TWISTED ALEXANDER POLYNOMIALS AND IDEAL POINTS GIVING SEIFERT SURFACES

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Abstract. The coefficients of twisted Alexander polynomials of a knot induce regular functions of the $SL_2(\mathbb{C})$-character variety. We prove that the function of the highest degree has a finite value at an ideal point which gives a minimal genus Seifert surface by Culler-Shalen theory. It implies a partial affirmative answer to a conjecture by Dunfield, Friedl and Jackson.

1. Introduction

The aim of this paper is to present an application of *twisted Alexander polynomials* to *Culler-Shalen theory* for knots, following a conjecture by Dunfield, Friedl and Jackson [DFJ, Conjecture 8.9].

In the notable work [CS] Culler and Shalen established a method to construct essential surfaces in a 3-manifold from an ideal point of the $SL_2(\mathbb{C})$-character variety. Their theory applies Bass-Serre theory [Se1, Se2] to the functional field of the representation variety. Twisted Alexander polynomials [Li, W], which are known to be essentially equal to certain Reidemeister torsion [KL, Kitan], are invariants of a 3-manifold associated to linear representations of the fundamental group. The torsion invariants generalize many properties of the Alexander polynomial, and were shown by Friedl and Vidussi [FV1, FV3] to detect fiberedness for 3-manifolds and the Thurston norms of irreducible ones which are not closed graph manifolds. We refer the reader to the expositions [Sh] and [FV2] for literature and related topics on Culler-Shalen theory and twisted Alexander polynomials respectively.

Let K be a null-homologous knot in a rational homology 3-sphere. We denote by $X^{irr}(K)$ the Zariski closure of the $SL_2(\mathbb{C})$-character variety of K. Dunfield, Friedl and Jackson [DFJ] showed that for each irreducible component X_0 in $X^{irr}(K)$ certain normalizations of twisted Alexander polynomials induce an invariant $\mathcal{T}_K^{X_0} \in \mathbb{C}[X_0][t, t^{-1}]$ called the *torsion polynomial function* of K. The invariant $\mathcal{T}_K^{X_0}$ satisfies that $\deg \mathcal{T}_K^{X_0} \leq 4g(K) - 2$ and that $\mathcal{T}_K^{X_0}(\chi)(t^{-1}) = \mathcal{T}_K^{X_0}(\chi)(t)$ for $\chi \in X_0$, where $g(K)$ is the genus of K (cf. [FK1, Theorem 1.1], [FKK, Theorem 1.5]). For a curve C in X_0 we denote by $\mathcal{T}_K^{C} \in \mathbb{C}[C][t, t^{-1}]$ the restriction of $\mathcal{T}_K^{X_0}$ to C, and by $c(\mathcal{T}_K^{C}) \in \mathbb{C}[C]$ the coefficient function in $\mathcal{T}_K^{C}$ of the highest degree $2g(K) - 1$. It is known that if K is a fibered knot, then $c(\mathcal{T}_K^{C})$ is the constant function with value 1 (cf. [C, FK1, GKM]).

Conjecture 1.1 ([DFJ, Conjecture 8.9])**.** *If an ideal point χ of a curve C in $X^{irr}(K)$ gives a Seifert surface of K, then the leading coefficient of $\mathcal{T}_K^{C}$ has a finite value at χ.*

In this paper we give a partial affirmative answer to Conjecture 1.1. The main theorem of this paper is as follows:

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Theorem 1.2. *If an ideal point χ of a curve C in $X^{irr}(K)$ gives a minimal genus Seifert surface of K, then $c(\mathcal{T}_K^C)(\chi)$ is finite.*

The statement of Theorem 1.2 is actually weaker than that of Conjecture 1.1 on the following two points:

(1) An essential Seifert surface is not necessarily of minimal genus.
(2) If $\deg \mathcal{T}_K^C < 4g(K) - 2$, then $c(\mathcal{T}_K^C)(\chi) = 0$ but the leading coefficient of $\mathcal{T}_K^C(\chi)$ is not necessarily finite.

Concerning (1) it should be remarked that classes of knots with a unique isotopy class of essential Seifert surfaces are known. For instance, Lyon [Ly, Theorem 2 and Corollary 2.1] constructed such a class of non-fibered knots containing p-twist knots with $|p| > 1$.

