TWISTED ALEXANDER POLYNOMIALS ON CURVES IN CHARACTER VARIETIES OF KNOT GROUPS

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ABSTRACT. For a fibered knot in the 3-sphere the twisted Alexander polynomial associated to an $SL(2,\mathbb{C})$-character is known to be monic. It is conjectured that for a nonfibered knot there is a curve component of the $SL(2,\mathbb{C})$-character variety containing only finitely many characters whose twisted Alexander polynomials are monic, i.e. finiteness of such characters detects fiberedness of knots. In this paper we discuss the existence of a certain curve component which relates to the conjecture when knots have nonmonic Alexander polynomials. We also discuss the similar problem of detecting the knot genus.

1. INTRODUCTION

The twisted Alexander polynomial was introduced by Lin [28] for knots in the 3-sphere and by Wada [37] for finitely presentable groups. It is a generalization of the Alexander polynomial and gives a powerful tool in low dimensional topology. One of the most notable applications is detecting fibered knots or more generally fibered 3-manifolds. To be more precise, Friedl and Vidussi showed in [11] that the twisted Alexander polynomials associated to finite representations determine whether knot complements and general irreducible 3-manifolds are fibered over the circle. Another important application is detecting the genus of knots. More generally, Friedl and Vidussi showed that the twisted Alexander polynomials associated to finite representations also determine the Thurston norms of irreducible 3-manifolds which are not closed graph manifolds [12]. For literature on the twisted Alexander polynomial and other related topics, we refer to the survey paper by Friedl and Vidussi [10].

In this paper we study the problems of detecting fiberedness and the genus $g(K)$ of a knot K by twisted Alexander polynomials from the viewpoint of the $SL(2,\mathbb{C})$-character variety of a knot group. In this point of view, we consider the regular functions on the character variety induced by the coefficients of the twisted Alexander polynomials associated to characters of a knot group. In particular, the regular function induced by the coefficients of the highest degree terms turns out to contain much information of a knot. We call a representation and its character monic if the highest coefficient of the associated twisted Alexander polynomial is one (see [14] for example). Moreover, we say that a representation and its character determines the knot genus if the degree of the associated twisted Alexander polynomial equals $4g(K)-2$. Using these terminologies, we can say that every $SL(2,\mathbb{C})$-representation (and its character) of a fibered knot is monic [13] and determines the knot genus [26].

It is natural to ask whether the converse is true. More precisely, one can ask if every $SL(2,\mathbb{C})$-character is monic for a knot, then the knot is fibered. Regarding detecting the knot genus, a natural question also arises: for a (possibly nonfibered) knot, does there

2010 *Mathematics Subject Classification*. Primary 57M27, Secondary 57M05, 57M25.
Key words and phrases. Twisted Alexander polynomial, character variety, fibered knot.

exist an irreducible $SL(2,\mathbb{C})$-character which determines the knot genus? However, only a few partial answers are known so far (see [31] for twist knots and [24] for 2-bridge knots). In fact, for 2-bridge knots it is shown that certain finiteness properties of a curve component in the character variety detect fiberedness and the genus [24]. More generally, we conjecture that for a nontrivial knot there is an irreducible component in the $SL(2,\mathbb{C})$-character variety which satisfies a certain finiteness condition (see Questions 3.1 and 4.1).

The purpose of the present paper is to give some evidence that the conjecture is true for a wide class of knots with nonmonic Alexander polynomials. More generally, we give several sufficient conditions which ensure the existence of a certain curve component in the character variety which relates to the conjecture mentioned above (see Sections 3 and 4). For instance, in Theorem 3.3 we show that if a knot K has the Alexander polynomial $\Delta_K(t)$ which is nonmonic and has a simple root, then there is a curve component of the $SL(2,\mathbb{C})$-character variety of the knot group of K which contains the character of an irreducible representation and has only finitely many monic characters.

Our criteria are sufficiently applicable and we can show the existence of such curves for all nonfibered prime knots with 10 or fewer crossings. The results stated in this paper use information of the Alexander polynomial, however there seems to be no a priori relation between finiteness properties of a curve component and the Alexander polynomial.

This paper is organized as follows. In Section 2, we quickly review some basic materials of the character variety and the twisted Alexander polynomials associated to $SL(2,\mathbb{C})$-representations. Here we also recall a conjecture of Dunfield, Friedl and Jackson [6] on the twisted Alexander polynomial of the holonomy representation for hyperbolic knots. In Section 3, we show finiteness of monic characters in curve components of the character varieties for a wide class of nonfibered knots. In particular, we show that our method can also be applied to satellite knots (hence nonhyperbolic knots). In Section 4, we apply the argument in Section 3 to the similar problem of detecting the knot genus. In Section 5, as an example, we give explicit computations of curves in the character variety and count the number of monic characters for a Montesinos knot with length 3.

2. PRELIMINARIES

2.1. **Character variety.** In this subsection, we review some basics on character varieties following [4]. Let G be a finitely generated group. The *variety of representations* of G is the set of $SL(2,\mathbb{C})$-representations: $R(G) = \mathrm{Hom}(G, SL(2,\mathbb{C}))$. Since G is finitely generated, $R(G)$ can be embedded in a product $SL(2,\mathbb{C}) \times \cdots \times SL(2,\mathbb{C})$ by mapping each representation to the image of a generating set. In this manner, $R(G)$ is an affine algebraic set whose defining polynomials are induced by the relations of a presentation of G. It is not hard to see that this structure is independent of the choice of presentations of G up to isomorphism.

