a homogeneous polynomial of order $2m$.

Lemma 2.6. $\rho(\omega) - \tilde{\rho}(\omega)$, $\rho(\omega)\tilde{\rho}(\omega)$ are homogeneous rational functions of ω.

Proof. To get more information from

$$M_s Q = H_s \ (s = 0, 1, 2, \ldots),$$

we remove Q from these equations, and make equations among H_s $(s = 0, 1, 2, \ldots)$ such that

$$H_{s+1} = S_s H_s.$$

To do so, we must find a matrix S_s satisfying

$$S_s M_s = M_{s+1}.$$

The following nilpotent matrix P lifts each row by 1 row :

$$P = (\delta_{k+1,j})_{k,j=0,1,\ldots,2m-1}.$$

So, we can assume the following form of S_s:

$$S_s := \begin{pmatrix} (\delta_{k+1,j})_{k=0,1,\ldots,2m-2,j=0,\ldots,2m-1} \\ \sigma_{2m}, \sigma_{2m-2}, \cdots, \sigma_2, \sigma_1 \end{pmatrix}.$$

Since $M_s = M_0 K^s$, we have

$$M_{s+1} = S_s M_s \quad \leftrightarrows \quad M_0 K = S_s M_0.$$

So, we can put $s=0$, and the equations for σ_ℓ $(\ell = 1, 2, \ldots, 2m)$ are written as follows: For $j = 0, 1, \ldots, m-1$ with $(u, v) = (\rho, -\tilde{\rho})$,

$$\begin{cases} \sum_{\ell=1}^{2m} c_{2m-\ell,j} u^{-\ell} \sigma_\ell(u,v) = c_{2m,j}, \\ \sum_{\ell=1}^{2m} c_{2m-\ell,j} v^{-\ell} \sigma_\ell(u,v) = c_{2m,j}. \end{cases}$$

Boman found the solutions $\sigma_\ell(u,v)$ as the following coefficients:

$$G(t) = (t-u)^m (t-v)^m = t^{2m} - \sum_{j=0}^{2m-1} t^j \, \sigma_{2m-j}(u,v).$$

So, $\sigma_1(u,v) = m(u+v)$, $\quad \sigma_2(u,v) = -\frac{m(m-1)}{2}(u^2+v^2) - m^2 uv$.
Since

$$S_0 H_\ell = H_{\ell+1} \;\; (\ell = 0, 1, \ldots),$$

we obtain the following equations from the $2m$-th component:

$$\sum_{k=0}^{2m-1} \sigma_{2m-k} h_{k+\ell} = h_{\ell+2m} \quad (\ell = 0, 1, \ldots).$$

We consider these equations for $\ell = 0, 1, \ldots, 2m-1$ as the equations for σ_ℓ $(\ell = 1, 2, \ldots, 2m)$. To do so, we must investigate the determinant of

$$W_0 := \left(h_{j+k}\right)_{j,k=0,1,\ldots,2m-1} = \Big(M_0 Q, M_1 Q, \cdots, M_{2m-1} Q\Big).$$

Since

$$\det W_0 = \det M_0 \cdot \det \Big(Q, KQ, K^2 Q, \cdots, K^{2m-1} Q\Big) = \det M_0 \cdot A \not\equiv 0$$

as seen before, σ_* is written as

$$\begin{pmatrix} \sigma_{2m} \\ \vdots \\ \sigma_1 \end{pmatrix} = W_0^{-1} \begin{pmatrix} h_{2m} \\ \vdots \\ h_{4m-1} \end{pmatrix}.$$

Therefore, $\rho(\omega) - \tilde{\rho}(\omega) = u + v = \sigma_1/m$, $\rho(\omega)\tilde{\rho}(\omega) = -uv = \frac{1}{m}(\sigma_2 + \frac{m-1}{2m}\sigma_1^2)$ are homogeneous rational functions of ω with degrees 1 and 2 respectively.