A generalization of Theorem 1.2 for general 3-manifolds will be discussed in a successive work [Kitay]. See [KKM, KM, Mo] for recent works on other conjectures by Dunfield-Friedl-Jackson.

This paper is organized as follows. Section 2 sets up notation and terminology, and provides a brief overview of Culler-Shalen theory. In particular, the precise meaning of 'an ideal point giving a surface' is described. In Section 3 we review some basics of Reidemeister torsion, and recalls properties of torsion polynomial functions. In this paper we mainly work with Reidemeister torsion rather than twisted Alexander polynomials, based on the equivalence. Finally, in Section 4 we prove Theorem 1.2.

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2. Culler-Shalen theory

We begin with briefly reviewing Culler-Shalen theory [CS, Sh]. For more details on character varieties we refer the reader to [LM].

2.1. **Character varieties and ideal points.**

Let M be a compact orientable 3-manifold. The algebraic group $SL_2(\mathbb{C})$ acts on the affine algebraic set $\mathrm{Hom}(\pi_1 M, SL_2(\mathbb{C}))$ by conjugation. The algebro-geometric quotient $X(M)$ of the action is called the $SL_2(\mathbb{C})$-*character variety* of M. We denote by $t\colon \mathrm{Hom}(\pi_1 M, SL_2(\mathbb{C})) \to X(M)$ the quotient map. For a representation $\rho\colon \pi_1 M \to SL_2(\mathbb{C})$ its *character* $\chi_\rho\colon \pi_1 M \to \mathbb{C}$ is given by $\chi_\rho(\gamma) = \mathrm{tr}\,\rho(\gamma)$ for $\gamma \in \pi_1 M$. The character variety $X(M)$ is known to be realized by the set of the characters χ_ρ of $SL_2(\mathbb{C})$-representations ρ, and $t(\rho) = \chi_\rho$ under the identification. For $\gamma \in \pi_1 M$ a *trace function* $I_\gamma\colon X(M) \to \mathbb{C}$ is defined by $I_\gamma(\chi_\rho) = \mathrm{tr}\,\rho(\gamma)$ for a representation $\rho\colon \pi_1 M \to SL_2(\mathbb{C})$, and it is known that the coordinate ring of $X(M)$ is generated by $\{I_\gamma\}_{\gamma \in \pi_1 M}$.

Let C be a curve in $X(M)$ which is not necessarily irreducible, and let $\widehat{C}$ be its smooth projective model. The points where the rational map $\widehat{C} \to C$ is undefined are called the *ideal points* of C.

Let K be a knot in a rational homology 3-sphere, and we denote by E its exterior. In the following we set $X(K) = X(E)$ and denote by $X^{irr}(K)$ the Zariski closure of the subset of $X(K)$ consisting of the characters of irreducible representations.

2.2. **Essential surfaces given by ideal points.** A non-empty properly embedded compact orientable surface S in M is called *essential* if for any component S_0 of S the homomorphism $\pi_1 S_0 \to \pi_1 M$ induced by the natural inclusion map is injective, and if no component of S is homeomorphic to S^2 or boundary parallel.

Let χ be an ideal point of a curve C in $X(M)$. There exists a curve D in $t^{-1}(C)$ such that $t|_D$ is not a constant map, and that $t|_D$ extends to a regular map $\widehat{D} \to \widehat{C}$ between the smooth projective models. We take a point $\tilde{\chi}$ of $\widehat{D}$ in the preimage of χ. Associated to the valuation of $\mathbb{C}(D)$ at $\tilde{\chi}$ Bass-Serre theory [Se1, Se2] gives a canonical action of $SL_2(\mathbb{C}(D))$ on a tree $T_{\tilde{\chi}}$ without inversions. Pulling back the action by the tautological representation $\pi_1 M \to SL_2(\mathbb{C}(D))$, we have an action of $\pi_1 M$ on $T_{\tilde{\chi}}$. Culler and Shalen [CS, Theorem 2.2.1] showed that the action is non-trivial, i.e., for any vertex of $T_{\tilde{\chi}}$ the stabilizer of the action is not whole the group $\pi_1 M$. Now essentially due to Stallings, Epstein and Waldhausen, there exists a map $f\colon M \to T_{\tilde{\chi}}/\pi_1 M$ such that $f^{-1}(P)$ is an essential surface, where P is the set of the middle points of edges. We say that χ *gives an essential surface* S if $S = f^{-1}(P)$ for some f as above.

