有界領域の体積関数の代数性に関するAgranovsky 達の結果について

(On the results of Agranovsky et-al. concerning algebraicity of the volume function of a bounded domain)

Kiyoomi Kataoka
Graduate School of Mathematical Sciences
the University of Tokyo

2023.3.8
「代数解析学の研究集会」
於：日本大学理工学部駿河台キャンパス タワースコラ (駿河台校舎) S504

Contents

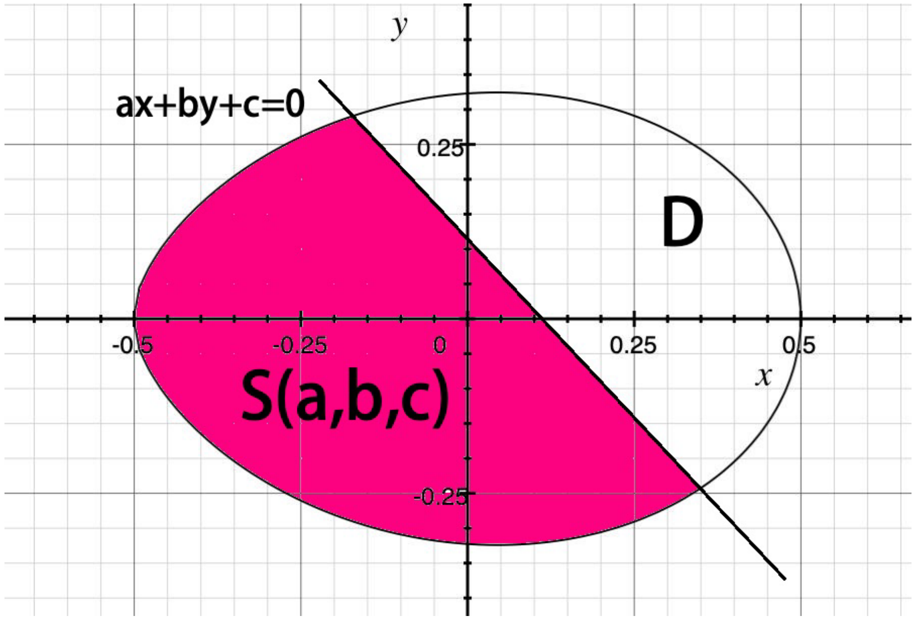
1. Newton's theorem and the Arnold conjecture on the volume functions of bounded convex domains in $\mathbb{R}^n$.

2. M.L.Agranovsky's theorems on the algebraic volume functions of bounded domains in $\mathbb{R}^n$ with odd n.

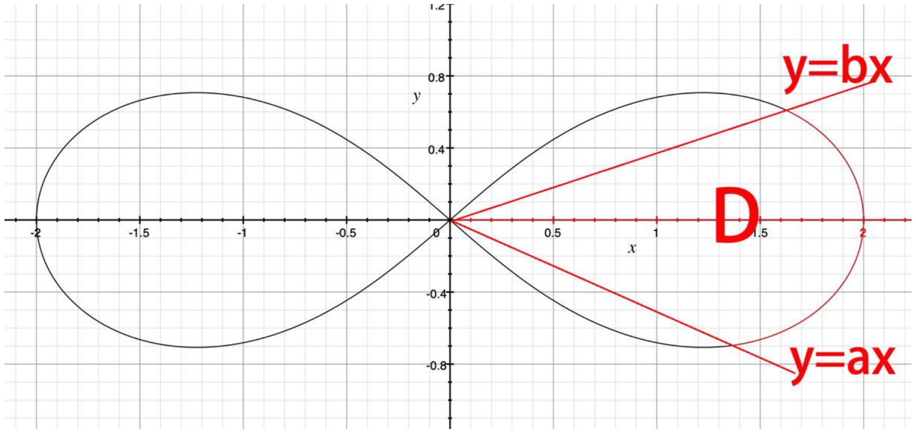
3. The inversion formula for Radon transformations, the Parseval-type formula and the proofs.

1. Newton's theorem and the Arnold conjecture.

1.1.Newton's theorem about ovals (lemma 28 of section VI of book 1 of Newton's Principia) *There is no convex smooth (meaning infinitely differentiable) curve such that the area $S(a, b, c)$ cut off by a line $ax + by = c$ is an algebraic function of a, b, and c.*



As for the assumption, the smoothness (no analytic singularity) of convex curves is necessary, because triangles and Bernoulli lemniscate $(x^2+y^2)^2 = 2\alpha^2(x^2-y^2)$ are algebraically integrable.



Indeed, for $\alpha > 0, -1 < a < b < 1$, put

$$D_{a,b} := \{(x,y) \in \mathbb{R}^2 \mid (x^2+y^2)^2 \le 2\alpha^2(x^2-y^2), ax \le y \le bx\}.$$

Then,

$$|D_{a,b}| = \alpha^2 \left(\frac{b}{1+b^2} - \frac{a}{1+a^2} \right).$$

1.2. The volume function for D and the Radon transform.

$D \subset \mathbb{R}^n$: a bounded open set such that $0 \in D$.