Lemma 2.7. $\rho(\omega)\tilde{\rho}(\omega)$, $\rho(\omega) - \tilde{\rho}(\omega)$ are homogeneous polynomials of ω with degrees 2 and 1 respectively.

proof. As for $\rho(\omega)\tilde{\rho}(\omega)$, we can write $\rho(\omega)\tilde{\rho}(\omega) = U(\omega)/V(\omega)$ (the irreducible fraction expression) by Lemma 2.6, where U, V are some homogeneous polynomials of ω. By the proof of Lemma 2.5, we have

$$(q_{m-1}\tilde{q}_{m-1})^m(\rho+\tilde{\rho})^{2m^2} \cdot \left(\frac{U(\omega)}{V(\omega)}\right)^{ms} \in \mathcal{P}_{m(2s+2m-1)} \quad (s = 0, 1, \ldots).$$

So, by the similar argument as before, we obtain that $V(\omega)$ is a non-zero constant. Namely, $\rho(\omega)\tilde{\rho}(\omega)$ is a homogeneous

polynomial with degree 2.
Concerning $\rho - \tilde{\rho}$, we consider the Trace of the matrix K^ℓ:

$$\mathrm{Tr}(K^\ell) = \mathrm{Tr}\begin{pmatrix}(\rho I_m - N')^\ell & 0 \\ 0 & (-\tilde{\rho} I_m + N'')^\ell\end{pmatrix} = m\{\rho(\omega)^\ell + (-\tilde{\rho}(\omega))^\ell\}.$$

Since $M_0 K = S M_0$ by the definition of $S = S_0$, we have $S = M_0 K M_0^{-1}$ and so $\mathrm{Tr}(S^\ell) = \mathrm{Tr}(K^\ell) = m\{\rho(\omega)^\ell + (-\tilde{\rho}(\omega))^\ell\}$. Further, since $H_{\ell+1} = S H_\ell = S^{\ell+1} H_0$, putting

$$W_\ell := \left(h_{j+k+\ell}\right)_{j,k=0,1,\cdots,2m-1} = (H_\ell, \cdots, H_{\ell+2m-1}) = S^\ell W_0,$$

we have $S^\ell = W_\ell W_0^{-1}$. Therefore $\mathrm{Tr}(S^\ell)$ is the coefficient of $(-\lambda)^{m-1}$ of

$$\det\left(W_\ell W_0^{-1} - \lambda I_m\right).$$

Hence the denominator of $\rho(\omega)^\ell + (-\tilde{\rho}(\omega))^\ell$ is a divisor of $(\det W_0)^{2m}$ for any $\ell = 1, 2, 3, \ldots$. Let $\rho(\omega) - \tilde{\rho}(\omega) := X(\omega)/Y(\omega)$ (the irreducible fraction expression). Then,

$$\begin{aligned}
(X/Y)^2 &= \rho(\omega)^2 + \tilde{\rho}(\omega)^2 - 2\rho\tilde{\rho}, \\
(X/Y)^4 &= \rho(\omega)^4 + \tilde{\rho}(\omega)^4 - 4\rho\tilde{\rho}(X/Y)^2 - 2(\rho\tilde{\rho})^2, \\
&\vdots
\end{aligned}$$

Since $\rho\tilde{\rho}$ is a polynomial, the denominators of right-sides remain as divisors of $(\det W_0)^{2m}$. On the other hand the denominators of the left sides increase if Y is not a constant when we consider $(X/Y)^{2^s}$ $(s = 1, 2, 3, \ldots)$. Contradiction! So Y is a non-zero constant, and $\rho(\omega) - \tilde{\rho}(\omega)$ is a homogeneous polynomial with degree 1. This completes the proof of Lemma 2.7.

