CHARACTER VARIETIES OF HIGHER DIMENSIONAL REPRESENTATIONS AND SPLITTINGS OF 3-MANIFOLDS

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Abstract. In 1983 Culler and Shalen established a way to construct essential surfaces in a 3-manifold from ideal points of the SL_2-character variety associated to the 3-manifold group. We present in this article an analogous construction of certain kinds of branched surfaces (which we call *essential tribranched surfaces*) from ideal points of the SL_n-character variety for a natural number n greater than or equal to 3. Further we verify that such a branched surface induces a nontrivial presentation of the 3-manifold group in terms of the fundamental group of a certain 2-dimensional complex of groups.

0. Introduction

In their notable work [CS83] Culler and Shalen established a method to construct *essential surfaces* in a 3-manifold from information of the $SL_2(\mathbb{C})$-character variety of its fundamental group. The method is based upon the interplay among hyperbolic geometry, the theory of incompressible surfaces and the theory on the structure of subgroups of the special linear group SL(2) of degree 2. Culler–Shalen theory provides a basic and powerful tool in low-dimensional topology, and it has given fundamentals for many significant breakthroughs; for example, Culler and Shalen themselves proved the generalised Smith conjecture as a special case of their main results in [CS83]. Meanwhile, Morgan and Shalen [MS84, MS88a, MS88b] proposed new understandings of Thurston's results: the characterisation of 3-manifolds with the compact space of hyperbolic structures [Th86] and a compactification of the Teichmüller space of a surface [Th88]. Further Culler, Gordon, Luecke and Shalen [CGLS87] proved the cyclic surgery theorem on Dehn fillings of knots. We refer the reader to the exposition [Sh02] for more literature and related topics on Culler and Shalen's theory.

Meanwhile, it is expected from the representation theoretic view that one can obtain much more fruitful information of a 3-manifold and its fundamental group, considering not only 2-dimensional representations but also *higher dimensional representations*. Indeed, recently there have been many interesting results in low dimensional topology derived from higher dimensional representations of surface groups and 3-manifold groups. Against the background of such situations, we attempt in this article to extend Culler and Shalen's theory for *higher dimensional representations* of 3-manifold groups. Fortunately, the main algebraic tools which Culler and Shalen used to establish their theory—the $SL_2(\mathbb{C})$-character varieties and the Bruhat–Tits trees— are naturally generalised to higher dimensional representations—the $SL_n(\mathbb{C})$-character varieties and the Bruhat–Tits buildings (associated to $SL(n)$). Therefore, as in classical Culler–Shalen theory, we can find from an ideal point of the $SL_n(\mathbb{C})$-character variety of the 3-manifold group $\pi_1(M)$ a nontrivial action of $\pi_1(M)$ on the Bruhat–Tits building, and obtain a map from M to the quotient of the Bruhat–Tits building by the action of $\pi_1(M)$. In the $SL_2(\mathbb{C})$-case, the

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resultant quotient complex is of dimension 1 (a graph), and thus Culler and Shalen just had to consider the midpoints of their edges, and pulled back them to obtain surfaces (which one could modify to be essential by several local surgeries). In the $\mathrm{SL}_n(\mathbb{C})$-case, however, the resultant quotient complex is of dimension $n-1$, and thus we have to consider a certain 1-subcomplex of the first barycentric subdivision of its 2-skelton. Pulling back such a 1-complex, one naturally encounters the notion of [*essential*] *tribranched surfaces*, which is a generalisation of the notion of [essential] surfaces in the sense that "[essential] tribranched surfaces without any branched points are [essential] surfaces" (for detailed explanation, see Section 2.1). It is worth pointing out that, even in the $\mathrm{SL}_n(\mathbb{C})$-case, one can sometimes obtain essential surfaces without any branched points; see Question 6.1 in Section 6 for details. Moreover, we obtain as a byproduct a nontrivial presentation of the 3-manifold group induced from an essential tribranched surface, which identifies the 3-manifold group as the fundamental group of a certain *2-complex of groups* (see Section 2 for details).

We here explain our strategy to construct an essential tribranched surface in more detail. Let M be a compact, connected, irreducible and orientable 3-manifold. We suppose that the $\mathrm{SL}_n(\mathbb{C})$-character variety $X_n(M)$ of $\pi_1(M)$ is of positive dimension, and let $\tilde{x}$ be an ideal point of an affine algebraic curve C in $X_n(M)$. By construction $X_n(M)$ is obtained as the (geometric invariant theoretical) quotient of the affine algebraic set $\mathrm{Hom}(\pi_1(M), \mathrm{SL}_n(\mathbb{C}))$ by the conjugate action of $\mathrm{SL}_n(\mathbb{C})$, and we may take a lift D of C in $\mathrm{Hom}(\pi_1(M), \mathrm{SL}_n(\mathbb{C}))$. Let $\tilde{y}$ be a "lift" of $\tilde{x}$, which is an ideal point of the affine curve D. We denote by $\mathbb{C}(D)$ the field of rational functions on D. The construction of an essential tribranched surface from $\tilde{x}$ is divided into the following three steps. Firstly, on the basis of the theory of *Bruhat–Tits buildings* elaborated by Iwahori and Matsumoto [IM65], and Bruhat and Tits [BT72, BT84], we may associate to the ideal point $\tilde{y}$ a canonical action of $\mathrm{SL}_n(\mathbb{C}(D))$ on an $(n-1)$-dimensional Euclidean building $\mathcal{B}_{n,\widetilde{D},\tilde{y}}$ (see Section 4.2 for details). Pulling back this canonical action by the tautological representation $\pi_1(M) \to \mathrm{SL}_n(\mathbb{C}(D))$, we obtain an action of $\pi_1(M)$ on $\mathcal{B}_{n,\widetilde{D},\tilde{y}}$. Secondly, we prove that this action is *nontrivial*, that is, the isotropy subgroup at each vertex of $\mathcal{B}_{n,\widetilde{D},\tilde{y}}$ with respect to this action is a proper subgroup of $\pi_1(M)$. The important point to note here is that in the case of $n = 2$ this step is an algebraic heart of Culler and Shalen's original work [CS83, Theorem 2.2.1]. Thirdly, we show that one can construct an essential tribranched surface in general from a nontrivial action of $\pi_1(M)$ on a Euclidean building. In this step we consider certain modifications of classical techniques due to Stallings and Waldhausen for constructing an essential surface as a *dual* of a nontrivial action of $\pi_1(M)$ on a tree.

Now let $\mathcal{B}^{(2)}_{n,\widetilde{D},\tilde{y}}$ denote the 2-skeleton of the Bruhat–Tits building $\mathcal{B}_{n,\widetilde{D},\tilde{y}}$ and let $Y(\mathcal{B}^{(2)}_{n,\widetilde{D},\tilde{y}})$ denote the 1-dimensional subcomplex of the first barycentric subdivision of $\mathcal{B}^{(2)}_{n,\widetilde{D},\tilde{y}}$ consisting of all the barycentres of 1- and 2-simplices and all the edges connecting them. The main theorem of this article is as follows:

Main Theorem (Theorem 4.9)**.** *Let n be a natural number greater than or equal to 3, and assume that the boundary ∂M of M is non-empty when n is strictly greater than 3. Then, for each ideal point $\tilde{x}$ of an affine curve in $X_n(M)$, there exists an essential tribranched surface Σ in M such that the inverse image of Σ in the universal cover $\widetilde{M}$ of M coincides with the inverse image of $Y(\mathcal{B}^{(2)}_{n,\widetilde{D},\tilde{y}})$ under some $\pi_1(M)$-equivariant piecewise-linear map $\tilde{f} \colon \widetilde{M} \to \mathcal{B}^{(2)}_{n,\widetilde{D},\tilde{y}}$.*

The assumption on the boundary of M comes from a certain technical reason required in the proof of the main result. See the proof of Theorem 4.7 for details.

This article is organised as follows. In Section 1 we give a brief exposition on complexes of groups. Section 2 is devoted to introduce the notion of essential tribranched surfaces and to describe splittings of the 3-manifold groups induced by essential tribranched surfaces. In Section 3 we collect fundamentals on Bruhat–Tits buildings, in particular, for the special linear groups. In Section 4 the main theorem stated above is proved. We first review several standard facts on $\mathrm{SL}_n(\mathbb{C})$-character varieties in Section 4.1. We then show in Section 4.2 that the action of the 3-manifold group on the Bruhat–Tits building associated to an ideal point of the $\mathrm{SL}_n(\mathbb{C})$-character variety $X_n(M)$ is nontrivial, and construct an essential tribranched surface from such a nontrivial action in Section 4.3. Section 5 provides an application of the theory of this article to small Seifert manifolds. In Section 6 we raise several questions to be further studied.

The contents of Sections 1, 2.2, 2.3 and 2.4 (concerning complexes of groups associated to essential tribranched surfaces) are rather independent of other parts of this article, and hence readers who are only interested in the construction of nontrivial essential tribranched surfaces may skip these sections and proceed to Section 4.

Note. After the first version of this article appeared on the arXiv, Question 6.1 in Section 6 was solved affirmatively in a much stronger form by Friedl, Nagel and the second-named author [FKN18]. In fact, based on the construction of essential tribranched surfaces developed in this article, the breakthroughs of Agol [Ag13] and Wise [Wi11] on the separability of subgroups in a 3-manifold group and the subsequent works of Przytycki and Wise [PW14a, PW14b], they proved that *every* connected essential surface (without any branched points) in M is detected by an ideal point of a rational curve in $X_n(M)$ for some natural number n as in Main Theorem.

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Contents

1. Preliminaries on complexes of groups

The theory of *graphs of groups* due to Hyman Bass and Jean-Pierre Serre [Se77] has been generalised to the theory of *complexes of groups* of higher dimension. Historically, Stephen M. Gersten and John Robert Stallings first considered a special case of complexes of groups of dimension 2, namely *triangles of groups*, and proved that "non-positively curved" triangles of groups were developable [St91]. The general theory of complexes of groups were developed independently by Jon Michael Corson [Co92] (mainly for 2-complexes of groups) and André Haefliger [Hae91] (for complexes of any dimension). We shall briefly recall the definitions of complexes of groups and their fundamental groups. We here adopt a combinatorial description proposed by Haefliger [Hae91] (see also [BH99, Chapter III.$\mathcal{C}$]) rather than a topological description based upon the concept of *complexes of spaces* especially when we define the fundamental groups of complexes of groups (see [Co92] for details of the latter approach). One of the great virtues of the combinatorial approach is that one may explicitly describe generators and relations of the fundamental group of a complex of groups, as we shall see later in Section 1.2.

1.1. **Scwols associated to Δ-complexes.** First let us recall from [Hat01, Appendix, Simplicial CW Structures] or [Hae91, Section 1] the notion of a *Δ-complex*[1] (or a *simplicial CW complex*). Roughly speaking, it is a CW complex each of whose open cells has an affine structure. More precisely, a CW complex X is called an unordered Δ-complex if, for each open n-cell σ, there exists a set of $(n+1)!$ continuous maps (called *orderings* of σ) from the standard n-simplex Δ^n into X such that

- (Δ1) the restriction of every ordering to the interior of Δ^n is a homeomorphism onto σ;
- (Δ2) two orderings of σ differ by an affine isomorphism on Δ^n;
- (Δ3) the composition of an affine injection $\Delta^k \hookrightarrow \Delta^n$ from Δ^k onto a k-face of Δ^n with an ordering of σ is an ordering for an open k-cell of X.

A continuous map $f\colon Y \to X$ of Δ-complexes is called *simplicial* if it sends each n-cell σ of Y onto a k-cell τ of X for some k so that the composition of an ordering of σ and f coincides with the composition of an affine surjection $\Delta^n \to \Delta^k$ and an ordering of τ.

The *barycentre* of an n-cell σ of a Δ-complex X is well defined as the image of the barycentre of the standard simplex Δ^n under an arbitrary ordering. Therefore we can consider the *barycentric subdivision* X' of X, which is again a Δ-complex. Note that every cell of X' has a natural

[1] To be precise, what we call a Δ-complex in this article is the one referred as an *unordered* Δ*-complex* in [Hat01].

ordering; namely, if we denote the barycentre of a cell σ of X by v_σ, the 0-cells $v_{\sigma_1}, v_{\sigma_2}, \ldots, v_{\sigma_{k+1}}$ appearing in the boundary of a k-cell τ' of X' are labeled by $k+1$ cells $\sigma_1, \sigma_2, \ldots, \sigma_{k+1}$ of X of distinct dimension, which equips τ' with an ordering.