A representation $\rho\colon G \to SL(2,\mathbb{C})$ is said to be *abelian* if $\rho(G)$ is an abelian subgroup of $SL(2,\mathbb{C})$. A representation ρ is called *reducible* if there exists a proper invariant subspace in $\mathbb{C}^2$ under the action of $\rho(G)$. This is equivalent to saying that ρ can be conjugated to a representation by upper triangular matrices. It is easy to see that every abelian representation is reducible, but the converse does not hold. Namely there is a reducible nonabelian representation in general. When ρ is not reducible, it is called *irreducible*. We denote the subset of $R(G)$ consisting of irreducible $SL(2,\mathbb{C})$-representations by $R^{\mathrm{irr}}(G)$.

Given a representation $\rho \in R(G)$, its *character* is the map $\chi_\rho\colon G \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \mathrm{tr}\,(\rho(\gamma))$ for $\gamma \in G$. We denote the set of all characters by $X(G)$. For a given element $\gamma \in G$, we define the map $\tau_\gamma : X(G) \to \mathbb{C}$ by $\tau_\gamma(\chi) = \chi(\gamma)$. Then it is known that $X(G)$ is an affine algebraic set which embeds in $\mathbb{C}^N$ with coordinates $(\tau_{\gamma_1}, \ldots, \tau_{\gamma_N})$ for some $\gamma_1, \ldots, \gamma_N \in G$. This affine algebraic set is called the *character variety* of G. We note that the set $\{\gamma_1, \ldots, \gamma_N\}$ can be chosen to contain a generating set of G. The projection $t\colon R(G) \to X(G)$ given by $t(\rho) = \chi_\rho$ is surjective. We denote the Zariski closure of $t(R^{\mathrm{irr}}(G))$ by $X^{\mathrm{irr}}(G)$.

Let $E_K = S^3 \backslash \mathrm{int}(N(K))$, the exterior of a knot K in the 3-sphere. For a knot group $G(K) = \pi_1 E_K$, we write $R(K) = R(G(K))$, $R^{\mathrm{irr}}(K) = R^{\mathrm{irr}}(G(K))$, $X(K) = X(G(K))$ and $X^{\mathrm{irr}}(K) = X^{\mathrm{irr}}(G(K))$ for simplicity.

Let $\eta_\lambda\colon G(K) \to SL(2,\mathbb{C})$ be the abelian representation given by $\mu \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where μ denotes a meridian of K. By results of Burde [1] and de Rham [5], there is a reducible nonabelian representation $\rho_\lambda\colon G(K) \to SL(2,\mathbb{C})$ such that $\chi_{\eta_\lambda} = \chi_{\rho_\lambda}$ if and only if λ^2 is a root of the Alexander polynomial $\Delta_K(t)$.

Here let us recall the following results, due to Heusener, Porti and Suárez Peiró [19] (see also Shors [36]), and due to Herald [17] and Heusener-Kroll [18], on the local description of a reducible character in $X^{\mathrm{irr}}(K)$. We denote by $Y(K)$ the curve component of $X(K)$ consisting of abelian characters.

Proposition 2.1. [19, Theorem 1.2] *If λ^2 is a simple root of $\Delta_K(t)$, there is a unique irreducible curve component X_0 of $X^{\mathrm{irr}}(K)$ such that $\chi_{\rho_\lambda} \in X_0 \cap Y(K)$.*

The following proposition is an immediate consequence of [17, Theorem 1] or [18, Theorem 1.1].

Proposition 2.2. *If the equivariant knot signature function $\sigma_K\colon U(1) \to \mathbb{Z}$ changes its value at $e^{2i\alpha}$ $(\alpha \in [0,\pi])$, then there is an irreducible component X_0 of $X^{\mathrm{irr}}(K)$ such that $\chi_{\rho_{e^{i\alpha}}} \in X_0 \cap Y(K)$.*

The equivariant knot signature function σ_K is also called the Levine-Tristram signature function (for example, see [18, Section 2.1] for the details). We note that in this paper the signature function σ_K is considered to be the *averaged signature function*. Namely, for $\omega \in U(1) = S^1$ the value $\sigma_K(\omega)$ is redefined to be the limit of the average of the values $\sigma_K(\omega_+)$ and $\sigma_K(\omega_-)$ where ω_+ and ω_- are points on S^1 approaching ω from opposite sides. It is known that $\sigma_K(1) = 0$, and the function σ_K is locally constant except at zeros of $\Delta_K(t)$. It is also known that if $e^{2i\alpha}$ is an odd multiple root of $\Delta_K(t)$, then σ_K changes its value at $e^{2i\alpha}$.

2.2. **Twisted Alexander polynomials.** Following Wada [37], we define the twisted Alexander polynomials. First we fix an epimorphism $\alpha\colon G(K) \to \langle t \rangle$ and a Wirtinger presentation

$$G(K) = \langle \gamma_1, \ldots, \gamma_n \,|\, r_1, \ldots, r_{n-1} \rangle.$$

For a given representation $\rho\colon G(K) \to GL(2,\mathbb{C})$, we can extend the group homomorphism $\alpha \otimes \rho\colon G(K) \to GL(2, \mathbb{C}[t^{\pm 1}])$ to a ring homomorphism $\Phi\colon \mathbb{Z}[G(K)] \to M(2, \mathbb{C}(t))$.