By Lemma 2.7, we have a vector $\alpha \in \mathbb{R}^n$ such that

$$\rho(\omega) - \rho(-\omega) = \alpha \cdot \omega.$$

Take a new coordinate system $x' = x - \frac{1}{2}\alpha$ for D. Then

$$\rho'(\omega) = \sup\{(x - \frac{1}{2}\alpha) \cdot \omega \mid x \in K\} = \rho(\omega) - \frac{1}{2}\alpha \cdot \omega.$$

Therefore

$$\rho'(\omega) - \rho'(-\omega) = \rho(\omega) - \frac{1}{2}\alpha \cdot \omega - (\rho(-\omega) + \frac{1}{2}\alpha \cdot \omega) = 0.$$

On the other hand, since $\rho'(\omega)^2 = \rho'(\omega)\rho'(-\omega)$ is a homogeneous polynomial $Z(\omega)$ with degree 2, we conclude that

$$\rho'(\omega) = \sqrt{Z(\omega)}.$$

Since $Z(\omega)$ should be a positive definite homogeneous polynomial with degree 2, under a suitable rotation of D, we get the form

$$Z(\omega) = \beta_1\omega_1^2 + \cdots + \beta_n\omega_n^2$$

with some positive constants $\beta_1, \ldots, \beta_n$. Then,

$$D = \text{Interior of} \bigcap_{\omega \in S^{n-1}} \left\{ x \cdot \omega \le \sqrt{Z(\omega)} \right\} = \left\{ x \in \mathbb{R}^n \mid \sum_{j=1}^{n} \frac{x_j^2}{\beta_j} - 1 < 0 \right\}.$$

3. A new proof of the Arnold conjecture and some related results

3.1. The volume function for D and the Radon transform of χ_D. Let $D(\ni 0)$ be a bounded domain of $\mathbb{R}^n$, and $\chi_D(x)$ be its characteristic function. Then,

$$R\chi_D(\omega, p) = |D \cap \{x \in \mathbb{R}^n \mid x \cdot \omega = p\}| \quad ((n-1)\text{-dimensional volume}).$$

Hence, the volume function for D is given by

$$V_D(\omega, p) = |D \cap \{x \cdot \omega < p\}| = \int_{-\rho(-\omega)}^{p} R\chi_D(\omega, s) ds.$$

Example 3.1.1. For $\mathbb{D} := \{x \in \mathbb{R}^n \mid |x| := \sqrt{x_1^2 + \cdots + x_n^2} < 1\}$,

$$R\chi_{\mathbb{D}}(\omega, p) = c_{n-1}(1 - p^2)_+^{(n-1)/2},$$

for some $c_{n-1} > 0$ because $\mathbb{D} \cap \{x \cdot \omega = p\}$ is an $(n-1)$-dimensional ball with radius $\sqrt{1-p^2}$. For any ellipsoidal region D, there is an affine bijective map:

$$x := \Phi(\tilde{x}) = \sum_{j=1}^{n} a_{ij}\tilde{x} + b_i : \mathbb{D} \xrightarrow{\sim} D, \quad \tilde{\omega}_j := \sum_{i=1}^{n} a_{ij}\omega_i, \ |\tilde{\omega}| := \sqrt{\sum_{j=1}^{n} \tilde{\omega}_j^2}.$$

Hence, we have

$$R\chi_D(\omega, p) = c_{n-1} |\det(a_{ij})| \cdot |\tilde{\omega}|^{-n} \Big(|\tilde{\omega}|^2 - (p - b \cdot \omega)^2 \Big)_+^{(n-1)/2}.$$

3.2. A.Koldobsky-A.Merkurjev-V.Yaskin's result: On polynomially integrable convex bodies, Advances in Mathematics 320 (2017), 876-886. They proved : *For an odd n and a C^∞ smooth* *convex ∂D, if the volume function $V_D(\omega, p)$ is a polynomial of p with degree$< N$* (N: independent of ω), *then D is an ellipsoidal region*.