3. Torsion invariants

We review basics of Reidemeister torsion and recall torsion polynomial functions introduced by Dunfield, Friedl and Jackson [DFJ]. For more details on torsion invariants we refer the reader to the expositions [Mi, N, T1, T2].

3.1. **Reidemeister torsion.** Let $C_* = (C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_0)$ be a finite dimensional chain complex over a commutative field $\mathbb{F}$, and let $c = \{c_i\}$ and $h = \{h_i\}$ be bases of C_* and $H_*(C_*)$ respectively. Choose bases b_i of $\operatorname{Im}\partial_{i+1}$ for each $i = 0, 1, \dots n$, and take a basis $b_i h_i b_{i-1}$ of C_i for each i as follows. Picking a lift of h_i in $\operatorname{Ker}\partial_i$ and combining it with b_i, we first obtain a basis $b_i h_i$ of C_i. Then picking a lift of b_{i-1} in C_i and combining it with $b_i h_i$, we obtain a basis $b_i h_i b_{i-1}$ of C_i. The *algebraic torsion* $\tau(C_*, c, h)$ is defined as:

$$\tau(C_*, c, h) := \prod_{i=0}^{n} [b_i h_i b_{i-1}/c_i]^{(-1)^{i+1}} \in \mathbb{F}^\times,$$

where $[b_i h_i b_{i-1}/c_i]$ is the determinant of the base change matrix from c_i to $b_i h_i b_{i-1}$. If C_* is acyclic, then we write $\tau(C_*, c)$. It can be easily checked that $\tau(C_*, c, h)$ does not depend on the choice of b_i and $b_i h_i b_{i-1}$.

Let (Y, Z) be a finite CW-pair. In the following when we write $C_*(\widetilde{Y}, \widetilde{Z})$, $\widetilde{Y}$ stands for the universal cover of Y and $\widetilde{Z}$ the pullback of Z by the universal covering map $\widetilde{Y} \to Y$. For a representation $\rho\colon \pi_1 Y \to GL(V)$ over a commutative field $\mathbb{F}$ we define the twisted homology group as:

$$H_i^\rho(Y, Z; V) := H_i(C_*(\widetilde{Y}, \widetilde{Z}) \otimes_{\mathbb{Z}[\pi_1 Y]} V).$$

If Z is empty, then we write $H_i^\rho(Y; V)$.

For an n-dimensional representation $\rho\colon \pi_1 Y \to GL(V)$ and a basis h of $H_*^\rho(Y, Z; V)$ the *Reidemeister torsion* $\tau_\rho(Y, Z; h)$ associated to ρ and h is defined as follows. We choose a lift $\tilde{e}$ in $\widetilde{Y}$

for each cell $e \subset Y \setminus Z$. Then

$$\tau_\rho(Y, Z; h) := \tau(C_*(\widetilde{Y}, \widetilde{Z}) \otimes_{\mathbb{Z}[\pi_1 Y]} V, \langle \tilde{e} \otimes 1 \rangle_e, h) \in \mathbb{F}^\times/(-1)^n \det \rho(\pi_1 Y).$$

If Z is empty or if $H^\rho_*(Y, Z; V) = 0$, then we drop Z or h in the notation $\tau_\rho(Y, Z; h)$. It can be easily checked that $\tau_\rho(Y, Z; h)$ does not depend on the choice of $\tilde{e}$ and is invariant under conjugation of representations. It is known that Reidemeister torsion is a simple homotopy invariant.