$f(x) \in \mathscr{D}'(\mathbb{R}^n)$ such that $\operatorname{supp} f \subset \overline{D}$ (the closure of D). Then the Radon transform of f:

$$Rf(\omega, p) := \int_{x\cdot\omega=p} f(x)dS = \int_{\mathbb{R}^n} \delta(x\cdot\omega-p)\, f(x)dx, \quad (\omega, p) \in S^{n-1}\times\mathbb{R}$$

Here dS is the $(n-1)$-dim measure on hyperplanes. Define

$$\rho(\omega) := \sup\{x \cdot \omega \mid x \in D\}.$$

Then

$$\operatorname{supp}\big(Rf(\omega, p)\big) \subset \{(\omega, p) \in S^{n-1} \times \mathbb{R} \mid -\rho(-\omega) \leq p \leq \rho(\omega)\}.$$

Let $\chi_D(x)$ be the characteristic function of D. Then, we have

$R\chi_D(\omega, p) = |D \cap \{x \in \mathbb{R}^n \mid x \cdot \omega = p\}|$ $\quad$ ($(n-1)$–dimensional volume),

where $(\omega, p) \in S^{n-1} \times \mathbb{R}$.

Hence the volume function for D is given by

$$V_D(\omega, p) = |D \cap \{x \cdot \omega < p\}| = \int_{-\infty}^{p} R\chi_D(\omega, s) ds.$$

Example 1.2.1. For $\mathbb{D} := \{x \in \mathbb{R}^n \mid |x| := \sqrt{x_1^2 + \cdots + x_n^2} < 1\}$,

$$R\chi_{\mathbb{D}}(\omega, p) = c_{n-1}(1 - p^2)_+^{(n-1)/2},$$

for some $c_{n-1} > 0$ because $\mathbb{D} \cap \{x \cdot \omega = p\}$ is an $(n-1)$-dimensional ball with radius $\sqrt{1 - p^2}$. Here, $(t)_+ = t$ $(t > 0), = 0$ $(t \leq 0)$.

For any ellipsoidal region D, there is an affine bijective map:

$$x := \Phi(\tilde{x}) = \sum_{j=1}^{n} a_{ij}\tilde{x}+b_i : \mathbb{D} \xrightarrow{\sim} D, \quad \tilde{\omega}_j := \sum_{i=1}^{n} a_{ij}\omega_i, \ |\tilde{\omega}| := \sqrt{\sum_{j=1}^{n} \tilde{\omega}_j^2}.$$

Hence, we have

$$R\chi_D(\omega, p) = c_{n-1}|\det(a_{ij})| \cdot |\tilde{\omega}|^{-n}\Big(|\tilde{\omega}|^2 - (p - b \cdot \omega)^2\Big)_+^{(n-1)/2}.$$

1.3. Arnold's conjecture on volume functions.

In a book "Arnold's Problem" 2nd edition, Springer-Verlag, Berlin, 2004, 656 pages by Vladimir I. Arnold:

Problem 1990-27(=1987-14) *An ovaloid in $\mathbb{R}^n$ (that is, a closed hypersurface bounding a convex body) is said to be algebraically integrable if the volume cut off by a hyperplane from this ovaloid is an algebraic function of the hyperplane. Do there exist algebraically integrable smooth ovaloids different from ellipsoid in $\mathbb{R}^n$ with odd n?* This is generalization of Newton 's theorem for higher dimensions.

1.4. Polynomially integrable domains.

A.Koldobsky-A.Merkurjev-V.Yaskin's result: On polynomially integrable convex bodies, Advances in Mathematics 320

(2017), 876-886. They proved : *For an odd n and a C^∞ smooth convex ∂D, if the volume function $V_D(\omega, p)$ is a polynomial of p with degree $< N$* (N: independent of ω), *then D is an ellipsoidal region*.

Similar results are obtained by two authors:

(1) M.L.Agranovsky: On polynomially integrable domains in Euclidean spaces, in: Complex Analysis and Dynamical Systems, New Trends and Open Problems, Birkhauser (2018), 1-21,

(2) J. Boman (a new approach): A hypersurface containing the support of a Radon transform must be an ellipsoid. II: The general case; J. Inverse Ill-Posed Probl. 2021; 29(3): 351—367.

Boman's theorem. Suppose that $f \not\equiv 0$, and that

$$\text{supp}\left(Rf(\omega,p)\right) \subset \Sigma_D := \{(\omega,p) \in S^{n-1}\times\mathbb{R} \mid p = \rho(\omega), \text{or } -\rho(-\omega)\}.$$

Further suppose that ∂D is a strictly convex C^2 boundary. Then, D is an ellipsoidal region. That is, after some translation and some rotation of coordinates, we have

$$D = \left\{x \in \mathbb{R}^n \mid \sum_{j=1}^{n} \frac{x_j^2}{\beta_j} - 1 < 0\right\}$$

with some $\beta_1, \ldots, \beta_n > 0$.