We consider the $(n-1) \times n$ matrix M whose (i,j)-entry is $\frac{\partial r_i}{\partial \gamma_j} \in \mathbb{Z}[G(K)]$, where $\frac{\partial}{\partial \gamma_j}$ denotes the Fox differential. For $1 \leq k \leq n$, we denote by M_k the $(n-1) \times (n-1)$ matrix obtained from M by removing the kth column, and by $\Phi(M_k)$ the matrix in $M(2(n-1), \mathbb{C}(t))$ obtained by taking the images of entries of M_k by Φ. Then the *twisted Alexander polynomial* $\Delta_{K,\rho}(t) \in \mathbb{C}(t)$ *associated with* $\rho\colon G(K) \to GL(2,\mathbb{C})$ is defined as

$$\Delta_{K,\rho}(t) = \frac{\det \Phi(M_k)}{\det \Phi(\gamma_k - 1)},$$

which is well-defined up to multiplication by ϵt^{2i} ($\epsilon \in \mathbb{C}^*$, $i \in \mathbb{Z}$). In the case that ρ is a nonabelian special linear representation $\rho\colon G(K) \to SL(2,\mathbb{C})$, we have $\Delta_{K,\rho}(t) \in \mathbb{C}[t^{\pm 1}]$ [26, Theorem 3.1] and it is well-defined up to multiplication by t^{2i} ($i \in \mathbb{Z}$). We note that if ρ and η are mutually conjugate $SL(2,\mathbb{C})$-representations, then $\Delta_{K,\rho}(t) = \Delta_{K,\eta}(t)$ holds. If ρ and $\eta\colon G(K) \to SL(2,\mathbb{C})$ are irreducible representations with $\chi_\rho = \chi_\eta$, then ρ is conjugate to η (see [4, Proposition 1.5.2]), and hence $\Delta_{K,\rho}(t) = \Delta_{K,\eta}(t)$. And if ρ and η are reducible, then they are determined by $\Delta_K(t)$ and hence $\Delta_{K,\rho}(t) = \Delta_{K,\eta}(t)$ (see the proof of [26, Theorem 3.1]). Therefore, we can define the twisted Alexander polynomial associated with $\chi \in X(K)$ to be $\Delta_{K,\rho}(t)$ where $\chi = \chi_\rho$ and we denote it by $\Delta_{K,\chi}(t)$.

2.3. **A conjecture of Dunfield, Friedl and Jackson.** We say a nonabelian representation $\rho\colon G(K) \to SL(2,\mathbb{C})$ (resp. a nonabelian character χ) is *monic* if $\Delta_{K,\rho}(t)$ (resp. $\Delta_{K,\chi}(t)$) is a monic polynomial, that is, its coefficient of the highest degree term is 1. Note that we do not allow -1 as the coefficient for monicness in this paper. It is well-known that every nonabelian representation of a fibered knot is monic [13, Theorem 3.1], and therefore so is every nonabelian character.

It is also well-known that the degrees of twisted Alexander polynomials give genus bounds from below [8]. In particular, in our settings we have

$$4g(K) - 2 \geq \deg \Delta_{K,\chi}(t),$$

for every $\chi \in X(K)$. If the equality holds, then we say that χ determines the genus. It is known that for any $\chi \in X(K)$ of a fibered knot, χ determines the genus [26, Theorem 3.2].

For a hyperbolic knot K, the holonomy representation $\bar{\rho}_0\colon G(K) \to PSL(2,\mathbb{C})$ has a lift $\rho_0\colon G(K) \to SL(2,\mathbb{C})$, see [4, Proposition 3.1.1] for the detail. Dunfield, Friedl and Jackson [6] presented the following conjecture, and confirmed it for all hyperbolic knots of 15 or fewer crossings. See also [6, Conjecture 1.13].

Conjecture 2.3. [6, Conjecture 1.7] *Let K be a hyperbolic knot and $\rho_0\colon G(K) \to SL(2,\mathbb{C})$ a lift of the holonomy representation. Then K is fibered if and only if χ_{ρ_0} is monic. Moreover, χ_{ρ_0} determines the knot genus $g(K)$.*

Recently, the third author confirmed the conjecture for twist knots [32], and the third author and Tran did for a certain wider class of 2-bridge knots [33]. These are the first infinite families of knots where Conjecture 2.3 is verified.

It is known that we can write the twisted Alexander polynomial $\Delta_{K,\chi}(t)$ without any ambiguity as

$$\Delta_{K,\chi}(t) = \sum_{j=0}^{4g-2} \psi_j(\chi) t^j.$$

with $\mathbb{C}$-valued functions ψ_j on $X^{\mathrm{irr}}(K)$ such that $\psi_k = \psi_{4g-2-k}$ ($0 \leq k \leq 2g-1$) where $g = g(K)$ (see [9, Theorem 1.5] and its proof). Then for a subvariety X_0 of $X^{\mathrm{irr}}(K)$ we say that ψ_n is *the coefficient of the highest degree term of $\Delta_{K,\chi}(t)$ on X_0* if $\psi_m \equiv 0$ for $m > n$ and $\psi_n \not\equiv 0$ on X_0. We end this section with the following useful proposition.

Proposition 2.4. *Let K be a knot of genus g and χ a character in a curve component X_0 of $X^{\mathrm{irr}}(K)$. We write the twisted Alexander polynomial $\Delta_{K,\chi}(t)$ as above:*

$$\Delta_{K,\chi}(t) = \sum_{j=0}^{4g-2} \psi_j(\chi) t^j.$$

Then each ψ_k is a regular function on X_0, and if $\psi_k \not\equiv c$ on X_0 for a constant $c \in \mathbb{C}$, then $\psi_k = c$ for finitely many points of X_0.

Proof. It follows from [6, Theorem 1.5] that ψ_k is a regular function. Therefore $\psi_k^{-1}(c)$ is a subvariety of the curve X_0 of codimension one. In particular, it consists of finitely many points. (Also see [6, Corollary 1.6].) □

3. FIBERING AND MONIC CHARACTERS

In this section we present some classes of nonfibered knots for which the following question is affirmatively solved, in particular, on a curve component.