3.2. **Torsion polynomial functions.** Let K be a null-homologous knot in a rational homology 3-sphere. We take an epimorphism $\alpha\colon \pi_1 E \to \langle t \rangle$, where $\langle t \rangle$ is the infinite cyclic group generated by the indeterminate t. For a representation $\rho\colon \pi_1 E \to GL_n(\mathbb{F})$ we define a representation $\alpha \otimes \rho\colon \pi_1 E \to GL_n(\mathbb{F}(t))$ by $\alpha \otimes \rho(\gamma) = \alpha(\gamma)\rho(\gamma)$ for $\gamma \in \pi_1 E$. If $H^{\alpha\otimes\rho}_*(E; \mathbb{F}(t)^n) = 0$, then the Reidemeister torsion $\tau_{\alpha\otimes\rho}(E)$ is defined, and is known by Kirk and Livingston [KL], and Kitano [Kitan] to be essentially equal to the *twisted Alexander polynomial* associated to α and ρ. Friedl and Kim [FK1, Theorem 1.1] showed that

$$\deg \tau_{\alpha\otimes\rho}(E) \leq n(2g(K) - 1)$$

(See also [FK2]). It is known by Cha [C], Friedl and Kim [FK1], and Goda, Kitano and Morifuji [GKM] that if K is a fibered knot, then

$$\deg \tau_{\alpha\otimes\rho}(E) = n(2g(K) - 1)$$

and $\tau_{\alpha\otimes\rho}(E)$ is represented by a fraction of monic polynomials in $\mathbb{F}[t, t^{-1}]$. See [FV2] for details on twisted Alexander polynomials and their precise relation with Reidemeister torsion.

Let X_0 be an irreducible component of $X^{irr}(K)$. Dunfield, Friedl and Jackson [DFJ, Theorem 1.5] showed that there exists an invariant $\mathcal{T}^{X_0}_K \in \mathbb{C}[X_0][t, t^{-1}]$ called the *torsion polynomial function* of X_0 such that the following are satisfied for $\chi_\rho \in X_0$:

(i) If $H^{\alpha\otimes\rho}_*(E; \mathbb{C}(t)^2) = 0$ then, $\mathcal{T}^{X_0}_K(\chi_\rho) = \tau_{\alpha\otimes\rho}(E) \in \mathbb{C}(t)/\langle t \rangle$.
(ii) If $H^{\alpha\otimes\rho}_*(E; \mathbb{C}(t)^2) \neq 0$ then, $\mathcal{T}^{X_0}_K(\chi_\rho) = 0$.
(iii) $\mathcal{T}^{X_0}_K(\chi_\rho)(t^{-1}) = \mathcal{T}^{X_0}_K(\chi_\rho)(t)$.

For a curve C in X_0 we denote by $\mathcal{T}^C_K \in \mathbb{C}[X_0][t, t^{-1}]$ the restriction of $\mathcal{T}^{X_0}_K$ to C, and by $c(\mathcal{T}^C_K) \in \mathbb{C}[C]$ the coefficient function in $\mathcal{T}^C_K$ of the highest degree $2g(K) - 1$.

4. Main theorem

Now we prove Theorem 1.2. We first prepare key lemmas for the proof.

4.1. **Lemmas.** Let K be a null-homologous knot in a rational homology 3-sphere and let S be a minimal genus Seifert surface of K. A tubular neighborhood of S is identified with $S \times [-1, 1]$. We set $N := E \setminus S \times (-1, 1)$, and denote by $\iota_\pm\colon S \to N$ the natural homeomorphisms such that $\iota_\pm(S) = S \times (\pm 1)$. Since the homomorphisms $\pi_1 S \to \pi_1 E$ and $\pi_1 N \to \pi_1 E$ induced by the natural inclusion maps are injective, in the following we regard $\pi_1 S$ and $\pi_1 N$ as subgroups of $\pi_1 E$.

Lemma 4.1. *Let $\rho\colon \pi_1 E \to GL_n(\mathbb{F})$ be an irreducible representation with $n > 1$ such that $H^{\alpha\otimes\rho}_*(E; \mathbb{F}(t)^n) = 0$. Then the following hold:*

(i) $H^\rho_0(S; \mathbb{F}^n) = H^\rho_0(N; \mathbb{F}^n) = H^\rho_2(N; \mathbb{F}^n) = 0$.
(ii) *If $\deg \tau_{\alpha\otimes\rho}(E) = n(2g(K) - 1)$, then $(\iota_\pm)_*\colon H^\rho_1(S; \mathbb{F}^n) \to H^\rho_1(N; \mathbb{F}^n)$ are isomorphisms.*

Proof. This lemma is proved by techniques developed in [FK1] together with [FKK, Proposition A.3] in terms of twisted Alexander polynomials. We give only the main steps of the proof with corresponding parts in the references, and the details are left to the reader.