Boman's new proof: His new proof is for the part:

$(D : \text{bdd}, C^2 \text{ strictly convex boundary}) + (V_D(\omega,p) : \text{polynomial in } p)$
$\Longrightarrow D : \text{an ellipsoidal region.}$

Proof. Since $R\chi_D(\omega,p) = \partial_p V_D(\omega,p)$, $R\chi_D(\omega,p)$ is a polynomial of p, whose degree is less than an integer N independent of ω. Therefore, for a sufficiently large integer $m > 0$, we have

$$0 = \partial_p^{2m} R\chi_D(\omega,p) = R(\Delta_x^m \chi_D)(\omega,p), \quad p \in (-\rho(-\omega), \rho(\omega)).$$

This is because for any distribution $f(x)$ with compact support, we have

$$\begin{aligned}(\partial_p)^{2m} Rf(\omega,p) &= \int_{\mathbb{R}^n} \delta^{(2m)}(x\cdot\omega - p) f(x)dx \\ &= \int_{\mathbb{R}^n} \delta(x\cdot\omega - p)\Delta_x^m f(x)dx = R((\Delta_x)^m f)(\omega,p).\end{aligned}$$

$g(x) = \Delta_x^m \chi_D(x)$ is a distribution with support in a compact set $\overline{D}$, and the support of its Radon transform is included in $\Sigma_D = \{p = \pm\rho(\pm\omega)\}$. Further $\Delta_x^m \chi_D(x) \not\equiv 0$ because its Fourier transform $(-|\xi|^2)^m \mathscr{F}[\chi_D](\xi) \not\equiv 0$. Therefore, by Boman's theorem, we conclude that D is an ellipsoidal region.

2. M.L.Agranovsky's theorems on the algebraic volume functions of bounded domains in $\mathbb{R}^n$ with odd n.

2.1. Agranovsky's results 1.

In "On polynomially integrable domains in Euclidean spaces, in: Complex Analysis and Dynamical Systems, New Trends and Open Problems, Birkhauser (2018), 1-21", he obtained :

Theorem 2, *There are no polynomially integrable domain with C^2-smooth boundary in $\mathbb{R}^n$ with even n.*

Theorem 5, *If a smoothly bounded domain D in $\mathbb{R}^n$ (with n odd) is polynomially integrable, then it is convex.*

Theorem 7, *he got a weaker version of A.Koldobsky-A.Merkurjev-V.Yaskin's result.*

2.2. Agranovsky's result 2.

In "On algebraically integrable bodies. In: Contemporary Mathematics, Functional Analysis and Geometry. Selim Krein Centennial, AMS, Providence RI, 33–44 (2019)":

Let $n \geq 3$ be odd and $D \subset \mathbb{R}^n$ be a bounded domain with infinitely smooth boundary ∂D. Further suppose that D is an algebraically integrable domain, free of real singularities, then, D is a polynomially integrable domain. Hence, D is an ellipsoidal region.

His arguments (for n:odd) are as follows:

$(D$: bdd, C^∞ boundary$) + (V_D(\omega, p)$: algebraic in $p)$
$\Longrightarrow (D$: bdd, C^∞ boundary$) + (V_D(\omega, p)$: polynomial in $p)$
$\Longrightarrow (D$: bdd, convex, C^∞ boundary$) + (V_D(\omega, p)$: polynomial in $p)$
$\Longrightarrow D$: an ellipsoidal region.

2.3. The precise definition of an algebraic volume function.

Definition 2.3.1. $V_D(\omega, p)$ is said to be algebraic in p if there is a polynomial $Q(\omega, p, w)$ in p, w given by

$$Q(\omega, p, w) = \sum_{j=0}^{N} q_j(\omega, p) w^j, \; q_j(\omega, p) = \sum_{k=0}^{k_j} q_{jk}(\omega) p^k \;\; (j = 0, \ldots, N),$$

where $q_{jk}(\omega) \in C^0(S^{n-1})$ such that $Q(\omega, p, V_D(\omega, p)) = 0$ on $S^{n-1} \times (-\delta, \delta)$ for some small $\delta > 0$. Further we assume the following conditions (i), (ii) on the discriminant $\mathrm{Disc}_Q(\omega, p)$ of Q.

Discriminant Conditions:
(i) $\mathrm{Disc}_Q(\omega, p) \neq 0$ on $S^{n-1} \times \{p \in \mathbb{C} \mid \mathrm{Im}\, p = 0\}$, (ii) $d(\omega) \neq$

0 $(\forall \omega \in S^{n-1})$ for the highest coefficient $d(\omega)$ of $\mathrm{Disc}_Q(\omega, p)$ in p.

In general, the discriminant of a polynomial

$$P(w) := a_0 + a_1 w + \cdots + a_N w^N = a_N(w - \beta_1) \cdots (w - \beta_N)$$

is defined by

$$a_N^{2N-2} \prod_{i<j} (\beta_i - \beta_j)^2,$$

which is the resultant of $P(w), P'(w)$. On the other hand ours is

$$\mathrm{Disc}_Q(\omega, p) := q_N(\omega, p)^{2N-1} \prod_{i<j} (w_i(\omega, p) - w_j(\omega, p))^2,$$

where $\{w_i(\omega, p)\}_{i=1}^N$ are all the roots of $Q(\omega, p, w) = 0$ in w.