Question 3.1. *For a nonfibered knot K, is there an irreducible component of $X^{\mathrm{irr}}(K)$ with finitely many monic characters?*

Combined with Proposition 2.4, Conjecture 2.3 implies an affirmative answer to Question 3.1 for hyperbolic knots since the characters of lifts of the holonomy representations are known to be contained in unique curve component. This unique curve component of a hyperbolic knot is called the *canonical component*. The third author solved the question affirmatively for twist knots [31], and later the first and third authors did for 2-bridge knots [24] in a strong sense. More precisely, the following theorem holds from [24, Theorem 4.3, Remark 2.3]. Let $X_0 \subset X^{\mathrm{irr}}(K)$ be an irreducible component. We say X_0 satisfies *Property* (F) if X_0 contains finitely many monic characters and an abelian character.

Theorem 3.2. [24, Theorem 4.3] *For a nonfibered 2-bridge knot, there is a curve component of $X^{\mathrm{irr}}(K)$ satisfying Property (F).*

At the present, except some special cases, we do not know if the curve component appeared in Theorem 3.2 is the canonical one (see [24, Theorem 4.6, Remark 6.5]).

Our first theorem in this paper is the following.

Theorem 3.3. *Let K be a knot such that $\Delta_K(t)$ is nonmonic and has a simple root. Then there is a curve component of $X^{\mathrm{irr}}(K)$ satisfying Property (F).*

Proof. By Proposition 2.1, there is a reducible character χ_{ρ_λ} and an irreducible curve component X_λ of $X^{\mathrm{irr}}(K)$ such that $\Delta_K(\lambda^2) = 0$ and $\chi_{\rho_\lambda} \in X_\lambda \cap Y(K)$. Since the twisted Alexander polynomial associated with ρ_λ is given by

$$\Delta_{K,\rho_\lambda}(t) = \frac{\Delta_K(\lambda t)\Delta_K(\lambda^{-1}t)}{(t-\lambda)(t-\lambda^{-1})}$$

(see [24, Remark 3.1 (iv)]), $\Delta_{K,\rho_\lambda}(t)$ is monic if and only if $\Delta_K(t)$ is monic. Hence, by the assumption, the coefficient of the highest degree term of the twisted Alexander polynomial on X_λ is not the constant one and the assertion follows from Proposition 2.4. This completes the proof. □

As an immediate corollary, if $\Delta_K(t)$ is irreducible over $\mathbb{Q}$ and nonmonic, the knot K satisfies the assumption of Theorem 3.3. It is known that a prime knot K of 10 or fewer crossings is fibered if and only if $\Delta_K(t)$ is monic. Then it can be checked that for all non 2-bridge and nonfibered prime knots with 10 or fewer crossings their Alexander polynomials have a simple root, although they might have nontrivial multiple factors in $\Delta_K(t)$. Hence, by Theorems 3.2 and 3.3, nonfibered prime knots with 10 or fewer crossings have curve components which satisfy Property (F).

Theorem 3.3 can also be applied to satellite knots, and it shows that Question 3.1 also makes sense for nonhyperbolic knots. Let $\tilde{K}$ be a knot embedded in a standard solid torus $\tilde{V} = S^1 \times D^2 \subset S^3$. We assume that $\tilde{K}$ is not isotopic to $S^1 \times \{0\}$ nor is contained in any 3-ball in $\tilde{V}$. Let h be a homeomorphism from $\tilde{V}$ onto a closed tubular neighborhood of a nontrivial knot $\hat{K}$ which maps a longitude of $\tilde{V}$ onto a longitude of $\hat{K}$. The image $K = h(\tilde{K})$ is called a *satellite knot* with *companion knot* $\hat{K}$ and *pattern* $(\tilde{V}, \tilde{K})$. The *winding number* of $\tilde{K}$ in $\tilde{V}$ is the nonnegative integer n such that the homomorphism $H_1\tilde{K} \to H_1\tilde{V} \cong \mathbb{Z}$ induced by the inclusion has the image $n\mathbb{Z}$. Under the notations above, it is known that the Alexander polynomial of a satellite knot K, with pattern $\tilde{K}$, companion $\hat{K}$ and the winding number n satisfies the following:

$$\Delta_K(t) = \Delta_{\tilde{K}}(t) \cdot \Delta_{\hat{K}}(t^n).$$

Hence, by Theorem 3.3, for a satellite knot K with pattern $\tilde{K}$ and the *winding number zero* such that $\Delta_{\tilde{K}}(t)$ is nonmonic and has a simple root, there is a curve component satisfying Property (F).

Recall that a knot K is called *small* if the exterior E_K contains no closed embedded essential surface. It is known that torus knots [23], 2-bridge knots [15, 16] and Montesinos knots with length 3 [34, Corollary 4] are small. It is also known that some knots of braid index 3 or 4 are small (see [7] and [30]).

Theorem 3.4. *Let K be a small knot such that $\Delta_K(t)$ is nonmonic. If the equivariant knot signature function σ_K is not identically zero, then there is a curve component of $X^{\text{irr}}(K)$ satisfying Property (F).*

Proof. Suppose that σ_K changes its value at $e^{2i\alpha}$. By Proposition 2.2, there is a reducible character $\chi_{\rho_{e^{i\alpha}}}$ and an irreducible component X_0 of $X^{\text{irr}}(K)$ such that $\chi_{\rho_{e^{i\alpha}}} \in X_0 \cap Y(K)$. Since K is small, X_0 is a curve [3, Section 2.4]. Now the similar argument as in the proof of Theorem 3.3 can work in this setting. □

Theorem 3.5. *Let K be a small knot such that $\Delta_K(t)$ is nonmonic. If there are a knot K' and an epimorphism $\phi\colon G(K) \to G(K')$ such that there is a component X_0' of $X^{\text{irr}}(K')$ satisfying Property (F), then there is a curve component X_0 of $X^{\text{irr}}(K)$ satisfying Property (F).*

Proof. It is straightforward to see that the regular map $\phi^*\colon X(K') \to X(K)$ induced by the epimorphism ϕ is injective. We set X_0 to be the image $\phi^*(X_0')$. It is a curve component of $X^{\text{irr}}(K)$ containing an abelian character since K is small. Moreover, the composition of an irreducible representation and an epimorphism of groups is also irreducible.