It follows from [FK1, Proposition 3.5] and [FKK, Proposition A.3] that $H_0^\rho(S;\mathbb{F}^n) = 0$. Since $H_*^{\alpha\otimes\rho}(E;\mathbb{F}(t)^n) = 0$, the long exact sequence in [FK1, Proposition 3.2] implies that

$$H_i^\rho(N;\mathbb{F}^n) = H_i^\rho(S;\mathbb{F}^n) = 0$$

for $i = 0, 2$, which proves (i).

It follows from Proof of [FK1, Theorem 1.1] that if $\deg \tau_{\alpha\otimes\rho}(E) = n(2g(K) - 1)$, then the inequalities in [FK1, Proposition 3.3] turn into equalities. Now (ii) follows from the proof of [FK1, Proposition 3.3]. □

Lemma 4.2. *Let $\rho\colon \pi_1 E \to GL_n(\mathbb{F})$ be an irreducible representation such that $H_*^{\alpha\otimes\rho}(E;\mathbb{F}(t)^n) = 0$. If $\deg \tau_{\alpha\otimes\rho}(E) = n(2g(K) - 1)$, then*

$$\tau_{\alpha\otimes\rho}(E) = \tau_\rho(N, S\times 1)\det(t\cdot id - (\iota_+)_*^{-1}\circ(\iota_-)_*),$$

where $(\iota_\pm)_*$ are the isomorphisms $H_1^\rho(S;\mathbb{F}^n) \to H_1^\rho(N;\mathbb{F}^n)$.

Proof. We pick a basis h of $H_1^\rho(S;\mathbb{F}^n)$. Since $H_1^{\alpha\otimes\rho}(S;\mathbb{F}(t)^n) = H_1^\rho(S;\mathbb{F}^n)\otimes\mathbb{F}(t)$ and $H_1^{\alpha\otimes\rho}(N;\mathbb{F}(t)^n) = H_1^\rho(N;\mathbb{F}^n)\otimes\mathbb{F}(t)$, h and $(\iota_+)_*(h)$ can be seen also as bases of $H_1^{\alpha\otimes\rho}(S;\mathbb{F}(t)^n)$ and $H_1^{\alpha\otimes\rho}(N;\mathbb{F}(t)^n)$ respectively. Taking appropriate triangulations of E, N and S and lift of simplices in the universal covers, we have the following exact sequences:

$$0 \to C_*(\widetilde{S})\otimes\mathbb{F}(t)^n \xrightarrow{t(\iota_+)*-(\iota_-)} C_*(\widetilde{N})\otimes\mathbb{F}(t)^n \to C_*(\widetilde{E})\otimes\mathbb{F}(t)^n \to 0,$$
$$0 \to C_*(\widetilde{S})\otimes\mathbb{F}^n \xrightarrow{(\iota_+)_*} C_*(\widetilde{N})\otimes\mathbb{F}^n \to C_*(\widetilde{N},\widetilde{S\times 1})\otimes\mathbb{F}^n \to 0,$$

where the local coefficients in the first and second sequences are understood to be induced by $\alpha\otimes\rho$ and ρ respectively. By the multiplicativity of Reidemeister torsion [Mi, Theorem 3.1] and Lemma 4.1 we have

$$\tau_{\alpha\otimes\rho}(N;(\iota_+)_*(h))\det(t\cdot id - (\iota_+)_*^{-1}\circ(\iota_-)_*) = \tau_{\alpha\otimes\rho}(S;h)\tau_{\alpha\otimes\rho}(E),$$
$$\tau_\rho(N;(\iota_+)_*(h)) = \tau_\rho(S;h)\tau_\rho(N, S\times 1).$$

By the functoriality of Reidemeister torsion [T1, Proposition 3.6] we have

$$\tau_{\alpha\otimes\rho}(N;(\iota_+)_*(h)) = \tau_\rho(N;(\iota_+)_*(h)),$$
$$\tau_{\alpha\otimes\rho}(S;h) = \tau_\rho(S;h).$$

The desired formula now follows from the above equalities. □

Lemma 4.3. *There exists a regular function f of $X(N)$ such that*

$$f(\chi_\rho) = \tau_\rho(N, S\times 1)$$

for a representation $\rho\colon \pi_1 N \to GL_n(\mathbb{F})$ satisfying that $H_*^\rho(N, S\times 1;\mathbb{F}^n) = 0$.