Definition 1.1 (The scwol $\mathcal{X}$ associated to X)**.** For a Δ-complex X, we define $V(\mathcal{X})$ (the *vertex set*) and $E(\mathcal{X})$ (the *edge set*) as the set of 0-cells and 1-cells of the first barycentric subdivision X' of X, respectively. Note that each vertex $v_\sigma \in V(\mathcal{X})$ is labeled by a cell σ of $\mathcal{X}$. For an edge $e \in E(\mathcal{X})$ with endpoints v_σ and v_τ satisfying $\dim \sigma > \dim \tau$, we define the *initial vertex* $i(e)$ and the *terminal vertex* $t(e)$ of e as $i(e) = v_\sigma$ and $t(e) = v_\tau$ respectively, which endow the quadruple $\mathcal{X} = (V(\mathcal{X}), E(\mathcal{X}), i, t)$ with a structure of a directed graph. We finally impose the *law of composition* on $E(\mathcal{X})$ as follows;

> a pair (a, b) of edges is called *composable* if $i(a) = t(b)$ holds, and in the case the composition $c = ab$ of a and b is defined as a unique edge c with $i(c) = i(b)$ and $t(c) = t(a)$ such that a, b and c form the boundary of a 2-cell of X'.

We call the directed graph $\mathcal{X} = (V(\mathcal{X}), E(\mathcal{X}), i, t)$ equipped with the law of composition on $E(\mathcal{X})$ the *scwol associated to the* Δ*-complex* X.

Figure 1 illustrates the scwol structure (namely the associated directed graph and the law of composition of edges) on the standard 2-simplex Δ^2. Here circles, squares and a triangle indicate barycentres of 0-cells, 1-cells and a 2-cell of Δ^2 respectively.

Remark 1.2. Indeed $\mathcal{X}$ fulfills all the axioms of a s̲mall c̲ategory w̲ith o̲ut l̲oops (abbreviated as *scwol*) introduced in [BH99, Chapter III.$\mathcal{C}$ Definitions 1.1]. Bridson and Haefliger develop their theory on complexes of groups over abstract scwols in [BH99, Chapter III.$\mathcal{C}$], but for our purpose it suffices to consider "geometric scwols" obtained as in Definition 1.1. Therefore we decided to adopt the description of [Hae91] and avoid dealing with generalities on abstract scwols.

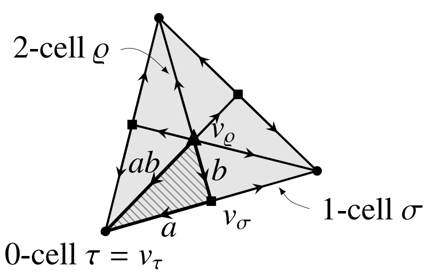

FIGURE 1. The scwol associated to the standard 2-simplex Δ^2

Now we introduce the *edge path fundamental group* of the scwol $\mathcal{X} = (V(\mathcal{X}), E(\mathcal{X}), i, t)$ associated to a Δ-complex X. We briefly summarise basic notion on edge paths in our context, but for details see [ST67, Section 4.4] (for simplicial complexes) and [Ge08, Chapter 3] (for CW complexes). We denote by $E^{\pm}(\mathcal{X})$ the set of symbols a^+ and a^-, where a belongs to the edge set $E(\mathcal{X})$. The symbols a^+ and a^- mean "following the edge a backward and forward" respectively[2]; namely the initial and terminal vertices of a^+ and a^- for $a \in E(\mathcal{X})$ as $i(a^+) = t(a^-) = t(a)$ and

[2]Here we adopt the convention introduced in [BH99, Chapter III.$\mathcal{C}$, Section 1.6]. Note that the opposite convention is adopted in [Hae91, Section 3.1].

$t(a^+) = i(a^-) = i(a)$, and regard $E^{\pm}(\mathcal{X})$ as the set of *oriented edges* of the first barycentric subdivision X' of X. We also set $(a^{\pm})^{-1} = a^{\mp}$ (double sign in the same order). An *edge path* in $\mathcal{X}$ is a finite sequence $l = (e_1, \ldots, e_n)$ of elements of $E^{\pm}(\mathcal{X})$ satisfying $t(e_j) = i(e_{j+1})$ for each $1 \leq j \leq n-1$. The vertices $i(e_1)$ and $t(e_n)$ are called the *initial* and *terminal* vertices of the edge path l, and denoted by $i(l)$ and $t(l)$ respectively. We define the *concatenation* $l * l'$ of two edge paths $l = (e_1, \ldots, e_m)$ and $l' = (e'_1, \ldots, e'_n)$ by $l * l' = (e_1, \ldots, e_m, e'_1, \ldots, e'_n)$ when they satisfy $t(l) = i(l')$, and also define the *inverse edge path* l^{-1} of an edge path $l = (e_1, \ldots, e_n)$ by $l^{-1} = (e_n^{-1}, e_{n-1}^{-1}, \ldots, e_1^{-1})$. An edge path is called an *edge loop* when its initial vertex coincides with its terminal vertex; in this case the initial (and hence also terminal) vertex is called its *base vertex*.

Now assume that X is connected, and consider the set of all edge loops with base vertex v_0. We endow it with an equivalence relation $\sim$ generated by the following two elementary relations (E1) and (E2) (called *simple equivalences*):

(E1) $(e_1, \ldots, e_{j-1}, e_j, e_{j+1}, e_{j+2}, \ldots, e_n) \sim (e_1, \ldots, e_{j-1}, e_{j+2}, \ldots, e_n)$ if e_{j+1} coincides with e_j^{-1};

(E2) for each composable pair (a, b) of edges in $\mathcal{X}$, we impose

$$(e_1, \ldots, e_{i-1}, e_i = a^+, e_{i+1} = b^+, e_{i+2}, \ldots, e_m) \sim (e_1, \ldots, e_{i-1}, (ab)^+, e_{i+2}, \ldots, e_m)$$

and

$$(e_1, \ldots, e_{j-1}, e_j = b^-, e_{j+1} = a^-, e_{j+2}, \ldots, e_n) \sim (e_1, \ldots, e_{j-1}, (ab)^-, e_{j+2}, \ldots, e_n).$$

The set $\pi_1^{\mathrm{edge}}(\mathcal{X}, v_0)$ of all the equivalence classes of edge loops with base vertex v_0 is indeed equipped with a group structure, whose group law is induced by the concatenation of edge loops: $[c] * [c'] = [c * c']$. For an edge loop c, the inverse of $[c]$ is given by the equivalence class $[c^{-1}]$ of the inverse edge loop c^{-1} of c, and the unit element is given by the equivalence class of the *constant loop* $[c_{v_0}]$ *at* v_0 (by definition the constant loop c_{v_0} corresponds to the "empty word" $c_{v_0} = ()$ of $E^{\pm}(\mathcal{X})$, both of whose initial and terminal vertices are defined as v_0). We call $\pi_1^{\mathrm{edge}}(\mathcal{X}, v_0)$ the *edge path fundamental group* of the scwol $\mathcal{X}$ at v_0. We can construct an isomorphism $\pi_1^{\mathrm{edge}}(\mathcal{X}, v_0) \xrightarrow{\sim} \pi_1^{\mathrm{edge}}(\mathcal{X}, v'_0); [c] \mapsto [l_{v_0, v'_0}^{-1} * c * l_{v_0, v'_0}]$ as in the case of usual fundamental groups, where l_{v_0, v'_0} is an edge path with initial vertex v_0 and terminal vertex v'_0 (since we assume that X is connected, such l_{v_0, v'_0} always exists; obviously this isomorphism heavily depends on the choice of an edge path l_{v_0, v'_0} and is far from canonical). We end this subsection by remarking that the edge path fundamental group $\pi_1^{\mathrm{edge}}(\mathcal{X}, v_0)$ is canonically isomorphic to the usual (or topological) fundamental group of the Δ-complex X.

Proposition 1.3. *Let X be a connected Δ-complex and v_0 a 0-cell of X'. Let $\pi_1^{\mathrm{top}}(|X|, v_0)$ denote the fundamental group (in the usual sense) of the underlying topological space $|X|$ of X. Then there exists a canonical isomorphism $\pi_1^{\mathrm{edge}}(\mathcal{X}, v_0) \xrightarrow{\sim} \pi_1^{\mathrm{top}}(|X|, v_0)$.*

Proof. Indeed $\pi_1^{\mathrm{edge}}(\mathcal{X}, v_0)$ is no other than the edge path fundamental group $\pi_1^{\mathrm{edge}}(X', v_0)$ of the Δ-complex X', but the underlying topological space of X is the same as that of its first barycentric subdivision X'. Therefore we obtain $\pi_1^{\mathrm{edge}}(\mathcal{X}, v_0) \xrightarrow{\sim} \pi_1^{\mathrm{top}}(|X'|, v_0) = \pi_1^{\mathrm{top}}(|X|, v_0)$ due to [Ge08, Theorem 3.4.1]. □

1.2. **Complexes of groups.** A *complex of groups* $G(X) = (\mathcal{X}, \{G_{v_\sigma}, \{\psi_a\}, \{g_{ab}\}\})$ over a Δ-complex X consists of for types of data: the scwol $\mathcal{X}$ associated to X, a group G_{v_σ} for each vertex v_σ of $\mathcal{X}$ (labeled by a cell σ of X) called the *local group* at v_σ, an injective group homomorphism $\psi_a\colon G_{i(a)} \to G_{t(a)}$ for each edge a of $\mathcal{X}$, and a specific element $g_{a,b}$ of $G_{t(a)}$, called a *twisting element*, for each composable pair (a,b) of edges in $\mathcal{X}$. We impose the following two constraints on these data:

(CG1) (twisted commutativity) the equality $g_{a,b}\psi_{ab}(x)g_{a,b}^{-1} = \psi_a \circ \psi_b(x)$ holds for each composable pair (a,b) of edges in $\mathcal{X}$ and every element x of $G_{i(b)}$;

(CG2) (cocycle condition) the equality $\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}$ holds for a,b,c in $E(\mathcal{X})$ such that both (a,b) and (b,c) are composable.

We define the *dimension* of a complex of groups $G(X)$ as the dimension of the Δ-complex X.

Remark 1.4. We here remark that for a complex of groups *of dimension at most* 2, the cocycle condition (CG2) is just the empty condition (simply because there is no triple (a,b,c) of edges such that both (a,b) and (b,c) are composable). Later we shall mainly study complexes of groups associated to essential tribranched surfaces in a 3-manifold (see Section 2.2). Obviously by construction they are of dimension at most 2, and hence we do not have to consider the cocycle condition (CG2) whenever we are concerned with complexes of groups associated to essential tribranched surfaces.

A *morphism* $\phi\colon G(X) \to G$ from a complex of groups $G(X)$ to a group G consists of group homomorphisms $\{\phi_{v_\sigma}\colon G_{v_\sigma} \to G\}_{v_\sigma \in V(\mathcal{X})}$ and specific elements $\{\phi(a)\}_{a\in E(\mathcal{X})}$ (called *twisting elements*) of G satisfying

$$\phi(a)(\phi_{i(a)}(x))\phi(a)^{-1} = \phi_{t(a)} \circ \psi_a(x) \quad \text{for each } a \in E(\mathcal{X}) \text{ and } x \in G_{i(a)}$$

and

$$\phi_{t(a)}(g_{a,b})\phi(ab) = \phi(a)\phi(b) \quad \text{for each } a,b \in E(\mathcal{X}) \text{ with } i(a) = t(b).$$

Next we introduce the notion of the *fundamental group of a complex of groups*. Let $G(X)$ be a complex of groups over a Δ-complex X. A $G(X)$*-path* in $\mathcal{X}$ is a finite sequence $l = (g_0, e_1, g_1, \ldots, g_{n-1}, e_n, g_n)$ where $(e_1,\ldots,e_n)$ is an edge path in $\mathcal{X}$, g_0 is an element of the local group $G_{i(e_1)}$ at $i(e_1)$ and g_j is an element of the local group $G_{t(e_j)}$ at $t(e_j)$ for each $1 \le j \le n$. For $G(X)$-paths we define their initial and terminal vertices, concatenations and inverse paths similarly to those of edge paths as follows.

Initial and terminal vertices: for a $G(X)$-path $l = (g_0, e_1, g_1, \ldots, e_n, g_n)$, set $i(l) = i(e_1)$ and $t(l) = t(e_n)$.

Concatenation: for $G(X)$-paths $l = (g_0, e_1, g_1, \ldots, e_m, g_m)$ and $l' = (g'_0, e'_1, g'_1, \ldots, e'_n, g'_n)$ satisfying $t(l) = i(l')$, set $l * l' = (g_0, e_1, g_1, \ldots, e_m, g_m g'_0, e'_1, g'_1, \ldots, e'_n, g'_n)$.

Inverse path: for a $G(X)$-path $l = (g_0, e_1, g_1, \ldots, e_n, g_n)$, define the inverse $G(X)$-path l^{-1} of l as $l^{-1} = (g_n^{-1}, e_n^{-1}, g_{n-1}^{-1}, \ldots, e_1^{-1}, g_0^{-1})$.

Now let $FG(X)$ be the *universal group associated to* $G(X)$ which is defined by the following generators and relations.

Generators: elements of all local groups G_{v_σ} and elements of $E^{\pm}(\mathcal{X})$.