3. The inversion formula for Radon transformations, the Parseval-type formula and the proofs.

3.1. The inversion formula for Radon transformations.

$$\delta(x) = \lim_{\epsilon\to+0} \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi-\epsilon|\xi|}d\xi = \frac{1}{(2\pi)^n}\int_{|\xi|=1} \frac{(n-1)!dS(\xi)}{(+0-ix\cdot\xi)^n}$$
$$= \frac{(n-1)!}{(-2\pi i)^n}\int_{S^{n-1}} \frac{1}{(x\cdot\xi+i0)^n}dS(\xi).$$

Hence,

$$\int_{S^{n-1}} \frac{1}{(x\cdot\xi\pm i0)^n}dS(\xi) = \frac{(\mp 2\pi i)^n}{(n-1)!}\delta(x).$$

So we have

$$\int_{S^{n-1}} \left(\frac{1}{(x\cdot\xi-i0)^n} - \frac{1}{(x\cdot\xi+i0)^n}\right) dS(\xi) = \frac{(1-(-1)^n)(2\pi i)^n}{(n-1)!}\delta(x).$$

Since

$$\int_{S^{n-1}} (\partial_p^{n-1} Rf)(\xi, x\cdot\xi) dS(\xi)$$
$$= \int_{S^{n-1}} dS(\xi) \int_{\mathbb{R}^n} (-1)^{n-1} \delta^{(n-1)}((y-x)\cdot\xi) f(y) dy,$$

and

$$\delta^{(n-1)}(t) = \frac{(-1)^{n-1}(n-1)!}{2\pi i}\left(\frac{1}{(t-i0)^n} - \frac{1}{(t+i0)^n}\right),$$

Theorem 3.1.1. For n odd, and supp f: compact, then,

$$\int_{S^{n-1}} (\partial_p^{n-1} Rf)(\omega, x\cdot\omega) dS(\omega) = (-1)^{(n-1)/2} 2^n \pi^{n-1} f(x).$$

For n even, this integral vanishes, instead we have

$$\int_{S^{n-1}} dS(\xi) \int_{-\infty}^{\infty} \frac{1}{s+i0} \partial_p^{n-1} Rf(\xi, x\cdot\xi - s) ds = (-2\pi i)^n f(x).$$

Theorem 3.1.2.(Parseval type formula for Radon transforms.)
For odd n, $f(x) \in C_0^\infty(\mathbb{R}^n)$, and $g(x) \in \mathscr{D}'(\mathbb{R}^n)$ with compact support, we have

$$\int_{\mathbb{R}^n} f(x)g(x)dx = \frac{1}{2(2\pi i)^{n-1}} \int_{S^{n-1}\times\mathbb{R}} \partial_p^{n-1} Rf(\omega,p)\cdot Rg(\omega,p)dp\, dS(\omega).$$

Proof. Since n is odd, we have

$$f(x) = \frac{1}{2(2\pi i)^{n-1}} \int_{S^{n-1}} (\partial_p^{n-1} Rf)(\omega, x\cdot\omega)dS(\omega),$$

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)g(x)dx &= \frac{1}{2(2\pi i)^{n-1}} \int_{\mathbb{R}^n} g(x)dx \int_{S^{n-1}} (\partial_p^{n-1} Rf)(\omega, x\cdot\omega)dS(\omega) \\ &= \frac{1}{2(2\pi i)^{n-1}} \int_{S^{n-1}} dS(\omega) \int_{\mathbb{R}^n} (\partial_p^{n-1} Rf)(\omega, x\cdot\omega)\cdot g(x)dx \\ &= \frac{1}{2(2\pi i)^{n-1}} \int_{S^{n-1}} dS(\omega) \int_{\mathbb{R}} (\partial_p^{n-1} Rf)(\omega, p)\cdot Rg(\omega,p)dp\end{aligned}$$

$$= \frac{1}{2(2\pi i)^{n-1}} \int_{S^{n-1}\times\mathbb{R}} (\partial_p^{n-1} Rf)(\omega, p) \cdot Rg(\omega, p) dp\, dS(\omega).$$

3.2. Proof of $(V_D(\omega, p)$:algebraic in $p \Rightarrow$ polynomial in $p)$

Lemma 3.2.1. For odd n, $\partial_p^{n+2} V_D(-\omega, -p) = \partial_p^{n+2} V_D(\omega, p)$ and

$$\int_{S^{n-1}} \partial_p^{n+2} V_D(\omega, x\cdot\omega) dS(\omega) = 0 \quad (\forall x \in D).$$

Proof. Since $V_D(\omega, p) + V_D(-\omega, -p) = |D|$, we have the first equality. Further, apply the inversion formula to $f(x) = \Delta_x \chi_D(x)$. Then,

$$\int_{S^{n-1}} (\partial_p^{n-1} R(\Delta_x \chi_D))(\omega, x\cdot\omega) dS(\omega)$$
$$= (-1)^{(n-1)/2} 2^n \pi^{n-1} \Delta_x \chi_D(x) = 0 \quad (x \in D).$$

Since $R(\Delta_x \chi_D)(\omega, p) = \partial_p^2 R\chi_D(\omega, p) = \partial_p^3 V_D(\omega, p)$, we have the last equality.

Under the discriminant condition for $Q(\omega, p, w)$, $V_D(\omega, p)$ becomes holomorphic in a neighborhood of $\{p \in \mathbb{C} \mid \mathrm{Im}\, p = 0\}$, in particular $\partial_p^{n+2} V_D(\omega, p)$ has a power series expansion at $p = 0$:

$$\partial_p^{n+2} V_D(\omega, p) = \sum_{j=0}^{\infty} \beta_j(\omega) p^j.$$

From now on, we consider the L^2-inner product for functions α, β on $S^{n-1} = \{\omega \in \mathbb{R}^n \mid |\omega| = 1\}$:

$$\langle \alpha, \beta \rangle := \int_{S^{n-1}} \alpha(\omega) \overline{\beta(\omega)} dS(\omega).$$

Lemma 3.2.2. If $V_D(\omega, p)$ is holomorphic at $p = 0$, then

$$\beta_j \perp \bigcup_{k=0}^{j+1} \mathcal{P}_k \; (j = 0, 1, 2, \ldots),$$

where $\mathcal{P}_\ell :=$ {homogeneous polynomials of $\omega \in \mathbb{R}^n$ with degree ℓ}.