An abelian character in X_0 can be written as χ_{η_λ} by an abelian representation η_λ. Since $\Delta_K(t)$ is not monic, neither is $\Delta_{K,\chi_{\eta_\lambda}}(t)$. Hence the coefficient of the highest degree term of $\Delta_{K,\chi}(t)$ on X_0 is not the constant one. Now the theorem follows from Proposition 2.4. □

The following corollary is an immediate consequence of Theorems 3.2, 3.3, 3.4 and 3.5.

Corollary 3.6. *Let K be a small knot. If there are a knot K' satisfying one of the following:*

(i) K' is a nonfibered 2-bridge knot,
(ii) $\Delta_{K'}(t)$ is nonmonic and has a simple root,
(iii) $\Delta_{K'}(t)$ is nonmonic and $\sigma_{K'}$ is not identically zero,

and an epimorphism $\phi\colon G(K) \to G(K')$, then there is a curve component X_0 of $X^{\text{irr}}(K)$ which satisfies Property (F).

It is well-known that if there is an epimorphism $\phi\colon G(K) \to G(K')$, then $\Delta_{K'}(t)$ divides $\Delta_K(t)$. Moreover, it is also known that $\Delta_{K'}(t)$ is nonmonic for a nonfibered 2-bridge knot K'. Hence, in the above theorem, it turns out that $\Delta_K(t)$ is nonmonic.

From [25, Section 8.2] (see also [22], [35, Section 9]), we see that for a given 2-bridge knot K' there exists a Montesinos knot K with length 3 such that $G(K)$ admits an epimorphism to $G(K')$. Namely there are infinitely many small knots which satisfy the assumption of Corollary 3.6.

4. Detecting genus

In this section we consider the following analogous question to Question 3.1 on detecting the knot genus.

Question 4.1. *For a nontrivial knot K, is there a component of $X^{irr}(K)$ where all but finitely many characters determine the knot genus?*

As was mentioned in the introduction, every character of a fibered knot determines the knot genus. Hence, Question 4.1 has an obvious positive answer for hyperbolic fibered knots since the canonical components satisfy the condition. Moreover Conjecture 2.3 implies an affirmative answer to the question for hyperbolic nonfibered knots. The first and third authors solved the question affirmatively for 2-bridge knots [24] in a strong sense. Let $X_0 \subset X^{\mathrm{irr}}(K)$ be an irreducible component. We say X_0 satisfies *Property* (G) if all but finitely many characters in X_0 determine the knot genus and X_0 contains an abelian character.

Theorem 4.2. [24, Theorem 4.4] *For a 2-bridge knot, there is a curve component of $X^{irr}(K)$ which satisfies Property (G).*

As before we do not know if the curve component appeared in Theorem 4.2 is the canonical one. Furthermore it is nontrivial whether there is a curve component which satisfies Property (G) even for a hyperbolic fibered knot.

Analogous arguments for Theorems 3.3, 3.4, 3.5 and Corollary 3.6 present an affirmative answer to Question 4.1 for similar classes of knots.

Theorem 4.3. *Let K be a knot with $\deg \Delta_K(t) = 2g(K)$ and $\Delta_K(t)$ has a simple root. Then there is a curve component X_0 of $X^{irr}(K)$ which satisfies Property (G).*

Proof. As in the proof of Theorem 3.3, by Proposition 2.1 there is a curve component X_λ of $X^{\mathrm{irr}}(K)$ containing a reducible character χ_{ρ_λ} such that

$$\Delta_{K,\chi_{\rho_\lambda}}(t) = \frac{\Delta_K(\lambda t)\Delta_K(\lambda^{-1}t)}{(t-\lambda)(t-\lambda^{-1})},$$

which determines the genus by the assumption. Hence the coefficient of the highest degree term of $\Delta_{K,\chi}(t)$ on X_λ is not identically zero, and the theorem follows from Proposition 2.4. □

By Theorems 4.2, 4.3 and an analogous argument to Property (F) we can check that all prime knots with 10 or fewer crossings, except the following seven 3-bridge knots, have curve components which satisfy Property (G):

$$\begin{aligned}
&\Delta_{8_{10}}(t) = \Delta_{10_{143}}(t) = (t^2 - t + 1)^3 \\
&\Delta_{8_{20}}(t) = \Delta_{10_{140}}(t) = (t^2 - t + 1)^2 \\
&\Delta_{10_{99}}(t) = (t^2 - t + 1)^4 \\
&\Delta_{10_{123}}(t) = (t^4 - 3t^3 + 3t^2 - 3t + 1)^2 \\
&\Delta_{10_{137}}(t) = (t^2 - 3t + 1)^2.
\end{aligned}$$

Note that it is known that for all prime knots with 10 or fewer crossings, $\deg \Delta_K(t) = 2g(K)$.