3.3. M.L.Agranovsky's results:

1. On polynomially integrable domains in Euclidean spaces, in: Complex Analysis and Dynamical Systems, New Trends and Open Problems, Birkhauser (2018), 1-21. He proved:

In Theorem 2, *There are no polynomially integrable domain with C^2-smooth boundary in $\mathbb{R}^n$ with even n*.

In Theorem 5, *If a smoothly bounded domain D in $\mathbb{R}^n$ (with n odd) is polynomially integrable, then it is convex.*

In Theorem 7, *he got a weaker version of A.Koldobsky-A.Merkurjev-V.Yaskin's result.*

2. On algebraically integrable bodies. In: Contemporary Mathematics, Functional Analysis and Geometry. Selim Krein Centennial, AMS, Providence RI, 33–44 (2019): *Let $n \geq 3$ be odd and $D \subset \mathbb{R}^n$ be a bounded domain with infinitely smooth boundary ∂D. Further suppose that D is an algebraically integrable domain, free of real singularities, then, D is a polynomially integrable domain. Hence, D is an ellipsoidal region.*

His arguments (for n:odd) are as follows:

$(D : \text{bdd}, C^\infty \text{ boundary}) + (V_D(\omega, p) : \text{algebraic in } p)$
$\Longrightarrow (D : \text{bdd}, C^\infty \text{ boundary}) + (V_D(\omega, p) : \text{polynomial in } p)$
$\Longrightarrow (D : \text{bdd, convex}, C^\infty \text{ boundary}) + (V_D(\omega, p) : \text{polynomial in } p)$
$\Longrightarrow D : \text{an ellipsoidal region.}$

3.4. Boman's new proof: His new proof is for the part:

$(D : \text{bdd}, C^2 \text{ strictly convex boundary}) + (V_D(\omega, p) : \text{polynomial in } p)$

$\Longrightarrow D$: an ellipsoidal region.

Proof. Since $R\chi_D(\omega,p) = \partial_p V_D(\omega,p)$, $R\chi_D(\omega,p)$ is a polynomial of p, whose degree is less than an integer N independent of ω. Therefore, for a sufficiently large integer $m>0$, we have

$$0 = \partial_p^{2m} R\chi_D(\omega,p) = R(\Delta_x^m \chi_D)(\omega,p), \quad p \in (-\rho(-\omega),\rho(\omega)).$$

For any distribution $f(x)$ with compact support, we have

$$\begin{aligned}(\partial_p)^{2m} Rf(\omega,p) &= \int_{\mathbb{R}^n} \delta^{(2m)}(x\cdot\omega - p) f(x)dx = \int_{\mathbb{R}^n} \delta(x\cdot\omega - p)\Delta_x^m f(x)dx \\ &= R((\Delta_x)^m f)(\omega,p).\end{aligned}$$

$g(x) = \Delta_x^m \chi_D(x)$ is a distribution with support in a compact set $\overline{D}$, and the support of its Radon transform is included in $\Sigma_D = \{p = \pm\rho(\pm\omega)\}$. Further $\Delta_x^m \chi_D(x) \not\equiv 0$ because its Fourier transform $(-|\xi|^2)^m \mathscr{F}[\chi_D](\xi) \not\equiv 0$. Therefore, by Theorem 2.1, we conclude that D is an ellipsoidal region.

Appendix (Theorems of M.L.Agranovsky).

In this section, D is a bounded domain in $\mathbb{R}^n$ such that $0 \in D$.

4.1. The volume function $V_D(\omega, p)$ is algebraic $\Rightarrow$ polynomial.

Definition 4.1.1. $V_D(\omega, p)$ is said to be algebraic in p if there is a polynomial $Q(\omega, p, w)$ in p, w given by

$$Q(\omega, p, w) = \sum_{j=0}^{N} q_j(\omega, p) w^j, \; q_j(\omega, p) = \sum_{k=0}^{k_j} q_{jk}(\omega) p^k \;\; (j = 0, \ldots, N),$$

where $q_{jk}(\omega) \in C^0(S^{n-1})$ such that $Q(\omega, p, V_D(\omega, p)) = 0$ on $S^{n-1} \times (-\delta, \delta)$ for some small $\delta > 0$. Further we assume the following conditions (i), (ii) on the discriminant $\mathrm{Disc}_Q(\omega, p)$ of Q.