Proof By Lemma 3.2.1, we have

$$\sum_{j=0}^{\infty} \int_{S^{n-1}} \beta_j(\omega)(x \cdot \omega)^j dS(\omega) = 0 \quad (\forall x, |x| < \epsilon).$$

Since its j-th term is the homogeneous polynomial of x with degree j, we have

$$\int_{S^{n-1}} \beta_j(\omega)(x \cdot \omega)^j dS(\omega) = 0 \quad (\forall x \in \mathbb{R}^n, \forall j = 0, 1, \ldots).$$

Finite sums of $(x \cdot \omega)^j$ with $x \in \mathbb{R}^n$ generate any homogeneous polynomials of ω with degree j, and so we conclude

$$\beta_j(\omega) \perp \mathcal{P}_j \quad (j = 0, 1, 2, \ldots).$$

Further, for $\ell = 0, 1, 2, \ldots, [j/2]$ we have an imbedding:

$$\mathcal{P}_{j-2\ell} \ni P(\omega) \hookrightarrow |\omega|^{2\ell} P(\omega) \in \mathcal{P}_j.$$

So,

$$\beta_j(\omega) \perp \bigcup_{\ell=0}^{[j/2]} \mathcal{P}_{j-2\ell} \quad (j = 0, 1, 2, \ldots).$$

On the other hand, by the same lemma we obtain that $\partial_p^{n+2} V_D(\omega, p) = \sum_{j=0}^{\infty} \beta_j(\omega) p^j$ is an even function in (ω, p), and so

$$\beta_j(-\omega) = (-1)^j \beta_j(\omega) \quad (\forall j).$$

Since any $P(\omega) \in \mathcal{P}_k$ satisfies $P(-\omega) = (-1)^k P(\omega)$, for odd $j-k$ we have

$$\beta_j(-\omega)P(-\omega) = (-1)^{j-k}\beta_j(\omega)P(\omega) = -\beta_j(\omega)P(\omega).$$

Therefore its integral on S^{n-1} vanishes; that is,

$$\beta_j \perp \bigcup_{k-j=\text{odd}} \mathcal{P}_k \ .$$

Thus the proof is completed.

To obtain a global expression of $V_D(\omega, p)$ in p, we use an expansion of V_D by orthogonal polynomials of ω; that is, an expansion by spherical harmonic functions on $\mathbb{R}^n$. Let

$$\mathscr{H}_k := \{f(\omega) \in \mathcal{P}_k \mid \Delta_\omega f = 0\}.$$

Proposition 3.2.3. Let $\{Y_k^{(m)}(\omega)\}_m, (m = 0, \pm 1, \pm 2, ..., \pm k)$ be the orthonormal base of $\mathscr{H}_k$ with respect to the inner product on S^{n-1} for $k = 0, 1, 2, \ldots$. Then, for any $\alpha(\omega) \in L^2(S^{n-1})$, we have

$$\alpha(\omega) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \alpha_{k,m} Y_k^{(m)}(\omega), \quad \alpha_{k,m} = \int_{S^{n-1}} \alpha(\omega) \overline{Y_k^{(m)}(\omega)} dS(\omega).$$

So putting

$$V_{k,m}(p) := \int_{S^{n-1}} V_D(\omega, p) \cdot \overline{Y_k^{(m)}(\omega)} dS(\omega),$$

we have a global expression of V_D in p:

$$V_D(\omega, p) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} V_{k,m}(p)\, Y_k^{(m)}(\omega).$$

Lemma 3.2.4. Assume that $V_D(\omega, p)$ is holomorphic at $p = 0$, then, $V_{k,m}(p)$ is a polynomial of p with degree $\leq k + n$.

Proof. For holomorphic functions, complex differentiation commutes with integration. So we have for $|p| \ll 1$

$$\partial_p^{n+2} V_{k,m}(p) = \int_{S^{n-1}} \partial_p^{n+2} V_D(\omega, p) \cdot \overline{Y_k^{(m)}(\omega)} dS(\omega)$$

$$= \sum_{j=0}^{\infty} \int_{S^{n-1}} \beta_j(\omega) p^j \cdot \overline{Y_k^{(m)}(\omega)} dS(\omega) = \sum_{j=0}^{k-2} p^j \int_{S^{n-1}} \beta_j(\omega) \cdot \overline{Y_k^{(m)}(\omega)} dS(\omega).$$

This completes the proof.