Relations: we impose on the generators above the following four types of relations:

- the group relations for each G_{v_σ};
- $(a^{\pm})^{-1} = a^{\mp}$ for each edge a in $\mathcal{X}$ (double sign in the same order);
- $a^+b^+ = g_{a,b}(ab)^+$ for each composable pair (a,b) of edges in $\mathcal{X}$;
- $\psi_a(x) = a^+xa^-$ for each edge a in $\mathcal{X}$ and each element x of $G_{i(a)}$.

Then it is easy to check that the morphism $\iota\colon G(X) \to FG(X)$, which consists of group homomorphisms $\{\iota_{v_\sigma}\colon G_{v_\sigma} \to FG(X); g \mapsto g\}_{v_\sigma \in V(\mathcal{X})}$ and twisting elements $\{\iota(a) = a^+\}_{a \in E(\mathcal{X})}$, has a universal property among morphisms from $G(X)$ to groups. More specifically, for every morphism $\phi\colon G(X) \to G$ from $G(X)$ to a group G, we obtain a unique group homomorphism $F\phi\colon FG(X) \to G$ which satisfies $\phi = F\phi \circ \iota$ (see [BH99, Chapter III.$\mathcal{C}$ Section 3.2] for details).

We associate to each $G(X)$-loop $c = (g_0, e_1, g_1, \ldots, e_n, g_n)$ with base vertex v_0 an element $[c]$ of $FG(X)$ which is by definition the image of the word $g_0 e_1 g_1 \cdots e_n g_n$ in $FG(X)$. The image of $[\,\cdot\,]$ (as a map from the set of $G(X)$-loops with base vertex v_0 to $FG(X)$) is equipped with a group structure induced by concatenations, which we denote by $\pi_1(G(X), v_0)$ and call the *fundamental group* of $G(X)$. We remark that the definition of the fundamental group $\pi_1(G(X), v_0)$ of a complex of groups $G(X)$ (of higher dimension) introduced here is a direct generalisation of that of the fundamental group of a graph of groups due to Bass and Serre [Se77, Section 5.1].

1.3. **Group actions on Δ-complexes and developability.** Let G be a group and Y a Δ-complex. An action of G on Y is *simplicial* if every g in G induces a simplicial homeomorphism on Y. We assume in this article that every group action on a Δ-complex is simplicial. An action *without inversions* of G on Y is a simplicial action of G such that, if an element of G maps a cell ϱ of Y onto itself, its restriction to ϱ is the identity map.

When a group G acts on a Δ-complex Y without inversions, we readily see that the quotient $X = G\backslash Y$ is naturally a Δ-complex so that the projection $p\colon Y \twoheadrightarrow X = G\backslash Y$ is a simplicial map. We shall associate to this action a complex of groups $G(X)$ over X as follows; for each cell σ of X, we choose a lift $\tilde\sigma$ of σ to Y (that is, $\tilde\sigma$ is a cell of Y satisfying $p(\tilde\sigma) = \sigma$). Then $v_{\tilde\sigma} \in V(\mathcal{Y})$ is a lift of the vertex v_σ of $\mathcal{X}$ labeled by σ. For each $a \in E(\mathcal{X})$ with $i(a) = v_\sigma$, we can find a unique edge $\tilde a$ contained in $\tilde\sigma$ such that $p(\tilde a) = a$. Let us choose an element h_a of G satisfying $h_a t(\tilde a) = v_{\tilde\tau}$, where $\tau = t(a)$. We define the local group G_{v_σ} at a vertex v_σ of $\mathcal{X}$ as the isotropy subgroup $G_{\tilde\sigma}$ of G at $\tilde\sigma$ with respect to the group action of G on Y. For each edge a of $\mathcal{X}$, we define a group homomorphism $\psi_a\colon G_{i(a)} \to G_{t(a)}$ by $\psi_a(g) = h_a g h_a^{-1}$. Finally for each composable pair (a, b) of edges in $\mathcal{X}$, we define a twisting element $g_{a,b}$ as $h_a h_b h_{ab}^{-1}$. It is easy to verify that these data endow the quotient Δ-complex $X = G\backslash Y$ with a structure of a complex of groups, which we denote by $G(X)$. Moreover the set of natural inclusions $\{\phi_{v_\sigma}\colon G_{v_\sigma} \hookrightarrow G\}_{v_\sigma \in V(\mathcal{X})}$ and the specification of elements $\{\phi(a) = h_a\}_{a \in E(\mathcal{X})}$ of G define a morphism $\phi\colon G(X) \to G$. Consequently we see that to an action of group G on a Δ-complex Y are associated a complex of groups $G(X)$ (over $X = G\backslash Y$) and a morphism $\phi\colon G(X) \to G$.

Remark 1.5. The complex of groups $G(X)$ and the morphism $\phi\colon G(X) \to G$ constructed as above depend on choices of $\tilde\sigma$'s and h_a's, and thus they are not canonical. However, even if we choose a different lift $\tilde\sigma'$ for each cell σ of X and a different element h'_a for each edge a of $\mathcal{X}$, the resultant complex of groups $G'(X)$ is still isomorphic to $G(X)$ as complexes of groups; see [BH99, Chapter III.$\mathcal{C}$ Section 2.9 (2)] for details.

A complex of groups $G(X)$ over a Δ-complex X is called *developable* if it is obtained as a complex of groups associated to an action without inversions of a group G on a Δ-complex Y such that $X = G\backslash Y$. Unlike graphs of groups, complexes of groups of higher dimension are not always developable. The following proposition provides a necessary and sufficient condition for a complex of groups to be developable.

Proposition 1.6. *A complex of groups $G(X)$ is developable if and only if there exists a morphism from $G(X)$ to a certain group G which is injective on each local group G_{v_σ} of $G(X)$.*

Proof. See [Hae91, Theorem 4.1] or [BH99, Chapter III.$\mathcal{C}$ Corollary 2.15]. □

We end this section by summarising several facts concerning the covering space theory for complexes of groups.

Theorem 1.7. *Let $G(X)$ be a complex of groups over a connected Δ-complex X and v_0 a vertex of X.*

(1) *There exist a simply connected Δ-complex $\widetilde{X}$ (called the* universal cover of X) *on which $\pi_1(G(X), v_0)$ acts without inversions and a simplicial map $\widetilde{p}\colon \widetilde{X} \to X$ such that $\widetilde{p}$ induces a homeomorphism from $\pi_1(G(X), v_0)\backslash\widetilde{X}$ to X.*
(2) *Let G be a group, and suppose that $G(X)$ and a morphism $G(X) \to G$ are associated to an action without inversions of G on a Δ-complex Y with $X = G\backslash Y$. Then there are a surjection $\varphi\colon \pi_1(G(X), v_0) \to G$ and a φ-equivariant simplicial map $f\colon \widetilde{X} \to Y$ such that the induced map $X = \pi_1(G(X), v_0)\backslash\widetilde{X} \to G\backslash Y = X$ is the identity map on X (namely $f\colon \widetilde{X} \to Y$ is a covering map with the Galois group* $\operatorname{Ker}\varphi$*).*

Proof. See [Hae91, Theorem 4.1], or see [BH99, Chapter III.$\mathcal{C}$ Theorem 3.13] for (1) and [BH99, Chapter III.$\mathcal{C}$ 2.18] for (2). □

2. Essential tribranched surfaces and complexes of groups

We introduce in this section the notion of *tribranched surfaces* and *essential tribranched surfaces* which shall play key roles throughout this article. It is a certain generalisation of the concepts of surfaces (contained in a 3-manifold) and essential surfaces; see, for example, [Sh02, Definition 1.5.1] for the definition of essential surfaces. After proposing the definitions of tribranched surfaces and essential tribranched surfaces in Section 2.1, we observe that essential tribranched surfaces behave compatibly with the theory of complexes of groups in Sections 2.2 and 2.3 (as the notion of essential surfaces is well adapted to Bass and Serre's theory on graphs of groups [Se77] in the original work of Culler and Shalen [CS83]). Section 2.4 is an appendix, where we discuss essential surfaces whose associated complexes of groups are not developable.

2.1. Tribranched surfaces and essential tribranched surfaces.

Let M be a compact, connected, irreducible and orientable 3-manifold with possibly nonempty boundary. Let Σ be a compact subset of M such that the pair (M, Σ) is locally homeomorphic to $(\overline{\mathbb{H}}, Y \times [0, \infty))$, where $\overline{\mathbb{H}}$ and Y are defined by

$$\overline{\mathbb{H}} = \{\, (z, s) \in \mathbb{C} \times \mathbb{R} \mid s \geq 0 \,\}, \qquad Y = \{\, re^{\sqrt{-1}\theta} \in \mathbb{C} \mid r \in \mathbb{R}_{\geq 0} \text{ and } \theta = 0, \pm 2\pi/3 \,\}.$$

We denote by $C(\Sigma)$ the set of *branched points* of Σ corresponding to $\{0\}\times[0,\infty) \subset Y\times[0,\infty)$, by $S(\Sigma)$ the complement of a sufficiently small tubular neighbourhood of $C(\Sigma)$ in Σ, and by $M(\Sigma)$ the complement of a sufficiently small regular neighbourhood of Σ in M. The subsets $C(\Sigma)$ and $S(\Sigma)$ are a properly embedded 1-submanifold and a subsurface of M respectively.

Definition 2.1 (Tribranched surfaces)**.** Let (M, Σ) be as above. We call Σ a *tribranched surface in M* if the following two conditions are fulfilled;

(TBS1) the intersection of Σ and a sufficiently small tubular neighbourhood of $C(\Sigma)$ in M is homeomorphic to $Y \times C(\Sigma)$;
(TBS2) the subsurface $S(\Sigma)$ is orientable.

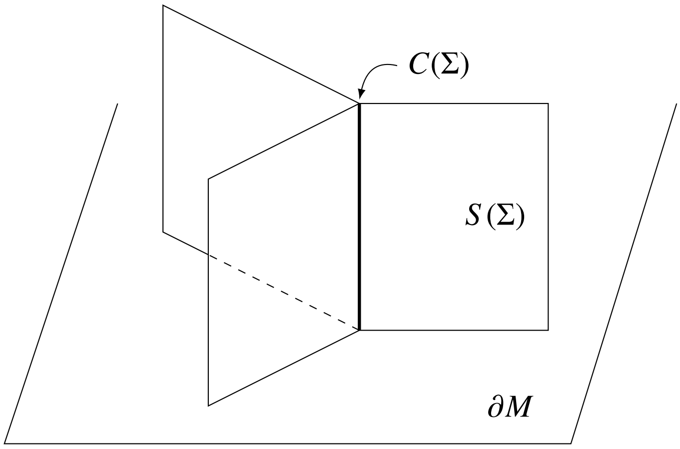

FIGURE 2. A local picture of a tribranched surface Σ

See Figure 2 for a local picture of a tribranched surface Σ. In the following, we will suppress the base point in the notation of fundamental groups unless specifically noted.

Now let us define essential tribranched surfaces. Note that if the boundary of $S \in \pi_0(S(\Sigma))$ appears in the boundary of a tubular neighbourhood U_C of $C \in \pi_0(C(\Sigma))$, we can define a natural homomorphism $\pi_1(C) \to \pi_1(S)$ as the composition $\pi_1(C) \xrightarrow{\sim} \pi_1(U_C \cap \partial S) \to \pi_1(S)$. Similarly if the boundary of $N \in \pi_0(M(\Sigma))$ appears in the boundary of a regular neighbourhood U_S of $S \in \pi_0(S(\Sigma))$, we can also define a natural homomorphism $\pi_1(S) \to \pi_1(N)$ as the composition $\pi_1(S) \xrightarrow{\sim} \pi_1(U_S \cap \partial N) \to \pi_1(N)$.

Definition 2.2 (Essential tribranched surfaces)**.** A tribranched surface Σ in M is said to be *essential* if the following three conditions are fulfilled, other than the conditions (TBS1) and (TBS2) of Definition 2.1;

- (ETBS1) for any component N of $M(\Sigma)$, the homomorphism $\pi_1(N) \to \pi_1(M)$ induced by the natural inclusion $N \hookrightarrow M$ is not surjective;
- (ETBS2) for any components C, S, N of $C(\Sigma)$, $S(\Sigma)$, $M(\Sigma)$ respectively, if the natural homomorphisms $\pi_1(C) \to \pi_1(S)$ and $\pi_1(S) \to \pi_1(N)$ are defined, they are injective;
- (ETBS3) no component of Σ is contained in a 3-ball in M or a collar of ∂M.

Remark 2.3. An essential surface (in the usual sense) in M is regarded as an essential tribranched surface without any branched points. Note that since M is irreducible, every embedded sphere in M is contained in a 3-ball. It follows that an essential triblanched surface in M contains no sphere component.

2.2. **Complexes of groups associated to essential tribranched surfaces.** It is well known that one may associate a *graph of groups* to an essential surface (without any branched points) embedded in a 3-manifold, which gives a *splitting* of the 3-manifold group; we refer the readers to [Sh02, Sections 1.4 and 1.5]. Then the concept of essential tribranched surfaces, which is a more general notion including essential surfaces, should be closely related to the theory of *complexes of groups* of higher dimension. Here we discuss the relation between them.