Proof. Let $\rho\colon \pi_1 N \to GL_n(\mathbb{F})$ be a representation such that $H_*^\rho(N, S\times 1;\mathbb{F}^n) = 0$. We take a finite 2-dimensional CW-pair (V, W) with $C_0(V, W) = 0$ which is simple homotopy equivalent to $(N, S\times 1)$. The differential map

$$C_2(\widetilde{V},\widetilde{W})\otimes_{\mathbb{F}[\pi_1 V]}\mathbb{F}^n \to C_1(\widetilde{V},\widetilde{W})\otimes_{\mathbb{F}[\pi_1 V]}\mathbb{F}^n$$

is represented by the matrix $\rho(A)$ obtained as follows from a matrix A in $\mathbb{Z}[\pi_1 V]$. We first consider the matrix whose (i, j)-entries are the image of that of A by ρ. Then we naturally forget the matrix structures of the entries to get a matrix $\rho(A)$ in $\mathbb{C}$. By the simple homotopy invariance and the definition of Reidemeister torsion we have

$$\tau_\rho(N, S \times 1) = \tau_\rho(V, W) = \det \rho(A).$$

It follows from basics of Linear algebra that $\det \rho(A)$ is written as a polynomial in $\{\mathrm{tr}\, \rho(A)^i\}_{i\in\mathbb{Z}}$, and that $\mathrm{tr}\, \rho(A)^i$ is as one in $\{\mathrm{tr}\, \rho(\gamma)\}_{\gamma\in\pi_1 V}$, which proves the lemma. □

The following lemma is a direct corollary of [CS, Theorem 2.2.1] and [CS, Proposition 2.3.1].

Lemma 4.4. *Suppose that an ideal point χ of a curve in $X^{irr}(K)$ gives an essential surface S. Then $I_\gamma(\chi) \in \mathbb{C}$ for $\gamma \in \pi_1 E$ represented by a loop in the complement of S.*

4.2. **Proof of the main theorem.**

Proof of Theorem 1.2. Let χ be an ideal point of a curve C in $X^{irr}(K)$ which gives a minimal genus Seifert surface S of K, and let $\rho\colon \pi_1 E \to SL_2(\mathbb{C})$ be an irreducible representation such that $\chi_\rho \in C$. If $H_*^{\alpha\otimes\rho}(E; \mathbb{C}(t)^2) = 0$ and if $\deg \tau_{\alpha\otimes\rho}(E) = 4g(K) - 2$, then by Lemma 4.2 we have

$$c(\mathcal{T}_K^C)(\chi_\rho) = \tau_\rho(N, S \times 1),$$

and so it follows from Lemma 4.3 that the function $c(\mathcal{T}_K^C)$ is in the subring of $\mathbb{C}[C]$ generated by I_γ for $\gamma \in \pi_1 N$. Since it follows from Lemma 4.4 that $I_\gamma(\chi) \in \mathbb{C}$ for $\gamma \in \pi_1 N$, we obtain $c(\mathcal{T}_K^C)(\chi) \in \mathbb{C}$, which completes the proof. □

References

[C] J. C. Cha, *Fibred knots and twisted Alexander invariants*, Trans. Amer. Math. Soc. **355** (2003), no. 10, 4187–4200.

[CS] M. Culler and P. B. Shalen, *Varieties of group representations and splittings of* 3*-manifolds*, Ann. of Math. (2) **117** (1983), no. 1, 109–146.

[DFJ] N. M. Dunfield, S. Friedl and N. Jackson, *Twisted Alexander polynomials of hyperbolic knots*, Experiment. Math. **21** (2012), 329–352.