Since $V_D(\omega, p)$ is an algebraic function of p, for any fixed ω, $V_D(\omega, p)$ extends analytically in p along any curve in $\mathbb{C} \backslash S_\omega$ starting from $p = 0$. Here

$$S_\omega := \{p \in \mathbb{C} \mid \mathrm{Disc}_Q(\omega, p) = 0\}$$

is a finite set. So, $V_{k,m}(p)$ also extends analytically in p along the same curve. Since $V_{k,m}(p)$ is a polynomial at $p = 0$, such an analytic extension is the same polynomial. The expansion formula

$$V_D(\omega, p) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} V_{k,m}(p)\, Y_k^{(m)}(\omega).$$

holds along such a curve, so we can conclude that $V_D(\omega, p)$ is an entire function of p. Further since

$$Q(\omega, p, w) = \sum_{j=0}^{N} q_j(\omega, p) w^j,\; q_j(\omega, p) = \sum_{k=0}^{k_j} q_{jk}(\omega) p^k \;\; (j = 0, \ldots, N),$$

setting $L := \max\{k_1, \ldots, k_N\}$, we know that the number of

$$\{p \in \mathbb{C} \mid Q(\omega, p, c) = 0\} \supset \{p \in \mathbb{C} \mid V_D(\omega, p) = c\}.$$

is at most L for generic $c \in \mathbb{C}$. By the great Picard theorem, we can conclude that $V_D(\omega, p)$ is a polynomial in p with degree $\leq L$.

Therefore,

$$V_D(\omega, p) = \sum_{j=0}^{L} \frac{\partial_p^j V_D(\omega, 0)}{j!} p^j.$$

3.3. The volume function $V_D(\omega, p)$ is polynomial $\Rightarrow$ D is convex.

(Or the convergence radius of $\sum_{j=0}^{\infty}(\partial^j V_D(\omega, 0)/\partial p^j) \cdot p^j$ is larger than the diameter of $D \Rightarrow D$ is convex.)

Assume that the volume function $V_D(\omega, p)$ is holomorphic at $p = 0$ and

$$V_D(\omega, p) = \sum_{k=0}^{\infty} \gamma_k(\omega) p^k.$$

Lemma 3.3.1. For $k \geq n+1$, $\gamma_k(\omega) \perp \mathcal{P}_{k-n}$.

Proof. Since $R\chi_D(\omega, p) = \partial_p V_D(\omega, p)$, in a neighborhood of $x = 0$ we have

$$
\begin{aligned}
1 = \chi_D(x) &= \frac{1}{2(2\pi i)^{n-1}} \int_{S^{n-1}} (\partial_p^{n-1} \partial_p V_D)(\omega, x \cdot \omega) dS(\omega) \\
&= \sum_{k=0}^{\infty} \frac{1}{2(2\pi i)^{n-1}} \int_{S^{n-1}} \gamma_k(\omega) \cdot (\partial_p^n p^k)|_{p = x \cdot \omega} dS(\omega) \\
&= \sum_{k=n}^{\infty} \frac{1}{2(2\pi i)^{n-1}} \frac{k!}{(k-n)!} \int_{S^{n-1}} \gamma_k(\omega) \cdot (x \cdot \omega)^{k-n} dS(\omega).
\end{aligned}
$$

Hence, for any $k - n > 0$,

$$\int_{S^{n-1}} \gamma_k(\omega) \cdot (x \cdot \omega)^{k-n} dS(\omega) = 0 \quad (\forall x \in \mathbb{R}^n).$$

Therefore,

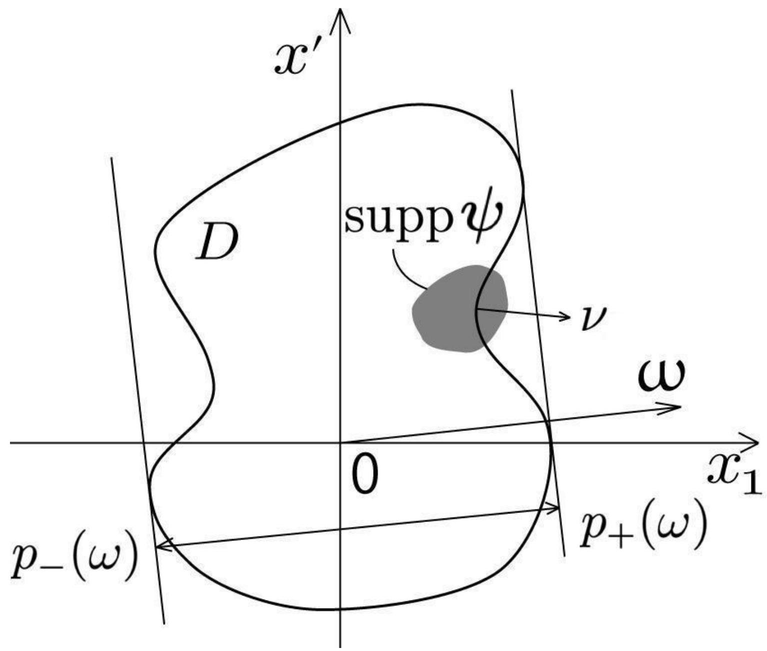
$$\int_{S^{n-1}} \gamma_k(\omega) \cdot R(\omega) dS(\omega) = 0 \quad (\forall R(\omega) \in \mathcal{P}_{k-n}).$$