Now let M be a compact, connected, irreducible and orientable 3-manifold. Suppose that M contains a tribranched surface Σ.

The dual 2-complex associated to Σ. In this paragraph we associate to the pair (M, Σ) a Δ-complex $X_\Sigma = X_{(M,\Sigma)}$ of dimension 2. The construction of X_Σ which we shall explain below is a natural generalisation of a well-known construction of the *dual graph* of a bicollared surface contained in a 3-manifold. The readers are referred to the exposition [Sh02, Section 1.4], for example, for details on the classical construction of dual graphs.

Recall that $C(\Sigma)$ denotes the set of branched points of Σ. Let C be a connected component of $C(\Sigma)$, and let D^2 (resp. $\mathring{D}^2$) denote the closed unit disk $\{ z \in \mathbb{C} \mid |z| \leq 1 \}$ (resp. the open unit disk $\{ z \in \mathbb{C} \mid |z| < 1 \}$). For each C, there exists a tubular neighbourhood $h_C \colon C \times D^2 \to M$ by virtue of the condition (TBS1) of tribranched surfaces; more specifically, h_C induces a homeomorphism of $C \times D^2$ onto a neighbourhood of C in M and satisfies $h_C(x, 0) = x$ for each point x of C. Furthermore $h|_{C\times(\mathring{D}^2\cap Y)}$ induces a homeomorphism of $C \times (\mathring{D}^2 \cap Y)$ onto a regular neighbourhood of C in Σ. We choose and fix such a tubular neighbourhood h_C for each connected component C of $C(\Sigma)$. We denote by U_C (resp. $\bar{U}_C$) the open tubular neighbourhood $h_C(C \times \mathring{D}^2)$ (resp. the closed tubular neighbourhood $h_C(C \times D^2)$) of C in M.

Next let S be an arbitrary connected component of $S(\Sigma) = \Sigma \setminus \bigcup_{C\in\pi_0(C(\Sigma))} U_C$. The condition (TBS2) combined with the theory of regular neighbourhoods provides us with a homeomorphism $h_S : S \times [-1, 1] \to M$ onto a bicollar neighbourhood of S in $M \setminus \bigcup_{C\in\pi_0(C(\Sigma))} U_C$; namely h_S satisfies $h_S(x, 0) = x$ for each point x of S and $\partial h_S(S \times [-1, 1])$ coincides with the intersection of $h_S(S \times [-1, 1])$ and $\partial M \cup \bigcup_{C\in\pi_0(C(\Sigma))} \partial \bar{U}_C$. We also choose and fix such a regular neighbourhood h_S for each connected component S of $S(\Sigma)$. We further assume that the closed sets $h_S(S \times [-1, 1])$ are pairwisely disjoint after replacing them by thinner ones if necessary. We denote by U_S (resp. $\bar{U}_S$) the subset $h_S(S \times (-1, 1))$ (resp. $h_S(S \times [-1, 1])$) of M which is an open (resp. a closed) bicollar neighbourhood of S in $M \setminus \bigcup_{C\in\pi_0(C(\Sigma))} U_C$.

We denote by $M(\Sigma)$ the complement of $\bigcup_{C\in\pi_0(C(\Sigma))} U_C \cup \bigcup_{S\in\pi_0(S(\Sigma))} U_S$ in M. Note that all of $\pi_0(C(\Sigma))$, $\pi_0(S(\Sigma))$ and $\pi_0(M(\Sigma))$ are finite sets due to the compactness of M. We thus obtain a partition of M into disjoint subsets:

$$M = \bigsqcup_{N\in\pi_0(M(\Sigma))} N \sqcup \bigsqcup_{\substack{S\in\pi_0(S(\Sigma))\\ t\in(-1,1)}} h_S(S \times \{t\}) \sqcup \bigsqcup_{\substack{C\in\pi_0(C(\Sigma))\\ s\in\mathring{D}^2}} h_C(C \times \{s\}). \tag{2.1}$$

We use the notation $x \sim_\Sigma y$ to indicate that both of two points $x, y \in M$ are contained in one of the disjoint subsets occurring in the right hand side of (2.1). Obviously $\sim_\Sigma$ defines an equivalence relation on M. Set $X_\Sigma = X_{(M,\Sigma)} = M/{\sim_\Sigma}$ and endow X_Σ with the quotient topology. One then easily observes that X_Σ has a natural structure of a Δ-complex of dimension 2 whose 0-cells, 1-cells and 2-cells are labeled by elements of $\pi_0(M(\Sigma))$, $\pi_0(S(\Sigma))$ and $\pi_0(C(\Sigma))$ respectively. It is straightforward to check that a 1-cell σ_S (labeled by an element S of $\pi_0(S(\Sigma))$) occurs in the boundary of a 2-cell σ_C (labeled by an element C of $\pi_0(C(\Sigma))$) if and only if the intersection of $\bar{U}_S$ and $\bar{U}_C$ is nonempty. Similarly a 0-cell σ_N (labeled by an element N of $\pi_0(M(\Sigma))$) occurs in the boundary of a 1-cell σ_S (resp. a 2-cell σ_C) if and only if the intersection of N and $\bar{U}_S$ (resp. $\bar{U}_C$) is nonempty. We call X_Σ the *dual* 2*-complex* associated to the tribranched surface Σ. Figure 3 illustrates a local picture of the dual 2-complex X_Σ. We denote the scwol associated to X_Σ by $\mathcal{X}_\Sigma$ (see Definition 1.1). Note that X_Σ is *connected* since M is connected by assumption and X_Σ is the image of a continuous quotient map $r_\Sigma \colon M \to X_\Sigma$.

The complexes of groups associated to essential tribranched surfaces. Now let us try to endow X_Σ with a structure of a complex of groups. As we shall see, the following construction works well only when Σ is *essential*. Set $\Pi_0(M, \Sigma) = \pi_0(C(\Sigma)) \sqcup \pi_0(S(\Sigma)) \sqcup \pi_0(M(\Sigma))$. Then, by

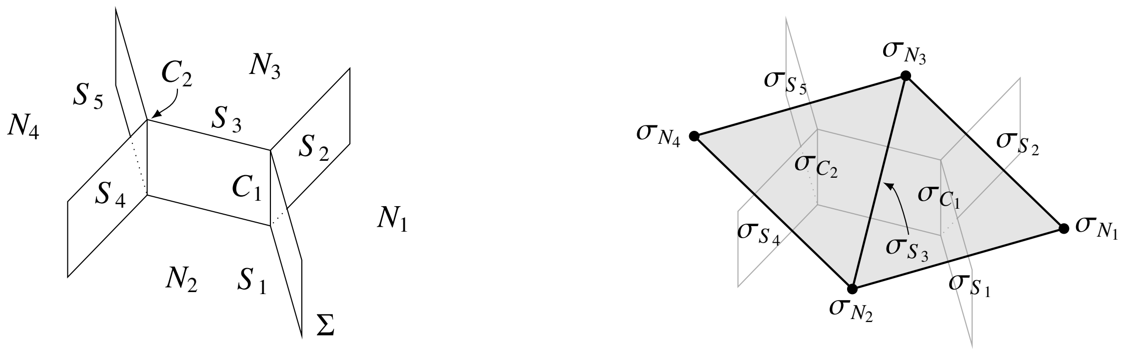

FIGURE 3. The dual 2-complex X_Σ associated to a tribranched surface Σ

construction, each cell of X_Σ corresponds to an element of $\Pi_0(M, \Sigma)$. In the following arguments, we often identify a cell of X_Σ with the corresponding element of $\Pi_0(M, \Sigma)$, and use the same symbol to indicate them. Let us choose and fix a point x_σ in each $\sigma \in \Pi_0(M, \Sigma)$ and define the local group $G^\Sigma_{v_\sigma}$ at $v_\sigma \in V(\mathcal{X}_\Sigma)$ as the fundamental group $\pi_1(\sigma, x_\sigma)$ (in the usual sense). We next associate a group homomorphism $\psi^\Sigma_a \colon G^\Sigma_{i(a)} \to G^\Sigma_{t(a)}$ to each edge a. Let σ and τ be elements of $\Pi_0(M, \Sigma)$ satisfying $i(a) = v_\sigma$ and $t(a) = v_\tau$. The existence of the edge a implies that the cell τ of X_Σ occurs in the boundary of the cell σ, and in particular the intersection of $\bar{U}_\sigma$ and $\bar{U}_\tau$ is nonempty as we have already remarked (with the convention $U_N = \bar{U}_N = N$ for each element N of $\pi_0(M(\Sigma))$). We may thus take a path $l_{\sigma,\tau} \colon [0, 1] \to \bar{U}_\sigma \cup \bar{U}_\tau$ satisfying $l_{\sigma,\tau}(0) = x_\sigma$ and $l_{\sigma,\tau}(1) = x_\tau$. We choose and fix such a path $l_{\sigma,\tau}$ for each edge a with $i(a) = v_\sigma$ and $t(a) = v_\tau$. We may readily verify that τ is a deformation retract of $\bar{U}_\sigma \cup \bar{U}_\tau$ by the definition of $\bar{U}_\tau$ as a tubular or bicollar neighbourhood, and therefore we may define a group homomorphism $\psi^\Sigma_a \colon G^\Sigma_{v_\sigma} \to G^\Sigma_{v_\tau}$ as the composition

$$G^\Sigma_{v_\sigma} = \pi_1(\sigma, x_\sigma) \to \pi_1(\bar{U}_\sigma \cup \bar{U}_\tau, x_\sigma) \xrightarrow{(\sharp)} \pi_1(\bar{U}_\sigma \cup \bar{U}_\tau, x_\tau) \xrightarrow{\sim} \pi_1(\tau, x_\tau) = G^\Sigma_{v_\tau} \tag{2.2}$$

where the first map is induced from the natural inclusion $\sigma \hookrightarrow \bar{U}_\sigma \cup \bar{U}_\tau$ and the last isomorphism is induced from a deformation retraction from $\bar{U}_\sigma \cup \bar{U}_\tau$ to τ. The middle map ($\sharp$) is the change of base points with respect to the path $l_{\sigma,\tau}$, or in other words, the map defined by $[c] \mapsto [l^{-1}_{\sigma,\tau} * c * l_{\sigma,\tau}]$. Here we define the *concatenation* $l_1 * l_2$ of two paths $l_1, l_2 \colon [0, 1] \to T$ in a topological space T with $l_1(1) = l_2(0)$ as follows:

$$l_1 * l_2(t) = \begin{cases} l_1(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ l_2(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

We finally define a twisting element $g^\Sigma_{a,b}$ for each composable pair (a, b) of edges in $\mathcal{X}_\Sigma$. Suppose that the vertices $i(b)$, $t(b)(= i(a))$ and $t(a)$ are labeled by elements C of $\pi_0(C(\Sigma))$, S of $\pi_0(S(\Sigma))$ and N of $\pi_0(M(\Sigma))$ respectively. Then we define $g^\Sigma_{a,b}$ as the image of $[l^{-1}_{S,N} * l^{-1}_{C,S} * l_{C,N}]$ under the map $\pi_1(N \cup \bar{U}_S \cup \bar{U}_C, x_N) \to \pi_1(N, x_N) = G^\Sigma_{v_N}$ induced by a deformation retraction from $N \cup \bar{U}_S \cup \bar{U}_C$ to N. The twisted commutativity (CG1)

$$g^\Sigma_{a,b} \psi^\Sigma_{ab}([c]) (g^\Sigma_{a,b})^{-1} = \psi^\Sigma_a \circ \psi^\Sigma_b([c]) \tag{2.3}$$

straightforwardly holds for each element $[c]$ of $G^\Sigma_C = \pi_1(C, x_C)$. We have now verified, combining Remark 1.4 with the calculations above, that $G(X_\Sigma) = (\mathcal{X}_\Sigma, \{G^\Sigma_{v_\sigma}\}, \{\psi^\Sigma_a\}, \{g^\Sigma_{a,b}\})$ satisfies all the conditions of complexes of groups over X_Σ except for injectivity of each ψ^Σ_a. If we further assume that the tribranched surface Σ under consideration is *essential*, we readily observe that every ψ_a^Σ is injective due to the condition (ETBS2) and the twisted commutativity (2.3). As a consequence $G(X_\Sigma)$ is indeed a 2-complex of groups over X_Σ when Σ is essential, which we call the *complex of groups associated to the essential tribranched surface* Σ.

2.3. **Splittings of 3-manifold groups induced by essential tribranched surfaces.** In this subsection we show that when M contains an essential tribranched surface Σ, the fundamental group $\pi_1(M)$ (in the usual sense) is presented as the fundamental group of the complex of groups $\overline{G}(X_\Sigma)$, which is a slight modification of $G(X_\Sigma)$.