[FK1] S. Friedl and T. Kim, *The Thurston norm, fibered manifolds and twisted Alexander polynomials*, Topology **45** (2006), no. 6, 929–953.

[FK2] S. Friedl and T. Kim, *Twisted Alexander norms give lower bounds on the Thurston norm*, Trans. Amer. Math. Soc. **360** (2008), no. 9, 4597–4618.

[FKK] S. Friedl, T. Kim and T. Kitayama, *Poincaré duality and degrees of twisted Alexander polynomials*, Indiana Univ. Math. J. **61** (2012), 147–192.

[FV1] S. Friedl and S. Vidussi, *Twisted Alexander polynomials detect fibered 3-manifolds*, Ann. of Math. (2) **173** (2011), no. 3, 1587–1643.

[FV2] S. Friedl and S. Vidussi, *A survey of twisted Alexander polynomials*, The mathematics of knots, 45–94, Contrib. Math. Comput. Sci., 1, Springer, Heidelberg, 2011.

[FV3] S. Friedl and S. Vidussi, *The Thurston norm and twisted Alexander polynomials*, to appear in J. Reine Angew. Math., arXiv:1204.6456.

[GKM] H. Goda, T. Kitano and T. Morifuji, *Reidemeister torsion, twisted Alexander polynomial and fibered knots*, Comment. Math. Helv. **80** (2005), no. 1, 51–61.

[KKM] T. Kim, T. Kitayama and T. Morifuji, *Twisted Alexander polynomials on curves in character varieties of knot groups*, Internat. J. Math. **24** (2013), no. 3, 1350022, 16 pp.

[KM] T. Kim and T. Morifuji, *Twisted Alexander polynomials and character varieties of* 2*-bridge knot groups*, Internat. J. Math. **23** (2012), no. 6, 1250022, 24 pp.

[KL] P. Kirk and C. Livingston, *Twisted Alexander invariants, Reidemeister torsion, and Casson-Gordon invariants*, Topology **38** (1999), no. 3, 635–661.

[Kitan] T. Kitano, *Twisted Alexander polynomial and Reidemeister torsion*, Pacific J. Math. **174** (1996), no. 2, 431–442.

[Kitay] T. Kitayama, *Twisted Alexander polynomials and incompressible surfaces given by ideal points*, in preparation.

[Li] X. S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sin. (Engl. Ser.) **17** (2001), no. 3, 361–380.

[LM] A. Lubotzky and A. R. Magid, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. **58** (1985), no. 336, xi+117 pp.

[Ly] H. C. Lyon, *Simple knots with unique spanning surfaces*, Topology **13** (1974), 275–279.

[Mi] J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966) 358–426.

[Mo] T. Morifuji, *On a conjecture of Dunfield, Friedl and Jackson*, C. R. Math. Acad. Sci. Paris **350** (2012), no. 19-20, 921–924.

[N] L. I. Nicolaescu, *The Reidemeister torsion of* 3*-manifolds*, de Gruyter Studies in Mathematics, 30, Walter de Gruyter & Co., Berlin, 2003. xiv+249 pp.

[Se1] J.-P. Serre, *Arbres, amalgames,* SL_2 (French), Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46, Société Mathematique de France, Paris, 1977, 189 pp, (1 plate).

[Se2] J.-P. Serre, *Trees*, Translated from the French by John Stillwell, Springer-Verlag, Berlin-New York, 1980. ix+142 pp.

[Sh] P. B. Shalen, *Representations of* 3*-manifold groups*, Handbook of geometric topology, 955–1044, North-Holland, Amsterdam, 2002.

[T1] V. Turaev, *Introduction to combinatorial torsions*, Notes taken by Felix Schlenk, Lectures in Mathematics ETH Zurich, Birkhauser Verlag, Basel, 2001. viii+123 pp.

[T2] V. Turaev, *Torsions of* 3*-dimensional manifolds*, Progress in Mathematics, 208, Birkhauser Verlag, Basel, 2002, x+196 pp.

[W] M. Wada, *Twisted Alexander polynomial for finitely presentable groups*, Topology **33** (1994), no. 2, 241–256.

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