Kirby calculus for null-homologous framed links in 3-manifolds

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In this talk, I plan to explain our recent results joint with T. Widmer on Kirby calculus for framed links in 3-manifolds [5, 6].

It is well known that every closed oriented 3-manifold can be obtained from S^3 by surgery along a framed link [8, 11]. Kirby [7] gave a criterion for two framed links in S^3 to give the same (i.e., orientation-preserving homeomorphic) results of surgery using two kinds of moves called *stabilization* and *handle slides*. One can use Kirby's theorem in order to define a 3-manifold invariant, by constructing framed link invariant which is invariant under the Kirby moves. For example, the Reshetikhin–Turaev invariant is constructed in this way.

Fenn and Rourke [3] gave a characterization of the equivalence relation on framed links in a closed oriented 3-manifold generated by Kirby's two types of moves. We generalized this result to 3-manifolds with boundary in [5]. As an application, we proved the following result for *null-homotopic framed links*, which are framed links whose components are null-homotopic.

Theorem 1 ([5]). *For null-homotopic framed links L and L' in a compact oriented 3-manifold M with connected boundary, the following conditions are equivalent.*

1. *L and L' are related by a sequence of stabilizations and handle-slides.*
2. *There exists an orientation-preserving homeomorphism $h\colon M_L \to M_{L'}$ relative to boundary satisfying the following commutative diagram*

$$\begin{array}{ccc} \pi_1(M_L) & \xrightarrow{\quad h_* \quad} & \pi_1(M_{L'}) \\ & {\scriptstyle e}\searrow \quad \swarrow{\scriptstyle e'} & \\ & \pi_1(M) & \end{array}$$

Here the surjective homomorphisms e (resp. e') are defined using the 4-manifold W_L (resp. $W_{L'}$) constructed by adding 2-handles on the cylinder $M \times [0,1]$ along $L \times \{1\}$ (resp. $L' \times \{1\}$).

For a more precise and general statement see [5, Theorem 3.1].

In [6], we considered *null-homologous framed links* in a 3-manifold. Here a framed link L is $\mathbb{Q}$-null-homologous if every component of L is $\mathbb{Q}$-null-homologous in the 3-manifold. Similarly, $\mathbb{Z}$-null-homologous framed links are defined. Based on our generalization of Fenn and Rourke's theorem, we proved the following result.

Theorem 2 ([6]). *Let M be a compact, connected, oriented 3-manifold with non-empty boundary. Let $P \subset \partial M$ be a subset containing exactly one point of each connected component of ∂M. Let L and L' be $\mathbb{Q}$-null-homologous framed links in M. Then the following conditions are equivalent.*

1. *L and L' are related by a sequence of stabilization, handle-slides, "$\mathbb{Q}$-null-homologous K_3-moves", and "IHX-moves".*
2. *There is an orientation-preserving homeomorphism $h\colon M_L \xrightarrow{\cong} M_{L'}$ restricting to the canonical identification $\partial M_L \cong \partial M_{L'}$ such that the following diagram commutes.*

$$\begin{array}{ccc} H_1(M_L, P; \mathbb{Q}) & \xrightarrow{\quad h_* \quad} & H_1(M_{L'}, P; \mathbb{Q}) \\ & {\scriptstyle e_L}\searrow \quad \swarrow{\scriptstyle e_{L'}} & \\ & H_1(M, P; \mathbb{Q}). & \end{array} \tag{1}$$

Here e_L and $e_{L'}$ are defined by using the 4-manifolds W_L and $W_{L'}$, respectively.

See [6, Theorem 1.1] for a more precise statement and the definition of $\mathbb{Q}$-null-homologous K_3-moves and IHX-moves. We note here that the IHX-moves are related to the IHX relations in the theory of finite type invariants of links and 3-manifolds.

We also consider a more special class of framed links, called *admissible framed links*. Here a framed link L is admissible if L is $\mathbb{Z}$-null-homologous and the linking matrix of L is diagonal with diagonal entries ± 1. Surgery along admissible framed links is studied e.g. in [1, 2, 9].

We have the following result.

Theorem 3 ([6]). *Let M and P be as in Theorem 2. Suppose that $H_1(M; \mathbb{Z})$ is torsion-free. Let L and L' be admissible framed links in M. Then the following conditions are equivalent.*

1. *L and L' are related by a sequence of stabilizations, "band-slides", "pair-moves", "admissible IHX-moves" and "lantern-moves".*
2. *There is an orientation-preserving homeomorphism $h\colon M_L \xrightarrow{\cong} M_{L'}$ restricting to the identification map $M_L \cong M_{L'}$ such that the following dia-*

gram commutes.

$$\begin{array}{ccc} H_1(M_L, P; \mathbb{Z}) & \xrightarrow{h_*} & H_1(M_{L'}, P; \mathbb{Z}) \\ & {\scriptstyle e_L} \searrow \quad \swarrow {\scriptstyle e_{L'}} & \\ & H_1(M, P; \mathbb{Z}). & \end{array} \tag{2}$$

See [6, Theorem 1.4] for a more precise statement. The case where $M = S^3$ has been proved in [4], where we do not need pair-moves, admissible IHX-moves and lantern-moves.

In the talk, I plan to discuss also some applications and generalizations of the results.

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