A morphism ϕ^Σ from $G(X_\Sigma)$ to the fundamental group $\pi_1(M)$. Recall that we have chosen and fixed a point x_σ in σ for each $\sigma \in \Pi_0(M, \Sigma)$ in the previous subsection. Let us further choose and fix a point x_0 in $M(\Sigma)$ and a path $l_\sigma \colon [0, 1] \to M$ for each element σ of $\Pi_0(M, \Sigma)$ such that $l_\sigma(0) = x_0$ and $l_\sigma(1) = x_\sigma$. We define a morphism $\phi^\Sigma \colon G(X_\Sigma) \to \pi_1(M, x_0)$ as follows. For each $\sigma \in \Pi_0(M, \Sigma)$, we define a group homomorphism $\phi^\Sigma_{v_\sigma} \colon G^\Sigma_{v_\sigma} \to \pi_1(M, x_0)$ as the composition

$$G^\Sigma_{v_\sigma} = \pi_1(\sigma, x_\sigma) \to \pi_1(M, x_\sigma) \xrightarrow{(\flat)} \pi_1(M, x_0), \tag{2.4}$$

where the first map is induced from the natural inclusion $\sigma \hookrightarrow M$ and the second map $(\flat)$ is the change of the base point with respect to the path l_σ, that is, the map defined as $[c] \mapsto [l_\sigma * c * l_\sigma^{-1}]$. We also associate an element $\phi^\Sigma(a)$ of $\pi_1(M, x_0)$ defined as $[l_\tau * l_{\sigma,\tau}^{-1} * l_\sigma^{-1}]$ to each edge a of $\mathcal{X}_\Sigma$ when $i(a)$ and $t(a)$ are denoted by v_σ and v_τ respectively. Then the twisted commutativity

$$\phi^\Sigma(a)\phi^\Sigma_{v_\sigma}([c])\phi^\Sigma(a)^{-1} = \phi^\Sigma_{v_\tau} \circ \psi^\Sigma_a([c]) \tag{2.5}$$

straightforwardly holds for each element $[c]$ of $G^\Sigma_{v_\sigma} = \pi_1(\sigma, x_\sigma)$ by the construction of $\phi^\Sigma(a)$. Furthermore, for each composable pair (a, b) of edges in $\mathcal{X}_\Sigma$ with $i(b) = v_C$, $t(b) = i(a) = v_S$ and $t(a) = v_N$, one readily verifies the equation $\phi^\Sigma_{v_N}(g^\Sigma_{a,b})\phi^\Sigma(ab) = \phi^\Sigma(a)\phi^\Sigma(b)$ by direct calculation. Therefore $\phi^\Sigma = (\{\phi^\Sigma_{v_\sigma}\}_{\sigma\in\Pi_0(M,\Sigma)}, \{\phi^\Sigma(a)\}_{a\in E(\mathcal{X}_\Sigma)})$ is a morphism from $G(X_\Sigma)$ to $\pi_1(M, x_0)$.

The quotient $\overline{G}(X_\Sigma)$ of $G(X_\Sigma)$ with respect to ϕ^Σ. For each $\sigma \in \Pi_0(M, \Sigma)$, set $\overline{G}^\Sigma_{v_\sigma} = \phi^\Sigma_{v_\sigma}(G^\Sigma_{v_\sigma})$. Then one readily sees from the twisted commutativity (2.5) that the kernel of $\phi^\Sigma_{i(a)}$ coincides with that of $\phi^\Sigma_{t(a)} \circ \psi_a$ for each $a \in E(\mathcal{X}_\Sigma)$; hence ψ_a induces an injection $\overline{\psi}_a \colon \overline{G}^\Sigma_{i(a)} \to \overline{G}^\Sigma_{t(a)}$. By construction, the twisted commutativity (CG1) clearly holds for $\{\overline{\psi}_a\}_{a\in E(\mathcal{X}_\Sigma)}$ if we define the twisting element $\overline{g}_{a,b}$ as $\overline{g}_{a,b} = \phi^\Sigma_{t(a)}(g_{a,b}) \in \overline{G}^\Sigma_{t(a)}$ for each composable pair (a, b) of edges in $\mathcal{X}_\Sigma$. Therefore $\overline{G}(X_\Sigma) := (\mathcal{X}_\Sigma, \{\overline{G}^\Sigma_{v_\sigma}\}, \{\overline{\psi}_a\}, \{\overline{g}_{a,b}\})$ is again a complex of groups over X_Σ, which we call the *quotient complex of* $G(X_\Sigma)$ *with respect to* ϕ^Σ. Note that since every local group $\overline{G}^\Sigma_{v_\sigma}$ is by definition a subgroup of $\pi_1(M, x_0)$, the induced morphism $\overline{\phi}^\Sigma \colon \overline{G}(X_\Sigma) \to \pi_1(M, x_0)$ satisfies the assumption of Proposition 1.6, and thus $\overline{G}(X_\Sigma)$ is developable. In the rest of this subsection, we prove the following theorem by explicitly constructing the "universal development" of $\overline{G}(X_\Sigma)$.

Theorem 2.4. *Let* Σ *be an essential tribranched surface contained in a compact, connected, irreducible and orientable* 3*-manifold* M. *Then the morphism* $\overline{\phi}^\Sigma \colon \overline{G}(X_\Sigma) \to \pi_1(M, x_0)$ *constructed as above induces an isomorphism* $\pi_1(\overline{G}(X_\Sigma), \sigma_0) \xrightarrow{\sim} \pi_1(M, x_0)$.

Geometric construction of a development. Consider the universal cover $\widetilde{M}$ of M and let $\widetilde{\Sigma}$ denote the preimage of Σ under the universal covering map $p_{\widetilde{M}}\colon \widetilde{M} \to M$. Then one readily shows by using the covering space theory that $\widetilde{\Sigma}$ is also a tribranched surface, and the preimage $C(\widetilde{\Sigma})$ of $C(\Sigma)$ under $p_{\widetilde{M}}$ coincides with the set of branched points of $\widetilde{\Sigma}$. Furthermore, for each connected component $\widetilde{C}$ of $C(\widetilde{\Sigma})$ in the preimage of $C \in \pi_0(C(\Sigma))$ under $p_{\widetilde{M}}$, there exists a unique tubular neighbourhood $h_{\widetilde{C}}\colon \widetilde{C} \times D^2 \to \widetilde{M}$ of $\widetilde{C}$ in $\widetilde{M}$ satisfying $p_{\widetilde{M}}(h_{\widetilde{C}}(x,t)) = h_C(p_{\widetilde{M}}(x),t)$. We define $U_{\widetilde{C}}$ as $h_{\widetilde{C}}(\widetilde{C} \times \mathring{D}^2)$ and set $S(\widetilde{\Sigma})$ as $\widetilde{\Sigma} \setminus \bigcup_{\widetilde{C}\in\pi_0(C(\widetilde{\Sigma}))} U_{\widetilde{C}}$. Then, for each connected component $\widetilde{S}$ of $S(\widetilde{\Sigma})$ in the preimage of $S \in \pi_0(S(\Sigma))$ under $p_{\widetilde{M}}$, there exists a unique regular neighbourhood $h_{\widetilde{S}}\colon \widetilde{S} \times [-1,1] \to \widetilde{M} \setminus \bigcup_{\widetilde{C}\in\pi_0(C(\widetilde{\Sigma}))} U_{\widetilde{C}}$ of $\widetilde{S}$ in $\widetilde{M} \setminus \bigcup_{\widetilde{C}\in\pi_0(C(\widetilde{\Sigma}))} U_{\widetilde{C}}$ satisfying $p_{\widetilde{M}}(h_{\widetilde{S}}(x,t)) = h_S(p_{\widetilde{M}}(x),t)$. We define $U_{\widetilde{S}}$ as $h_{\widetilde{S}}(\widetilde{S} \times (-1,1))$, and define $M(\widetilde{\Sigma})$ as the complement of $\bigcup_{\widetilde{C}\in\pi_0(C(\widetilde{\Sigma}))} U_{\widetilde{C}} \cup \bigcup_{\widetilde{S}\in\pi_0(S(\widetilde{\Sigma}))} U_{\widetilde{S}}$ in $\widetilde{M}$. We remark that $S(\widetilde{\Sigma})$ and $M(\widetilde{\Sigma})$ coincide with the preimages of $S(\Sigma)$ and $M(\Sigma)$ under $p_{\widetilde{M}}$ respectively. Now we can endow $\widetilde{M}$ with an equivalence relation $\sim_{\widetilde{\Sigma}}$ and construct a Δ-complex $X_{\widetilde{\Sigma}}$ of dimension 2 as the quotient space $X_{\widetilde{\Sigma}} = \widetilde{M}/\sim_{\widetilde{\Sigma}}$, in the completely same manner as the construction of X_Σ. Similarly to X_Σ, there exists a quotient map $r_{\widetilde{\Sigma}}\colon \widetilde{M} \to X_{\widetilde{\Sigma}}$, and it is easy to construct a continuous map $i_{\widetilde{\Sigma}}\colon X_{\widetilde{\Sigma}} \to \widetilde{M}$ such that $r_{\widetilde{\Sigma}} \circ i_{\widetilde{\Sigma}}$ is homotopic to the identity map on $X_{\widetilde{\Sigma}}$. The compositions of the induced maps

$$\pi_0(X_{\widetilde{\Sigma}}) \xrightarrow{i_{\widetilde{\Sigma},*}} \pi_0(\widetilde{M}) \xrightarrow{r_{\widetilde{\Sigma},*}} \pi_0(X_{\widetilde{\Sigma}}), \qquad \pi_1(X_{\widetilde{\Sigma}}) \xrightarrow{i_{\widetilde{\Sigma},*}} \pi_1(\widetilde{M}) \xrightarrow{r_{\widetilde{\Sigma},*}} \pi_1(X_{\widetilde{\Sigma}})$$

are thus the identity maps. Since $\widetilde{M}$ is simply connected, both of $\pi_0(X_{\widetilde{\Sigma}})$ and $\pi_1(X_{\widetilde{\Sigma}})$ are singletons, and hence we see that $X_{\widetilde{\Sigma}}$ is also *simply connected*. Note that the Δ-complex $X_{\widetilde{\Sigma}}$ admits an action of $\pi_1(M,x_0)$, which is indeed a simplicial action without inversions, induced from its natural action on $\widetilde{M}$ via deck transformations.

Proposition 2.5. *The 2-complex of groups $\overline{G}(X_\Sigma)$ and the morphism $\overline{\phi}^{\Sigma}\colon \overline{G}(X_\Sigma) \to \pi_1(M,x_0)$ are associated to the action of $\pi_1(M,x_0)$ on the Δ-complex $X_{\widetilde{\Sigma}}$ constructed as above.*

Proof. Recall that $p_{\widetilde{M}}\colon \widetilde{M} \to M$ denotes the universal covering of M. Take an arbitrary point $\tilde{x}_0$ from $p_{\widetilde{M}}^{-1}(x_0)$. For each $\sigma \in \Pi_0(M,\Sigma)$, let $\tilde{l}_\sigma$ denote a unique lift of l_σ to $\widetilde{M}$ satisfying $\tilde{l}_\sigma(0) = \tilde{x}_0$. We set $\tilde{x}_\sigma = \tilde{l}_\sigma(1)$ and denote by $\tilde{\sigma}$ a unique connected component of $p_{\widetilde{M}}^{-1}(\sigma)$ containing $\tilde{x}_\sigma$. Note that $\tilde{x}_\sigma$ is a lift of x_σ to $\widetilde{M}$. We shall verify that all the data of which the complex of groups $\overline{G}(X_\Sigma)$ consists (specifically the local groups $\overline{G}^{\Sigma}_{v_\sigma}$, the local homomorphisms $\overline{\psi}^{\Sigma}_a$ and the twisting elements $\overline{g}^{\Sigma}_{a,b}$) are obtained from the action of $\pi_1(M,x_0)$ on the Δ-complex $X_{\widetilde{\Sigma}}$.

Via the monodromy homomorphism and the parallel translation along $\tilde{l}_\sigma$, we may identify $\pi_1(M,x_0)$ with the automorphism group of $p_{\widetilde{M}}^{-1}(\{x_\sigma\})$. The isotropy subgroup $\pi_1(M,x_0)_{\tilde{\sigma}}$ of $\pi_1(M,x_0)$ at $\tilde{\sigma}$ is then identified with the group of covering automorphisms of $\tilde{\sigma} \to \sigma$, and the latter group coincides with the image of $\pi_1(\sigma,x_\sigma)$ in $\pi_1(M,x_0)$ under the map (2.4). We can thus conclude that

$$\overline{G}^{\Sigma}_{v_\sigma} = \phi^{\Sigma}_{v_\sigma}(G^{\Sigma}_{v_\sigma}) = \text{the image of } \pi_1(\sigma,x_\sigma) \text{ in } \pi_1(M,x_0) \text{ under the map (2.4)}$$

is the isotropy subgroup of $\pi_1(M,x_0)$ at $\tilde{\sigma}$ (or at $v_{\tilde{\sigma}}$) with respect to the natural action of $\pi_1(M,x_0)$ on $\widetilde{M}$ (or on $X_{\widetilde{\Sigma}}$).