Discriminant Conditions:
(i) $\mathrm{Disc}_Q(\omega, p) \neq 0$ on $S^{n-1} \times \{p \in \mathbb{C} \mid \mathrm{Im}\, p = 0\}$, (ii) $d(\omega) \neq 0$ $(\forall \omega \in S^{n-1})$ for the highest coefficient $d(\omega)$ of $\mathrm{Disc}_Q(\omega, p)$ in p.

In general, the discriminant of a polynomial

$$P(w) := a_0 + a_1 w + \cdots + a_N w^N = a_N (w - \beta_1) \cdots (w - \beta_N)$$

is defined by

$$a_N^{2N-2} \prod_{i<j} (\beta_i - \beta_j)^2,$$

which is the resultant of $P(w), P'(w)$. On the other hand ours is

$$\mathrm{Disc}_Q(\omega, p) := q_N(\omega, p)^{2N-1} \prod_{i<j} (w_i(\omega, p) - w_j(\omega, p))^2,$$

where $\{w_i(\omega,p)\}_{i=1}^N$ are all the roots of $Q(\omega,p,w)=0$ in w.

Key Lemma 4.1.2. For n odd, and $\operatorname{supp} f$: compact, then,

$$\int_{S^{n-1}}(\partial_p^{n-1}Rf)(\omega,x\cdot\omega)dS(\omega)=(-1)^{(n-1)/2}2^n\pi^{n-1}f(x).$$

This integral vanishes for n even.

We consider the L^2-inner product for functions α,β on $S^{n-1}=\{\omega\in\mathbb{R}^n \mid |\omega|=1\}$:

$$\langle\alpha,\beta\rangle:=\int_{S^{n-1}}\alpha(\omega)\overline{\beta(\omega)}dS(\omega).$$

To obtain a global expression of $V_D(\omega,p)$ in p, we use an expansion of V_D by orthogonal polynomials of ω; that is, an expansion by spherical harmonic functions on $\mathbb{R}^n$. Let

$$\mathscr{H}_k := \{f(\omega) \in \mathcal{P}_k \mid \Delta_\omega f = 0\}.$$

Key Lemma 4.1.3. Let $\{Y_k^{(m)}(\omega)\}_m, (m = 0, \pm 1, \pm 2, ..., \pm k)$ be the orthonormal base of $\mathscr{H}_k$ with respect to the inner product on S^{n-1} for $k = 0, 1, 2, \ldots$. Putting

$$V_{k,m}(p) := \int_{S^{n-1}} V_D(\omega, p) \cdot \overline{Y_k^{(m)}(\omega)} dS(\omega),$$

we have a global expression of V_D in p:

$$V_D(\omega, p) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} V_{k,m}(p)\, Y_k^{(m)}(\omega).$$

4.2. The volume function $V_D(\omega, p)$ is polynomial $\Rightarrow$ D is convex.

(Or the convergence radius of $\sum_{j=0}^{\infty}(\partial^j V_D(\omega, 0)/\partial p^j) \cdot p^j$ is larger than the diameter of $D \Rightarrow D$ is convex.)

Key Lemma 4.2.1.(Parseval type formula for Radon transforms.)

For odd n and suitable functions $f(x), g(x)$ on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} f(x)g(x)dx = \frac{1}{2(-4\pi^2)^{(n-1)/2}} \int_{S^{n-1}\times\mathbb{R}} \partial_p^{n-1} Rf(\omega, p)\cdot Rg(\omega, p)dp\, dS(\omega).$$

For example, $f \in C_0^\infty(\mathbb{R}^n), g \in \mathscr{D}'(\mathbb{R}^n)$ with compact support K.

ご清聴ありがとうございました！