Proof of 3.3. Let $\widehat{D}$ be the convex hull of D. Suppose that $\widehat{D} \neq D$. Choose a $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ such that

$$\mathrm{supp}(\psi) \subset \widehat{D}, \quad I := \int_{\partial D} \psi(x)\nu_1(x)dS(x) > 0,$$

where $\nu(x)$ is the outer unit normal vector for ∂D. We choose a coodinate $x = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$ as

$$\nu_1(x) > 0 \quad \text{on} \quad \mathrm{supp}(\psi).$$



$$
\begin{aligned}
I &= \int_{\partial D} \psi(x)\nu_1(x)dS(x) = \int_D \partial_{x_1}\psi(x)dx = \int_{\mathbb{R}^n} \partial_{x_1}\psi(x)\cdot\chi_D(x)dx \\
&= \frac{1}{2(2\pi i)^{n-1}} \int_{S^{n-1}\times\mathbb{R}} \partial_p^{n-1} R(\partial_{x_1}\psi)(\omega,p)\cdot R\chi_D(\omega,p)dp\,dS(\omega) \\
&= \frac{1}{2(2\pi i)^{n-1}} \int_{S^{n-1}\times\mathbb{R}} \partial_p^{n-1} (\omega_1\partial_p R\psi(\omega,p))\cdot R\chi_D(\omega,p)dp\,dS(\omega)
\end{aligned}
$$

$$= \frac{1}{2(2\pi i)^{n-1}} \int_{S^{n-1}\times\mathbb{R}} \omega_1 (\partial_p^n R\psi)(\omega,p) \cdot R\chi_D(\omega,p) dp\, dS(\omega).$$

Now we perform the integration by parts; move ∂_p^n from $R\psi(\omega,p)$ to $R\chi_D(\omega,p)$. Indeed, the integral end points in p are

$$p_+(\omega) := \sup\{x\cdot\omega \mid x \in D\}, \quad p_-(\omega) := \inf\{x\cdot\omega \mid x \in D\}.$$

On the other hand

$$\operatorname{supp}\psi(x) \subset \widehat{D} \subset \{x \in \mathbb{R}^n \mid p_-(\omega) < x\cdot\omega < p_+(\omega)\}.$$

So, for a sufficiently small $\epsilon > 0$, we have

$$\operatorname{supp}\psi(x) \subset \{x \in \mathbb{R}^n \mid p_-(\omega) + \epsilon < x\cdot\omega < p_+(\omega) - \epsilon\}.$$

Thus we obtain the following (we use the convergence radius of $V_D(\omega,p)$ at $p = 0$ is larger than the diameter of D!)

$$I = \frac{(-1)^n}{2(2\pi i)^{n-1}} \int_{S^{n-1}\times\mathbb{R}} \omega_1\, R\psi(\omega,p) \cdot (\partial_p^n R\chi_D)(\omega,p) dp\, dS(\omega)$$

$$= \frac{(-1)^n}{2(2\pi i)^{n-1}} \int_{S^{n-1}\times\mathbb{R}} \omega_1 \, R\psi(\omega,p) \cdot \partial_p^{n+1} V_D(\omega,p) dp \, dS(\omega)$$

$$= \frac{(-1)^n}{2(2\pi i)^{n-1}} \sum_{k=n+1}^{\infty} \frac{k!}{(k-n-1)!} \int_{S^{n-1}} \omega_1 \gamma_k(\omega) dS(\omega)$$

$$\times \int_{p_-(\omega)+\epsilon}^{p_+(\omega)-\epsilon} R\psi(\omega,p) \cdot p^{k-n-1} dp.$$

Since $R\psi(\omega,p) = 0$ $(p \notin [p_- + \epsilon, p_+ - \epsilon])$, we have

$$I = \frac{(-1)^n}{2(2\pi i)^{n-1}} \sum_{k=n+1}^{\infty} \frac{k!}{(k-n-1)!} \int_{S^{n-1}} \omega_1 \gamma_k(\omega) dS(\omega)$$

$$\times \int_{-\infty}^{\infty} R\psi(\omega,p) \cdot p^{k-n-1} dp.$$

We note here that

$$\varphi_k(\omega) := \int_{-\infty}^{\infty} R\psi(\omega,p) \cdot p^{k-n-1} dp = \int_{\mathbb{R}^n\times\mathbb{R}} \psi(x)\delta(x\cdot\omega - p) p^{k-n-1} dp \, dx$$

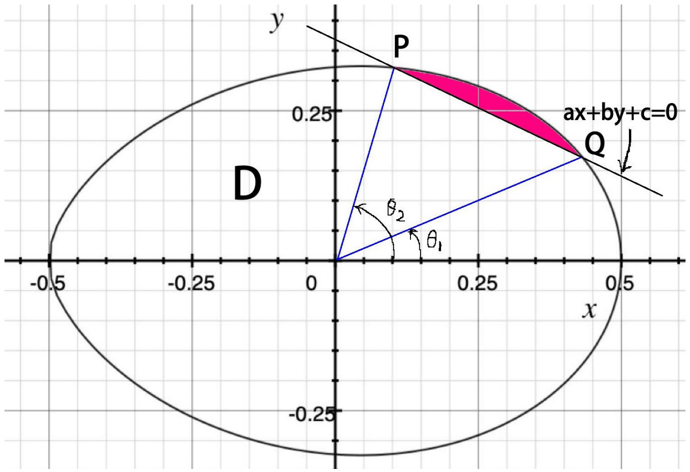
$$= \int_{\mathbb{R}^n} \psi(x)(x \cdot \omega)^{k-n-1} dx \in \mathcal{P}_{k-n-1}.$$

Therefore for $k \geq n+1$ we have

$$\int_{S^{n-1}} \omega_1 \gamma_k(\omega) dS(\omega) \int_{-\infty}^{\infty} R\psi(\omega, p) \cdot p^{k-n-1} dp$$
$$= \int_{S^{n-1}} \omega_1 \varphi_k(\omega) \cdot \gamma_k(\omega) dS(\omega) = 0.$$

This is because $\omega_1 \varphi_k(\omega) \in \mathcal{P}_{k-n}$ $(k \geq n+1)$. So $\gamma_k(\omega) \perp \omega_1 \varphi_k(\omega)$. Consequently, we have $I = 0$. Contradiction! Therefore D is convex.