The knots $8_{10}, 8_{20}, 10_{137}$, 10_{140} and 10_{143} are known to be Montesinos knots with length 3, hence they are small. On the other hand, the knots 10_{99} and 10_{123} are not small. In fact, we can construct essential surfaces in the exteriors as follows: The knots 10_{99} and 10_{123} are obtained as the closure of the 3-braids

$$\sigma_1^3\sigma_2^2\sigma_1^2\sigma_2^{-3}\sigma_1^{-2}\sigma_2^{-2},\ \sigma_1^{-3}\sigma_2^{-2}\sigma_1^2\sigma_2^3\sigma_1^2\sigma_2^{-2}.$$

Spheres with 3 holes separating σ_1 and σ_2 give tangle decompositions of the knots. Connecting 2 of such spheres by 3 tubes along the strands of the braids, we obtain embedded surfaces of genus 2, which can be checked to be essential. This construction is based on [29, Theorem 3.2]. One can also check that the knots 10_{99} and 10_{123} are not small in the list given in [2].

Theorem 4.4. *Let K be a small knot such that $\deg \Delta_K(t) = 2g(K)$. If the equivariant knot signature function σ_K is not identically zero, then there is a curve component of $X^{irr}(K)$ which satisfies Property (G).*

Proof. Suppose that σ_K changes its value at $e^{2i\alpha}$. By Proposition 2.2, there is a reducible character $\chi_{\rho_{e^{i\alpha}}}$ and an irreducible component X_0 of $X^{\mathrm{irr}}(K)$ such that $\chi_{\rho_{e^{i\alpha}}} \in X_0 \cap Y(K)$. Since K is small, X_0 is a curve. Now the same proof as that of Theorem 4.3 works. □

The Alexander polynomials $\Delta_K(t)$ of the knots 8_{10} and 10_{143} have no simple root, but we can check that their equivariant knot signature functions σ_K are not identically zero. Hence, by Theorem 4.4, these two knots have curve components which satisfy Property (G).

Theorem 4.5. *Let K be a small knot with $\deg \Delta_K(t) = 2g(K)$. If there are a knot K' and an epimorphism $\phi\colon G(K) \to G(K')$ such that there is a component X_0' of $X^{irr}(K')$ satisfying Property (G), then there is a curve component X_0 of $X^{irr}(K)$ which satisfies Property (G).*

Proof. We set X_0 to be the image of X_0' of the injection $X(K') \to X(K)$ induced by ϕ. Then X_0 is a curve component of $X^{\mathrm{irr}}(K)$ containing an abelian character χ_{η_λ} as in the proof of Theorem 3.5. Since $\deg \Delta_K(t) = 2g(K)$, the character χ_{η_λ} determines the genus. Hence the coefficient of the highest degree term of $\Delta_{K,\chi}(t)$ on X_0 is not identically zero, which proves the theorem by Proposition 2.4. □

The equivariant knot signature functions σ_K of the knots $8_{20}, 10_{137}$ and 10_{140} are identically zero, but they admit the following epimorphisms to 2-bridge knot groups (see [27, Theorem 1.1]):

$$G(8_{20}) \twoheadrightarrow G(3_1),\ G(10_{137}) \twoheadrightarrow G(4_1),\ G(10_{140}) \twoheadrightarrow G(3_1).$$

Therefore, by Theorems 4.2 and 4.5, these three knots have curve components which satisfy Property (G).

More generally, we obtain the following as an immediate corollary of Theorems 4.2, 4.3, 4.4 and 4.5.

Corollary 4.6. *Let K be a small knot with $\deg \Delta_K(t) = 2g(K)$. If there are a knot K' satisfying one of the following:*

(i) K' is a 2-bridge knot,

(ii) $\deg \Delta_{K'}(t) = 2g(K')$ *and* $\Delta_{K'}(t)$ *has a simple root,*
(iii) $\deg \Delta_{K'}(t) = 2g(K')$ *and* $\sigma_{K'}$ *is not identically zero,*

and an epimorphism $\phi\colon G(K) \to G(K')$*, then there is a curve component* X_0 *of* $X^{\mathrm{irr}}(K)$ *which satisfies Property* (G)*.*

The Alexander polynomials $\Delta_K(t)$ of the remained knots 10_{99} and 10_{123} have no simple root and their equivariant knot signature functions σ_K are identically zero. By [27, Theorem 1.1] the knot group $G(10_{99})$ admits an epimorphism to $G(3_1)$, but 10_{99} is not small. Moreover $G(10_{123})$ admits no epimorphism to the groups of knots of fewer crossings [27]. Therefore we can say nothing about the existence of curve components which satisfy Property (G) for these knots.

5. EXAMPLE

Let K be the knot 9_{35}, which is a nonfibered alternating knot of genus 1 with $\Delta_K(t) = 7t^2 - 13t + 7$. As in Figure 1, the knot K is the $(-3, -3, -3)$ pretzel knot which is a Montesinos knot with length 3, and so K is a small knot. By Theorem 3.3 there is a curve component of $X^{\mathrm{irr}}(K)$ with finitely many monic characters, and by Theorem 4.3 there is also one where all but finitely many characters determine the genus. Here we explicitly give such curve components.