Next we verify that, for an appropriate choice of $h_a \in \pi_1(M,x_0)$ for each edge a in $\mathcal{X}_\Sigma$, the equalities $\overline{\psi}_a(-) = h_a(-)h_a^{-1}$ and $\overline{g}_{a,b} = h_a h_b h_{ab}^{-1}$ hold. Let a be an edge in $\mathcal{X}_\Sigma$ and denote its initial and terminal vertices by v_σ and v_τ respectively. Let $\tilde{a}$ be a unique edge of $\mathcal{X}_{\widetilde{\Sigma}}$ which is a lift of a and satisfies $i(\tilde{a}) = v_{\tilde{\sigma}}$. We may identify $\tilde{a}$ with a unique lift $\tilde{l}_{\sigma,\tau}$ of $l_{\sigma,\tau}$ to $\widetilde{M}$ satisfying $\tilde{l}_{\sigma,\tau}(0) = \tilde{x}_\sigma$ up to homotopy. Then the parallel translation along the path $\tilde{l}_{\sigma,\tau}^{-1} * \tilde{l}_\sigma^{-1} * \tilde{l}_\tau$ defines an element $\tilde{h}_a = [l_{\sigma,\tau}^{-1} * l_\sigma^{-1} * l_\tau]$ of $\mathrm{Aut}(p_{\widetilde{M}}^{-1}(\{x_\tau\})) \cong \pi_1(M, x_\tau)$ satisfying $\tilde{h}_a t(\tilde{a}) = \widetilde{t(a)} = v_{\tilde{\tau}}$. We denote by $h_a = [l_\tau * l_{\sigma,\tau}^{-1} * l_\sigma^{-1}]$ the image of $\tilde{h}_a$ in $\pi_1(M, x_0)$ under the change of base points $\pi_1(M, x_\tau) \xrightarrow{(\flat)} \pi_1(M, x_0)$ appearing in (2.4). Then the image $\phi_{v_\sigma}^\Sigma(\xi)$ of an element ξ of $\pi_1(\sigma, x_\sigma)$ in $\pi_1(M, x_0)$ under the map (2.4) is none other than $[l_\sigma]\overline{\xi}[l_\sigma^{-1}]$ if we denote the image of ξ in $\pi_1(M, x_\sigma)$ by $\overline{\xi}$, and thus we may calculate as

$$h_a \phi_{v_\sigma}^\Sigma(\xi) h_a^{-1} = [l_\tau * l_{\sigma,\tau}^{-1} * l_\sigma^{-1}]([l_\sigma]\overline{\xi}[l_\sigma^{-1}])[l_\sigma * l_{\sigma,\tau} * l_\tau^{-1}] = [l_\tau * l_{\sigma,\tau}^{-1}]\overline{\xi}[l_{\sigma,\tau} * l_\tau^{-1}],$$

which is identified with the image of $[l_{\sigma,\tau}^{-1}]\xi[l_{\sigma,\tau}]$ in $\pi_1(\tau, x_\tau)$ under the map (2.4). Comparing with the definition (2.2) of ψ_a^Σ, we obtain the desired equality $\overline{\psi}_a^\Sigma(\phi_{v_\sigma}^\Sigma(\xi)) = h_a \phi_{v_\sigma}^\Sigma(\xi) h_a^{-1}$. Similarly we calculate as

$$\begin{aligned} h_a h_b h_{ab}^{-1} &= [l_N * l_{S,N}^{-1} * l_S^{-1}][l_S * l_{C,S}^{-1} * l_C^{-1}][l_C * l_{C,N} * l_N^{-1}] \\ &= [l_N * l_{S,N}^{-1} * l_{C,S}^{-1} * l_{C,N} * l_N^{-1}] \qquad \text{(as an element of } \pi_1(M, x_0)) \\ &= [l_{S,N}^{-1} * l_{C,S}^{-1} * l_{C,N}] = \overline{g}_{a,b}^\Sigma \qquad \text{(as an element of } \phi_{v_N}^\Sigma(\pi_1(N, x_N))) \end{aligned}$$

for composable edges a and b. Here C, S and N denote elements of $\pi_0(C(\Sigma))$, $\pi_0(S(\Sigma))$ and $\pi_0(M(\Sigma))$ respectively such that $i(b) = v_C$, $t(b) = i(a) = v_S$ and $t(a) = v_N$ hold. Moreover $\phi^\Sigma(a) = h_a$ trivially follows by the definitions of $\phi^\Sigma(a)$ and h_a. Therefore, under the specific choices of a lift $\tilde{\sigma}$ of each cell σ of X_Σ and an element $h_a = [l_\tau * l_{\sigma,\tau}^{-1} * l_\sigma^{-1}] \in \pi_1(M, x_0)$ for each edge a in $\mathcal{X}_\Sigma$ with $i(a) = v_\sigma$ and $t(a) = v_\tau$, the complex of groups $\overline{G}(X_\Sigma)$ is indeed the one associated to the action of $\pi_1(M, x_0)$ on the Δ-complex $X_{\widetilde{\Sigma}}$, and $\overline{\phi}^\Sigma$ is the associated morphism. □

Proof of Theorem 2.4. Since the induced morphism $\overline{\phi}^\Sigma \colon \overline{G}(X_\Sigma) \to \pi_1(M, x_0)$ satisfies the assumption of Proposition 1.6, the complex of groups $\overline{G}(X_\Sigma)$ over the connected Δ-complex X_Σ is developable and thus, by Theorem 1.7 (1), it admits a universal cover $\widetilde{X}_\Sigma$. On the other hand, we have constructed the simply connected Δ-complex $X_{\widetilde{\Sigma}}$ and the action without inversions of $\pi_1(M, x_0)$ on $X_{\widetilde{\Sigma}}$, to which the complex of groups $\overline{G}(X_\Sigma)$ and the morphism $\overline{\phi}^\Sigma \colon \overline{G}(X_\Sigma) \to \pi_1(M, x_0)$ are associated. Then Theorem 1.7 (2) implies that there are a surjection $\varphi \colon \pi_1(\overline{G}(X_\Sigma), v_0) \twoheadrightarrow \pi_1(M, x_0)$ and a φ-equivariant simplicial map $f \colon \widetilde{X}_\Sigma \to X_{\widetilde{\Sigma}}$ such that f is a covering map with Galois group $\mathrm{Ker}\,\varphi$; in particular $\pi_1(X_{\widetilde{\Sigma}}) = \mathrm{Ker}\,\varphi$ holds. The simply connectedness of $X_{\widetilde{\Sigma}}$ implies the triviality of $\mathrm{Ker}\,\varphi$, and we thus obtain an isomorphism $\pi_1(\overline{G}(X_\Sigma), v_0) \xrightarrow{\sim} \pi_1(M, x_0)$ as desired. □

Remark 2.6. Note that there exists a natural surjection

$$\pi_1(\overline{G}(X_\Sigma), v_0) \twoheadrightarrow \pi_1^{\mathrm{edge}}(\mathcal{X}_\Sigma, v_0) \xrightarrow{\sim} \pi_1^{\mathrm{top}}(|X_\Sigma|, x_0),$$

where the first map sends any elements of $\coprod_\sigma \overline{G}_{v_\sigma}^\Sigma$ to the unit, and the second isomorphism is due to Proposition 1.3. The rightmost group $\pi_1^{\mathrm{top}}(|X_\Sigma|, x_0)$ reflects topological information on the arrangement of the essential tribranched surface Σ in M (recall the construction of the Δ-complex X_Σ). Theorem 2.4 combined with the the natural surjection above implies that the

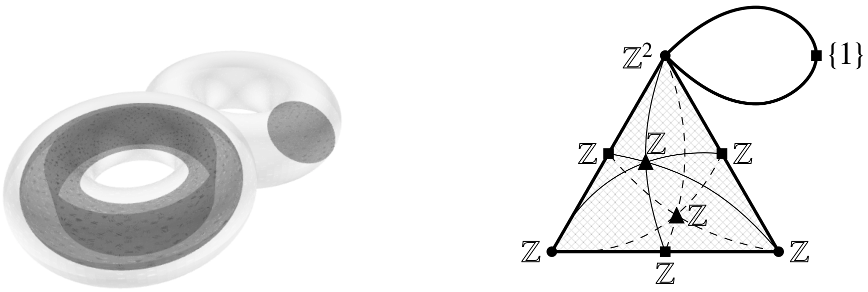

FIGURE 4. An essential tribranched surface which does not satisfy (ETBS4)

arrangement of Σ also affects the presentation of $\pi_1(M, x_0)$ as the fundamental group of the complex of groups $\overline{G}(X_\Sigma)$.

2.4. **Appendix: About injectivity on local groups.** Now let us consider the following additional condition on essential tribranched surfaces;

(ETBS4) for each component N of $M(\Sigma)$, the homomorphism $\pi_1(N) \to \pi_1(M)$ induced from the natural inclusion $N \hookrightarrow M$ is injective.

We readily see that $\overline{G}(X_\Sigma)$ coincides with $G(X_\Sigma)$ if an essential tribranched surface Σ satisfies the condition (ETBS4). In the case Theorem 2.4 implies that $\pi_1(M, x_0)$ can be completely recovered from the fundamental groups $\pi_1(\sigma, x_\sigma)$ of components $\sigma \in \Pi_0(M, \Sigma)$.

Unlike the case of essential surfaces (without any branched points), one cannot derive the condition (ETBS4) from the conditions (ETBS1), (ETBS2) and (ETBS3) of essential tribranched surfaces, and there does exist an essential tribranched surface which does not satisfy the condition (ETBS4). For example, consider two solid tori M_1 and M_2. Let Σ_1 be a tribranched surface isomorphic to $\Theta \times S^1$, where Θ is a theta graph, naturally embedded in the interior of M_1, and Σ_2 a meridian disk in M_2. Then the boundary-connected sum M of M_1 and M_2 contains an essential tribranched surface $\Sigma = \Sigma_1 \sqcup \Sigma_2$, which does not satisfy (ETBS4); indeed if we denote by N_0 a unique connected componen of $M \setminus \Sigma$ outside Σ_1, the image of its fundamental group $\pi_1(N_0) \cong \mathbb{Z}^2$ in $\pi_1(M)$ is cyclic because a meridian loop of N_0 is homotopic to the trivial loop in M. Theorem 2.4 combined with construction of the complex of groups $\overline{G}(X_\Sigma)$ implies that such a “redundant” loop does not contributes to the fundamental group of the total space M. Figure 4 illustrates the picture of Σ and the associated complex of groups $G(X_\Sigma)$; the base Δ-complex of $G(X_\Sigma)$ consists of two triangles identified along their boundaries and a circuit whose initial and terminal vertex corresponds to N_0. Here circles, squares and triangles indicate barycentres of 0-cells, 1-cells and 2-cells of X_Σ respectively.

The example above seems a bit artificial, because the handle body M of genus 2 contains a much simpler essential (tribranched) surface, namely the disjoint union of two meridian disks each in M_1 and M_2, which obviously satisfies the condition (ETBS4). Thus it would be interesting to ask for which class of 3-manifolds their fundamental groups have presentations associated to essential tribranched surfaces satisfying (ETBS4).

3. THE BRUHAT–TITS BUILDINGS $\mathcal{B}(\mathrm{SL}(n)_{/F})$ ASSOCIATED TO THE SPECIAL LINEAR GROUPS

Bruhat–Tits buildings are combinatorial and topological objects associated to reductive algebraic groups defined over non-archimedean valuated fields, which behave as Riemannian symmetric spaces in differential geometry; in particular they admit natural "transitive" actions of the algebraic groups (to be precise, the natural group actions on the Bruhat–Tits buildings are *strictly transitive*; see the end of Section 3.1 for the definition of strict transitivity). The theory of Bruhat–Tits buildings has its origin in the study of Nagayoshi Iwahori and Hideya Matsumoto on the generalised Bruhat decomposition of $\mathfrak{p}$-adic Chevalley groups [IM65], and then it has been elaborated by François Bruhat and Jacques Tits in a systematic and axiomatic way [BT72, BT84]. The Bruhat–Tits tree, which appears in the work of Culler and Shalen [CS83], is none other than the Bruhat–Tits building associated to the special linear group SL(2) of degree 2, and the Bruhat–Tits buildings associated to the special linear groups *of higher degree* play crucial roles in our extension of Culler and Shalen's results. In this section we shall summarise basic notion on Bruhat–Tits buildings and their fundamental properties, especially for the special linear groups.

3.1. **Euclidean buildings and their contractibility.** We first review the axiomatic definition of (Euclidean) buildings after Tits and basic properties of Euclidean buildings. Refer, for instance, to [AB08, Ga97] for details of the contents of this subsection.

Definition 3.1 (Chamber complexes)**.** Let Σ be an abstract simplicial complex of finite dimension (that is, every simplex of Σ is of finite dimension). We call Σ a *chamber complex* if the following two conditions are fulfilled:

(CC1) every maximal simplex of Σ has the same dimension n;
(CC2) every two maximal simplices C and C' are connected by a *gallery*; that is, there exists a sequence of maximal simplices $C_0 = C,\ C_1, \dots, C_r = C'$ of Σ such that C_{i-1} and C_i are adjacent for each $1 \leq i \leq r$.