Appendix 1. Newton's proof.



$$\left(\text{An ovaloid:}\left((x+0.5)^2+1.2y^2\right)^2=(x+0.5)^3+0.3(x+0.5)y^2\right)$$

Assume $0 \in D \subset \mathbb{R}^2$, and let $r(\theta)(>0)$ be a continuous function with period 2π such that $\partial D = \{(r(\theta)\cos\theta, r(\theta)\sin\theta) \mid \theta \in \mathbb{R}\}$.

We define the area function $S(\theta)$ of D by

$$S(\theta) := \int_0^\theta \frac{1}{2} r(\theta')^2 d\theta'.$$

Then it is sufficient to prove that $S(\theta)$ never be any algebraic function of $t = \tan\theta$.

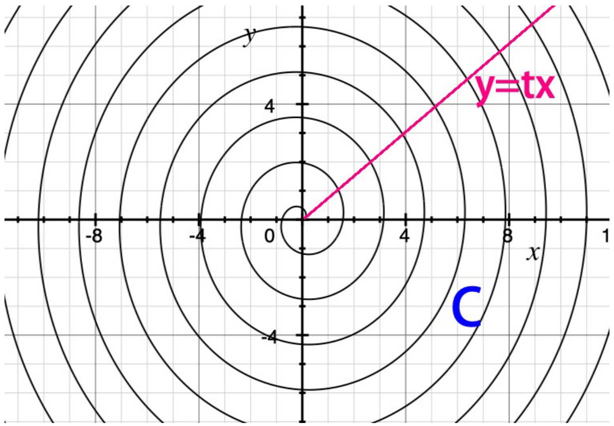
Proof. (Newton's proof) We assume that $S(\theta)$ is an algebraic function of $t = \tan\theta$. Hence we have a polynomial $R(s;t)$ of s:

$$R(s;t) = \sum_{j=0}^{J} \alpha_j(t) s^j, \quad R(S(\arctan t); t) = 0 \ \ (\forall t),$$

where $\alpha_j(t)$'s are polynomials of t such that $\alpha_J(t) \not\equiv 0$. Since ∂D has no analytic singularities, $S(\theta)$ is analytic in $\mathbb{R}$ with respect to θ. Consider the following spiral curve:

$$C := \{(S(\theta)\cos\theta, S(\theta)\sin\theta) \mid \theta \in \mathbb{R}\}$$

Then C is an analytic curve in $\mathbb{R}^2$.



Since

$$C \cap \{y = tx\} \subset \{(x, tx) \mid R(\sqrt{1+t^2}\,|x|; t) = 0\},$$

$C \cap \{y = tx\}$ is a finite set for any given $t \in \mathbb{R}$. However it is clear by the picture of C that $C \cap \{y = tx\}$ is an infinite set. Contradiction! $\square$

Appendix 2. Algebraically integrable domains in $\mathbb{R}^2$.

We consider only domains of the following type:

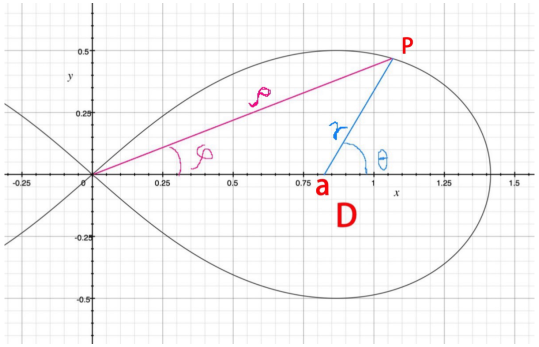
$$D = \{(x, y) \in \mathbb{R}^2 \mid P(x, y)^2 - Q(x, y) < 0\},$$

where $P(x, y)$ is a positive semi-definite second-order homogeneous polynomial, and $Q(x, y)$ is a homogeneous polynomial with degree ≤ 3.

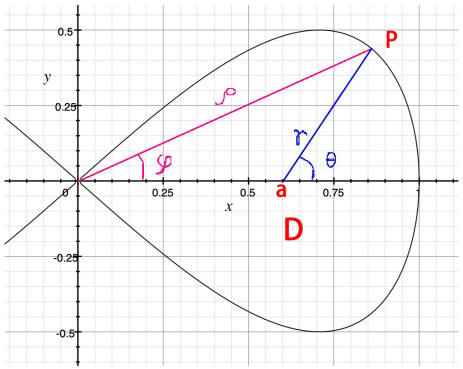
Theorem. If a connected component of D is algebraically integrable, then after a suitable linear coordinate transfomation, D is equal to either one of

$$\{(x^2 + y^2)^2 - 2(x^2 - y^2) < 0\}, \quad \{x^4 - (x^2 - y^2) < 0\}.$$

Bernoulli lemniscate: $(x^2 + y^2)^2 - 2(x^2 - y^2) = 0.$



Gerono (Huygens) lemniscate: $x^4 - (x^2 - y^2) = 0.$



ご清聴ありがとうございました！