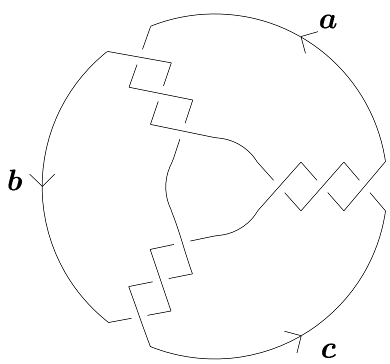

FIGURE 1. The knot 9_{35}

The knot K has a period 3, as is easy to see in Figure 1. In fact, K is the inverse image of one unknotted component of the 2-bridge link $12/5$ by the 3-fold branched covering map $S^3 \to S^3$ whose branching set is the other unknotted component. First, using the method of Hilden, Lozano and Montesinos-Amilibia [20, 21], we compute defining equations of the curve components of $X(K)$ which come from the character variety of the orbifold fundamental group of the quotient orbifold by the periodicity. By [20, Proposition 5.3] the nontrivial components of the character variety of the link $12/5$ are defined by

$r_6(y_1, y_2, v) = 0$, where $r_6(y_1, y_2, v)$ is inductively defined by

$$
\begin{aligned}
r_m(y_1, y_2, v) =& - y_1^{-1} y_2 t_{m-1} r_{m-3}(y_2, y_1, v) \\
&+ \frac{1}{2}(-2 + y_2^2 + y_1^{-1} y_2 t_{m-1}(2v - y_1 y_2)) r_{m-2}(y_1, y_2, v) \\
&+ \frac{1}{2}(-2y_1^{-1} y_2 t_{m-1} + y_1 y_2 t_{m-1} + 2v - y_1 y_2) r_{m-1}(y_2, y_1, v), \\
r_0(y_1, y_2, v) =& 0,\ r_1(y_1, y_2, v) = 1,\ r_2(y_1, y_2, v) = v, \\
t_1 =& 1,\ t_2 = -1,\ t_3 = 1,\ t_4 = -1,\ t_5 = 1.
\end{aligned}
$$

Here y_1, y_2 are the trace functions of two standard generators of the 2-bridge link group and v is the trace function of the product of these generators. A computation implies

$$r_6(y_1, y_2, v) = (v^2 - vy_1y_2 + y_1^2 + y_2^2 - 3)(v^3 - v^2 y_1 y_2 + vy_1^2 + vy_2^2 - v - y_1 y_2).$$

By [21, Theorem 3.1] it follows from the equation $r_6(y_1, y_2, v) = 0$ that $X(K)$ contains nontrivial curve components defined by

$$
\begin{aligned}
f(y, b, w) &= 0, \\
(b + 2)(wy - b - z) - w^2 &= 0, \\
b + 1 &= 0,
\end{aligned}
$$

where $f(y, b, w)$ is the polynomial obtained from $y_1^5 r_6(y_1, y_2, v)$ by the following change of variables:

$$
\begin{aligned}
y &= y_2, \\
b &= y_1^2 - 2, \\
w &= y_1 v.
\end{aligned}
$$

By substituting -1 for b, the equations become

$$
\begin{aligned}
(w^2 - wy + y^2 - 2)(w^3 - w^2 y + wy^2 - y) &= 0, \\
w^2 - wy + z - 1 &= 0.
\end{aligned}
$$

Taking the resultant in w, we have

$$(y^2 - z - 1)^2 (y^4 z - 2y^4 - 2y^2 z^2 + 5y^2 z - 2y^2 + z^3 - 3z^2 + 3z - 1) = 0,$$

where y, z are the trace functions of a meridian, and the product of it and its image by the periodic map, respectively. We denote by C, C' the curves defined by

$$
\begin{aligned}
y^2 - z - 1 &= 0, \\
y^4 z - 2y^4 - 2y^2 z^2 + 5y^2 z - 2y^2 + z^3 - 3z^2 + 3z - 1 &= 0,
\end{aligned}
$$

respectively.

Next we compute the restrictions of the regular function ψ_2 to C, C' induced by the highest degree terms of twisted Alexander polynomials as in Proposition 2.4. Taking the meridional elements a, b, c depicted in Figure 1, we have $G(K) = \langle a, b, c \mid r, s \rangle$, where

$$
\begin{aligned}
r &= a\bar{b}ab\bar{a}b\bar{c}b\bar{c}\bar{b}c\bar{b}, \\
s &= b\bar{c}bc\bar{b}c\bar{a}c\bar{a}\bar{c}a\bar{c}.
\end{aligned}
$$

Here we write $\bar{a}, \bar{b}, \bar{c}$ for a^{-1}, b^{-1}, c^{-1} respectively. Note that

$$y = \tau_a = \tau_b = \tau_c.$$

Since C, C' are symmetric with respect to the periodicity of $X(K)$ induced by that of K,

$$z = \tau_{ab} = \tau_{bc} = \tau_{ca}$$

on these curves. Since

$$\begin{aligned}
\frac{\partial r}{\partial a} =& 1 + a\bar{b} - a\bar{b}ab\bar{a}, \\
\frac{\partial r}{\partial b} =& - a\bar{b} + a\bar{b}a + a\bar{b}ab\bar{a} + a\bar{b}ab\bar{a}b\bar{c} - a\bar{b}ab\bar{a}b\bar{c}b\bar{c}\bar{b} - a\bar{b}ab\bar{a}b\bar{c}b\bar{c}\bar{b}cb, \\
\frac{\partial s}{\partial a} =& - b\bar{c}bc\bar{b}c\bar{a} - b\bar{c}bc\bar{b}c\bar{a}c\bar{a} + b\bar{c}bc\bar{b}c\bar{a}c\bar{a}\bar{c}, \\
\frac{\partial s}{\partial b} =& 1 + b\bar{c} - b\bar{c}bc\bar{b},
\end{aligned}$$