Here we say that maximal simplices C and C' of Σ are *adjacent* if C and C' are distinct and contain a common $(n-1)$-dimensional face. A maximal simplex of Σ is called a *chamber* of Σ. The *dimension* of Σ is defined as the (same) dimension n of a chamber of Σ. A chamber complex Σ of dimension n is said to be *thin* if every $(n-1)$-dimensional simplex of Σ is a face of exactly two chambers.

Definition 3.2 (Buildings)**.** Let Δ be an abstract simplicial complex. We call Δ a (*simplicial*) *thick building* of dimension n if there exists a family $\mathcal{A}$ of n-dimensional thin chamber subcomplexes of Δ and the pair $(\Delta, \mathcal{A})$ satisfies the following axioms:

(B0) the complex Δ is the union of all elements of $\mathcal{A}$, and each $(n-1)$-dimensional simplex of Δ is a face of at least three maximal simplices of Δ (which are of dimension n);
(B1) every two simplices of Δ lie in a common chamber subcomplex of Δ belonging to $\mathcal{A}$;
(B2) if Σ and Σ' are elements of $\mathcal{A}$ both of which contain two simplices σ and τ, there exists an isomorphism $\Sigma \xrightarrow{\sim} \Sigma'$ of chamber complexes which fixes all the vertices of σ and τ.

A thin chamber subcomplex Σ of Δ belonging to $\mathcal{A}$ is called an *apartment* of Δ, and a maximal simplex of Δ is called a *chamber* of Δ. Among families of thin chamber subcomplexes of Δ satisfying all the axioms (B0), (B1) and (B2), there exists a unique maximal one $\mathcal{A}^{\mathrm{cpl}}$ which is called the *complete system of apartments* of Δ. We can deduce from the conditions (B0), (B1), (B2) that the apartments of a thick building Δ are all isomorphic to some *Coxeter complex* $\Sigma(W, S)$ for a well-defined Coxeter system (W,S); refer to [AB08, Theorem 4.131] (see also [AB08, Sections 2 and 3] for details on Coxeter systems and Coxeter complexes).

It is well known that a building Δ of dimension n is a *colorable* chamber complex; namely there exists an I_{n+1}-valued function τ on the vertices of Δ such that the vertices of each chamber of Δ are mapped bijectively onto I_{n+1}, where I_{n+1} denotes a finite set of cardinality $n+1$. Such a function τ is called a *type function on* Δ (with values in I_{n+1}). We refer the reader to [AB08, Proposition 4.6] for details.

Definition 3.3 (Euclidean buildings)**.** A building Δ of dimension n is said to be a *Euclidean building* (or a *building of affine type*) if the geometric realisations of all apartments of Δ are isomorphic to a *Euclidean Coxeter complex* $\Sigma(W, S)$ of dimension n for a certain Euclidean Coxeter system (W, S).

We omit the definition of Euclidean Coxeter systems and Euclidean Coxeter complexes because we do not need their explicit definition in this article (see, for example, [AB08, Chapter 10] for details). As a simplicial complex, the Euclidean Coxeter complex $\Sigma(W, S)$ is homeomorphic to the Euclidean space $\mathbb{R}^n$ of dimension n tessellated by n-simplices; the type of tessellation is determined by the Euclidean Coxeter system (W, S).

Now let Δ be a Euclidean building. For arbitrary two points x and y of the geometric realisation $|\Delta|$ of Δ, there exists an apartment $\Sigma_{(x,y)}$ of Δ whose geometric realisation $|\Sigma_{(x,y)}|$ contains both of x and y due to the axiom (B1) of buildings. We equip $|\Sigma_{(x,y)}|$ with the standard Euclidean metric $d_{|\Sigma_{(x,y)}|}$, and define a real-valued function $d_{|\Delta|}$ on $|\Delta| \times |\Delta|$ by

$$d_{|\Delta|} \colon |\Delta| \times |\Delta| \to \mathbb{R}_{\geq 0} \ ; \ (x, y) \mapsto d_{|\Sigma_{(x,y)}|}(x, y).$$

Then $d_{|\Delta|}$ is a metric on the geometric realisation $|\Delta|$ of Δ which is well defined independently of the choice of an apartment $\Sigma_{(x,y)}$ due to the axiom (B2) of buildings. One readily checks that the topology of $|\Delta|$ determined by the metric $d_{|\Delta|}$ coincides with the weak topology endowed on $|\Delta|$. Bruhat and Tits have verified that the metric space $(|\Delta|, d_{|\Delta|})$ is a CAT(0) space; in particular $|\Delta|$ is *contractible* (refer to [BT72, Propositions 2.5.3. et 2.5.16] for details; see also [AB08, the proof of Theorem 11.16]). The contractibility of Euclidean buildings shall play a crucial role in the construction of tribranched surfaces in Section 4.2.

We shall end this subsection by presenting several notion concerning group actions on buildings. Let G be a group and Δ a building on which G acts. One easily verifies that the action of G on Δ induces actions of G both on the complete system of apartments $\mathcal{A}^{\mathrm{cpl}}$ of Δ and on the set of all the chambers of Δ. An action of a group G on a building Δ is said to be *strictly transitive* if G acts transitively on the set of all pairs (Σ, C) consisting of an apartment Σ (belonging to $\mathcal{A}^{\mathrm{cpl}}$) and a chamber C contained in Σ, and said to be *type-preserving* if an arbitrary element γ of G maps a vertex of Δ to one of the same type (with respect to a certain type function on Δ).

3.2. **Combinatorial construction of $\mathcal{B}(\mathrm{SL}(n)_{/F})$.** One of the most significant aspects in the theory of Euclidean buildings is the fact that one may associate in a canonical manner a Euclidean building [3] $\mathcal{B}(G_{/F})$ to a reductive algebraic group G defined over a non-archimedean valuated field F. Furthermore $\mathcal{B}(G_{/F})$ admits a natural, strictly transitive action of $G(F)$. The existence of such Euclidean buildings was first observed in the pioneering work of Iwahori and Matsumoto [IM65] for Chevalley groups (which are in particular split, semisimple and simply

[3] We remark that, for general reductive groups, the Bruhat–Tits buildings are *polysimplicial complexes* and not necessarily simplicial.

connected algebraic groups) defined over $\mathfrak{p}$-adic fields.[4] Then Bruhat and Tits established construction of such Euclidean buildings in [BT72, BT84] for general reductive algebraic groups. The Euclidean building $\mathcal{B}(G_{/F})$ attached to $G_{/F}$ is therefore called the *Bruhat–Tits building* associated to $G_{/F}$.

Bruhat and Tits's construction of $\mathcal{B}(G_{/F})$ utilising "valuated root data" is rather abstract and complicated, but limiting ourselves to the Bruhat–Tits building $\mathcal{B}(G_{/F})$ associated to the *special linear group* $G = \mathrm{SL}(n)$ defined over a discrete valuation field (which is a $\mathfrak{p}$-adic Chevalley group and thus has been already dealt with by Iwahori and Matsumoto in [IM65]), we may explicitly describe the combinatorial structure of $\mathcal{B}(G_{/F})$ and the effect of the action of $G(F)$ on $\mathcal{B}(G_{/F})$ without introducing any root datum. We propose in this subsection a combinatorial description of the Bruhat–Tits building $\mathcal{B}(\mathrm{SL}(n)_{/F})$ associated to the special linear group $\mathrm{SL}(n)_{/F}$, mainly following [Ga97, Chapter 19]. We shall only utilise the Bruhat–Tits buildings $\mathcal{B}(\mathrm{SL}(n)_{/F})$ associated to the special linear groups in our later applications.

Let F be a field equipped with a (normalised) discrete valuation $w\colon F^\times \to \mathbb{Z}$. We do not require that the base field F is complete with respect to the multiplicative valuation $|\cdot|_w$ associated to w (indeed we shall later apply results of this subsection to a case where the base field is not complete). We denote the valuation ring of F with respect to w by O_w. We fix a uniformiser ϖ_w of the discrete valuation field (F, w); in other words, we choose and fix a generator ϖ_w of the maximal ideal of O_w (which is known to be a principal ideal due to basic facts of valuation theory).

Let V_n denote an n-dimensional vector space over F equipped with a basis $\{e_1, \dots, e_n\}$. We identify V_n with $F^{\oplus n}$ (the F-vector space of n-dimensional column vectors) with respect to the specified basis $\{e_j\}_{j=1}^n$ and regard the special linear group $\mathrm{SL}_n(F)$ as a subgroup of $\mathrm{Aut}_F(V_n)$. An O_w-submodule L of V_n is called a *lattice* of V_n if L spans V_n over F: $\langle L\rangle_F = V_n$. Every lattice of V_n is then a free O_w-module of rank n by elementary divisor theory. Two lattices L and L' of V_n are said to be *homothetic* if there exists a nonzero element a of F such that L coincides with aL' (as an O_w-submodule of V_n). The homothety relation is an equivalence relation on the set of all lattices of V_n, and we define the vertex set $V(\mathcal{B}(\mathrm{SL}(n)_{/(F,w)}))$ of the Bruhat–Tits building $\mathcal{B}(\mathrm{SL}(n)_{/(F,w)})$ as the set of homothety classes of lattices of V_n. We say that two distinct elements v and v' of $V(\mathcal{B}(\mathrm{SL}(n)_{/(F,w)}))$ are *adjacent* if there exist lattices L and L' representing the homothety classes v and v' respectively such that

$$\varpi_w L' \subsetneq L \subsetneq L'$$

holds (as O_w-submodules of V_n). We then define $\mathcal{B}(\mathrm{SL}(n)_{/(F,w)})$ as an abstract simplicial complex each of whose simplices is a finite subset $\{v_1, \dots, v_r\}$ of $V(\mathcal{B}(\mathrm{SL}(n)_{/(F,w)}))$ consisting of vertices adjacent to each other; in other words, a set $\{v_1, \dots, v_r\}$ of r vertices of $\mathcal{B}(\mathrm{SL}(n)_{/(F,w)})$ forms an r-simplex if and only if there exists a lattice L_i representing v_i for each $1 \leq i \leq r$ such that

$$\varpi_w L_r \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_r$$

holds (after appropriate relabeling of the subindices). For an arbitrary F-basis $\mathbf{f} = \{f_1, \dots, f_n\}$ of V_n, consider a subcomplex $\Sigma_{\mathbf{f}}$ of $\mathcal{B}(\mathrm{SL}(n)_{/(F,w)})$ generated by the homothety classes of lattices of the form $\sum_{j=1}^n O_w \varpi_w^{m_j} f_j$ (each m_j takes an arbitrary integer). The subcomplex $\Sigma_{\mathbf{f}}$ is indeed

[4] More precisely, Iwahori and Matsumoto have constructed a (generalised) BN pair with respect to the *Iwahori subgroup* B of a $\mathfrak{p}$-adic Chevalley group in [IM65, Proposition 2.2, Theorem 2.22]. Although they have never mentioned buildings in [IM65], it is well known that one may associate buildings to such BN-pairs in a canonical way; see [AB08, Theorem 6.56] for example.

a thin chamber complex of dimension $n-1$. Denote by $\mathcal{A}$ the family of the subcomplexes $\Sigma_{\mathbf{f}}$ of $\mathcal{B}(\mathrm{SL}(n)_{/(F,w)})$ indexed by an F-basis $\mathbf{f}$ of V_n. Then we may readily verify that the pair $(\mathcal{B}(\mathrm{SL}(n)_{/(F,w)}), \mathcal{A})$ satisfies all the axioms (B0), (B1) and (B2) of buildings; see [Ga97, Chapter 19.2] for details. The special linear group $\mathrm{SL}_n(F)$ acts on the set of lattices of V_n in an obvious manner; namely, for a lattice $L = \sum_{j=1}^{n} \mathcal{O}_w f_j$ with an $\mathcal{O}_w$-basis $\{f_1, \dots, f_n\}$, we define gL as an $\mathcal{O}_w$-submodule of V_n spanned by $\{g(f_1), \dots, g(f_n)\}$ (here we regard g as an element of $\mathrm{Aut}_F(V_n)$). This defines an action of $\mathrm{SL}_n(F)$ on $V(\mathcal{B}(\mathrm{SL}(n)_{/(F,w)}))$, which is naturally extended to an action of $\mathrm{SL}_n(F)$ on $\mathcal{B}(\mathrm{SL}(n)_{/(F,w)})$. One of the significant features of the action of $\mathrm{SL}_n(F)$ on $\mathcal{B}(\mathrm{SL}(n)_{/(F,w)})$ is that it is a strictly transitive and type-preserving action. In particular, an element γ of $\mathrm{SL}_n(F)$ fixes *all the vertices* of a chamber C whenever γ stabilises C.