for an $SL(2,\mathbb{C})$-representation ρ,

$$\Delta_{K,\rho}(t) = \frac{\det(At+B)}{\det(\rho(c)t - I)},$$

where I denotes the 2×2 identity matrix and

$$\begin{aligned}
A =& \begin{pmatrix} -\rho(a\bar{b}ab\bar{a}) & \rho(a\bar{b}a) + \rho(a\bar{b}ab\bar{a}) + \rho(a\bar{b}ab\bar{a}b\bar{c}) \\ -\rho(b\bar{c}bc\bar{b}c\bar{a}) - \rho(b\bar{c}bc\bar{b}c\bar{a}c\bar{a}) & -\rho(b\bar{c}bc\bar{b}) \end{pmatrix}, \\
B =& \begin{pmatrix} I + \rho(a\bar{b}) & \rho(a\bar{b}) - \rho(a\bar{b}ab\bar{a}b\bar{c}b\bar{c}\bar{b}) - \rho(a\bar{b}ab\bar{a}b\bar{c}b\bar{c}\bar{b}cb) \\ \rho(b\bar{c}bc\bar{b}c\bar{a}c\bar{a}\bar{c}) & I + \rho(b\bar{c}) \end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
\psi_2(\chi_\rho) =& \det A \\
=& \det \begin{pmatrix} -\rho(a\bar{b}ab\bar{a}) & \rho(a\bar{b}a)+\rho(a\bar{b}ab\bar{a})+\rho(a\bar{b}ab\bar{a}b\bar{c}) \\ 0 & -\rho(b\bar{c}bc\bar{b}c\bar{a})(\rho(c\bar{a}b\bar{c})+\rho(a\bar{b})+\rho(a\bar{c})+\rho(b\bar{c})+\rho(c\bar{a})+\rho(c\bar{b})+I)) \end{pmatrix} \\
=& \det(\rho(c\bar{a}b\bar{c}) + \rho(a\bar{b}) + \rho(a\bar{c}) + \rho(b\bar{c}) + \rho(c\bar{a}) + \rho(c\bar{b}) + I) \\
=& \frac{1}{2}((\mathrm{tr}\,(\rho(c\bar{a}b\bar{c}) + \rho(a\bar{b}) + \rho(a\bar{c}) + \rho(b\bar{c}) + \rho(c\bar{a}) + \rho(c\bar{b}) + I))^2 \\
&- \mathrm{tr}\,((\rho(c\bar{a}b\bar{c}) + \rho(a\bar{b}) + \rho(a\bar{c}) + \rho(b\bar{c}) + \rho(c\bar{a}) + \rho(c\bar{b}) + I)^2)) \\
=& 2(\mathrm{tr}\,\rho(a\bar{b}))^2 + 2(\mathrm{tr}\,\rho(b\bar{c}))^2 + 2(\mathrm{tr}\,\rho(c\bar{a}))^2 \\
&+ 4\mathrm{tr}\,\rho(a\bar{b})\mathrm{tr}\,\rho(b\bar{c}) + 4\mathrm{tr}\,\rho(b\bar{c})\mathrm{tr}\,\rho(c\bar{a}) + 4\mathrm{tr}\,\rho(c\bar{a})\mathrm{tr}\,\rho(a\bar{b}) \\
&- \mathrm{tr}\,\rho(a\bar{b}c\bar{a}b\bar{c}) - \mathrm{tr}\,\rho(a\bar{b}a\bar{b}) - \mathrm{tr}\,\rho(b\bar{c}b\bar{c}) - \mathrm{tr}\,\rho(c\bar{a}c\bar{a}) \\
&- 2\mathrm{tr}\,\rho(a\bar{b}a\bar{c}) - 2\mathrm{tr}\,\rho(b\bar{c}b\bar{a}) - 2\mathrm{tr}\,\rho(c\bar{a}c\bar{b}) - 3 \\
=& (2\tau_{a\bar{b}}^2 + 2\tau_{b\bar{c}}^2 + 2\tau_{c\bar{a}}^2 + 4\tau_{a\bar{b}}\tau_{b\bar{c}} + 4\tau_{b\bar{c}}\tau_{c\bar{a}} + 4\tau_{c\bar{a}}\tau_{a\bar{b}} - \tau_{a\bar{b}c\bar{a}b\bar{c}} \\
&- \tau_{a\bar{b}a\bar{b}} - \tau_{b\bar{c}b\bar{c}} - \tau_{c\bar{a}c\bar{a}} - 2\tau_{a\bar{b}a\bar{c}} - 2\tau_{b\bar{c}b\bar{a}} - 2\tau_{c\bar{a}c\bar{b}} - 3)(\chi_\rho).
\end{aligned}$$

Here by trace identities we have

$$\begin{aligned}
\tau_{a\bar{b}} = \tau_{b\bar{c}} = \tau_{c\bar{a}} &= x, \\
\tau_{a\bar{b}a\bar{b}} = \tau_{b\bar{c}b\bar{c}} = \tau_{c\bar{a}c\bar{a}} &= x^2 - 2, \\
\tau_{a\bar{b}a\bar{c}} = \tau_{b\bar{c}b\bar{a}} = \tau_{c\bar{a}c\bar{b}} &= x^2 - x, \\
\tau_{a\bar{b}c\bar{a}b\bar{c}} &= -x^3 + 3x^2 - 2
\end{aligned}$$

on C, C', where we set $x = y^2 - z$. Consequently, we obtain

$$\psi_2 = x^3 + 6x^2 + 6x + 5$$

on C, C'.

It is easy to check that ψ_2 is the constant function with value 18 on C. In particular, there is no monic character in C and every character in C determines the knot genus. A straightforward computation implies that the number of monic characters in C' is 6 and that all but 2 characters in C' determine the genus.

ACKNOWLEDGMENTS

The authors would like to thank Hiroshi Matsuda for telling us various constructions of closed essential surfaces in knot complements and the literature [7, 29, 30]. They would also like to thank the anonymous referee for helpful comments. The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2012R1A1A2012001747 and No. 20120000341). The second author was supported by JSPS Research Fellowships for Young Scientists. The third author was partially supported by Grant-in-Aid for Scientific Research (No. 23540076), the Ministry of Education, Culture, Sports, Science and Technology, Japan.

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