In order to see that it is type-preserving, one has only to check that an association of a value $\tau(v) = (w(\det g_v) \bmod n)$ to each vertex v of $\mathcal{B}(\mathrm{SL}(n)_{/(F,w)})$ defines a type function τ on $\mathcal{B}(\mathrm{SL}(n)_{/(F,w)})$ with values in $\mathbb{Z}/n\mathbb{Z}$. Here g_v is an element of $\mathrm{Aut}_F(V_n)$ satisfying $L = g_v(L_0)$ for a certain lattice L representing v, and L_0 denotes the standard lattice of V_n defined as $L_0 = \sum_{j=1}^{n} \mathcal{O}_w e_j$. Then the type of a vertex of $\mathcal{B}(\mathrm{SL}(n)_{/(F,w)})$ does not change under the action of an element γ of $\mathrm{SL}_n(F)$ since one has

$$\tau(\gamma v) = (w(\det(\gamma g_v)) \mod n) = (w(\det \gamma) \mod n) + \tau(v) = \tau(v)$$

by using $\det(\gamma) = 1$.

Remark 3.4. The Bruhat–Tits building $\mathcal{B}(\mathrm{GL}(n)_{/(F,w)})$ associated to the *general linear group* $\mathrm{GL}(n)_{/(F,w)}$ is the same one as $\mathcal{B}(\mathrm{SL}(n)_{/(F,w)})$. However, the natural action of $\mathrm{GL}_n(F)$ on $\mathcal{B}(\mathrm{GL}(n)_{/(F,w)})$ does *not* preserve the type function $\tau(v) = (w(\det g_v) \bmod n)$ introduced above since the $\mathbb{Z}$-valued function $w \circ \det$ on $\mathrm{GL}_n(F)$ takes arbitrary value (indeed $\mathrm{GL}_n(F)$ acts transitively on the vertex set $V(\mathcal{B}(\mathrm{GL}(n)_{/(F,w)}))$). In order to guarantee that the natural action on the Bruhat–Tits building is type-preserving, we deal with the Bruhat–Tits building associated to the special linear group $\mathrm{SL}(n)$ rather than the Bruhat–Tits building associated to the general linear group $\mathrm{GL}(n)$. We shall effectively utilise the type-preserving property of the action when we consider the quotient complex $\mathcal{B}_{n,\widetilde{D},\tilde{y}}/\pi_1(M, x_0)$ in Section 4.3.

Example 3.5 (Bruhat–Tits trees). In the case where n equals 2, the construction of $\mathcal{B}(\mathrm{SL}(2)_{/(F,w)})$ explained above is none other than the classical construction of the *Bruhat–Tits tree* associated to $\mathrm{SL}(2)_{/F}$, which is, for example, presented in [Se77, Chapitre II, Section 1]. Note that the Bruhat–Tits trees play crucial roles in the original work of Culler and Shalen [CS83].

4. Construction of essential tribranched surfaces

We shall establish our construction of essential tribranched surfaces in this section. There are two technical hearts in the construction. One is to obtain a *nontrivial* type-preserving action of the 3-manifold group on the Bruhat–Tits building associated to the special linear group $\mathrm{SL}(n)$ by utilising geometry of character varieties of higher degree. After a brief review on character varieties of higher degree in Section 4.1, we explain how to obtain such a nontrivial action in Section 4.2. The other is to construct a non-empty tribranched surfaces from such a nontrivial action. In Section 4.3, we put this procedure in practice, and then modify the obtained tribranched surfaces to be *essential* by certain local surgeries.

4.1. $\mathrm{SL}_n(\mathbb{C})$**-character variety.** We begin with briefly reviewing the $\mathrm{SL}_n(\mathbb{C})$-*character variety* of a finitely generated group. See for instance Lubotzky and Magid [LM85] for basic results on representations and character varieties.

Let π be a finitely generated group. We denote by $R_n(\pi)$ the set $\mathrm{Hom}(\pi, \mathrm{SL}_n(\mathbb{C}))$ of all the $\mathrm{SL}_n(\mathbb{C})$-representations of π, which is an affine algebraic set. The algebraic group $\mathrm{SL}(n)_{/\mathbb{C}}$ acts on $R_n(\pi)$ by conjugation. We denote by $X_n(\pi)$ the geometric invariant theoretical quotient of $R_n(\pi)$ with respect to this action, which is called the $\mathrm{SL}_n(\mathbb{C})$-*character variety* of π. We define the *character* $\chi_\rho \colon \pi \to \mathbb{C}$ of an $\mathrm{SL}_n(\mathbb{C})$-representation $\rho\colon \pi \to \mathrm{SL}_n(\mathbb{C})$ as $\chi_\rho(\gamma) = \mathrm{tr}\,\rho(\gamma)$ for each element γ in π. The set of $\mathbb{C}$-valued points $X_n(\pi)$ of $X_n(\pi)$ coincides with that of characters χ_ρ of $\mathrm{SL}_n(\mathbb{C})$, and under this identification the quotient map $R_n(\pi)(\mathbb{C}) \to X_n(\pi)(\mathbb{C})$ is regarded as the map which sends ρ to χ_ρ. For an element γ of π, we define the *invariant functions* $\sigma_{\gamma,i}\colon X_n(\pi)(\mathbb{C}) \to \mathbb{C}$ associated to γ for $i = 1, \dots, n-1$ as the (sign-modified) coefficients of the characteristic polynomial of $\rho(\gamma)$

$$\det(tE_n - \rho(\gamma)) = t^n - \sigma_{\gamma,1}(\chi_\rho)t^{n-1} + \cdots + (-1)^{n-1}\sigma_{\gamma,n-1}(\chi_\rho)t + (-1)^n,$$

where E_n is the identity matrix of size n. Note that each coefficient $(-1)^i\sigma_{\gamma,i}(\chi_\rho)$ depends only on the equivalent class of ρ, and is a regular functions on $X_n(\pi)(\mathbb{C})$. We set $I_\gamma = \sigma_{\gamma,1}$ for each γ and call it the *trace function* associated to γ.

The following theorem is a direct consequence of the result of Claudio Procesi [Pr76].

Theorem 4.1 (Procesi, [Pr76, Theorem 3.4 (a)])**.** *Let $\gamma_1, \dots, \gamma_m$ be a generator system of π. Then the trace functions $\{I_{\gamma_{i_1}\cdots\gamma_{i_k}}\}_{1\le i_1,\dots,i_k\le m}^{1\le k\le 2^n-1}$ give affine coordinates of $X_n(\pi)$.*

For a compact 3-manifold M we abbreviate $X_n(\pi_1(M))$ as $X_n(M)$ to simplify notation.

Remark 4.2. Let M be an orientable, finite-volume, hyperbolic 3-manifold with l torus cusps. Then we may consider a lift $\rho_0\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ of the monodromy representation with respect to the hyperbolic structure of M [CS83, Proposition 3.1.1]. Menal-Ferrer and Porti [MFP12a, MFP12b] showed for general n the following facts;

i) the character variety $X_n(M)$ is smooth at the $\mathbb{C}$-valued point $\chi_{\iota_n\circ\rho_0}$;
ii) the irreducible component of $X_n(M)$ containing $\chi_{\iota_n\circ\rho_0}$ is of dimension $l(n-1)$.

Here $\iota_n\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$ denotes a rational irreducible representation of highest weight $n-1$, which is well known to be a unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension n (up to equivalences). They also gave explicit local coordinates around $\chi_{\iota_n\circ\rho_0}$ [MFP12b]. When n equals 2, these results had been already proved by Kapovich [Ka01] (see also Bromberg [Br04]).

Definition 4.3 (Ideal points)**.** Suppose that $X_n(\pi)$ is of positive dimension and let us take an affine curve C contained in $X_n(\pi)$. Let $\widetilde{C} \to C$ denote a desingularisation of a projective completion of C, so that $\widetilde{C}$ is a smooth projective model of C. A $\mathbb{C}$-valued point $\tilde{x}$ of $\widetilde{C}$ is called an *ideal point of* C if the birational map $\widetilde{C} \to C$ above is undefined at $\tilde{x}$.

Note that the notion of ideal points does not depend on the choices of projective completions and desingularisations in the definition (see [CS83, Section 1.3] for details). We also remark that there are only finitely many ideal points of C on $\widetilde{C}$.

4.2. **Nontrivial actions on Bruhat–Tits buildings.** We discuss in this subsection how to obtain a nontrivial, type-preserving action of a finitely generated group π on a Euclidean building. Such a nontrivial action gives rise to a nontrivial splitting of π, which shall play a central role in the construction of tribranched surfaces when π is a 3-manifold group. Similarly to the arguments in [CS83, Section 2.2], we utilise geometry of the character variety associated to π in order to obtain such an action.

Assume that the character variety $X_n(\pi)$ is of positive dimension and consider an affine curve C in $X_n(\pi)$. Then we may take a lift D of C in $R_n(\pi)$. Namely D is an affine curve contained in the inverse image of C under the natural projection $\mathrm{pr}_n\colon R_n(\pi) \to X_n(\pi)$ such that the restriction $\mathrm{pr}_n|_D$ is not a constant morphism. The projection $\mathrm{pr}_n|_D\colon D \to C$ induces a (surjective) regular morphism $\mathrm{pr}_n|_{\widetilde{D}}\colon \widetilde{D} \to \widetilde{C}$ on the smooth projective models of C and D, which sends the ideal points of $\widetilde{D}$ to those of $\widetilde{C}$.

Recall that, by the definition of $R_n(\pi)$, each $\mathbb{C}$-valued point y of $R_n(\pi)$ corresponds to an $\mathrm{SL}_n(\mathbb{C})$-representation $\rho_y\colon \pi \to \mathrm{SL}_n(\mathbb{C})$. We denote by $\mathbb{C}[R_n(\pi)]$ the ring of regular functions of $R_n(\pi)$. Let $\rho_{\mathrm{taut}}\colon \pi \to \mathrm{SL}_n(\mathbb{C}[R_n(\pi)])$ denote the tautological representation of π; namely $\rho_{\mathrm{taut}}(\gamma)$ is a regular $\mathrm{SL}_n(\mathbb{C})$-valued function on $R_n(\pi)$ for each element γ of π whose evaluation at a $\mathbb{C}$-valued point y of $R_n(\pi)$ is $\rho_y(\gamma)$. Let $\rho_{\widetilde{D}}\colon \pi \to \mathrm{SL}_n(\mathbb{C}(D))$ denote the composition of the tautological representation $\rho_{\mathrm{taut}}\colon \pi \to \mathrm{SL}_n(\mathbb{C}[R_n(\pi)])$ with

$$\mathrm{SL}_n(\mathbb{C}[R_n(\pi)]) \to \mathrm{SL}_n(\mathbb{C}[D]) \hookrightarrow \mathrm{SL}_n(\mathbb{C}(D)),$$

where the first map is induced by the natural embedding $D \hookrightarrow R_n(\pi)$. In the construction of $\rho_{\widetilde{D}}$, we identify $\mathbb{C}(D)$ with the field of rational functions of $\widetilde{D}$ due to the fact that $\widetilde{D}$ is birational to D (this gives justification to the notation $\rho_{\widetilde{D}}$). We call $\rho_{\widetilde{D}}$ the *tautological representation associated to the affine curve D*. Now recall that a $\mathbb{C}$-valued point y of the smooth projective curve $\widetilde{D}$ (possibly an ideal point of D) determines a discrete valuation $w_y\colon \mathbb{C}(D)^\times \to \mathbb{Z}; f \mapsto \mathrm{ord}_y(f)$ on the field of rational functions $\mathbb{C}(D)$ of $\widetilde{D}$ (that is, the *order function* at y; see [Mu91, Definition (1.32)] for details). The *Bruhat–Tits building associated to* $(\widetilde{D}, y)$ is then defined as $\mathcal{B}_{n,\widetilde{D},y} = \mathcal{B}(\mathrm{SL}(n)_{/(\mathbb{C}(D),w_y)})$, which admits a canonical action of $\mathrm{SL}_n(\mathbb{C}(D))$. We thus obtain an action of π on the Bruhat–Tits building $\mathcal{B}_{n,\widetilde{D},y}$

$$\pi \xrightarrow{\rho_{\widetilde{D}}} \mathrm{SL}_n(\mathbb{C}(D)) \xrightarrow{\text{canonical}} \mathrm{Aut}(\mathcal{B}_{n,\widetilde{D},y})$$

which is automatically type-preserving as we have already remarked in Section 3.2.

The following theorem is a generalisation of Culler and Shalen's "Fundamental Theorem" [CS83, Theorem 2.2.1] for representations of π of higher dimension.

Theorem 4.4. *Let* $\mathrm{pr}_n|_{\widetilde{D}}\colon \widetilde{D} \to \widetilde{C}$ *be as above and let* y *be a* $\mathbb{C}$*-valued point of* $\widetilde{D}$*. Set* $x = \mathrm{pr}_n|_{\widetilde{D}}(y)$*. Then the invariant functions* $\sigma_{\gamma,i}$ *associated to an element* γ *of* π *are holomorphic at* x *for all* i *if and only if* γ *fixes some vertex of the Bruhat–Tits building* $\mathcal{B}_{n,\widetilde{D},y}$ *associated to* $(\widetilde{D}, y)$*.*

Proof. We first claim that, for each i, $\sigma_{\gamma,i}$ is holomorphic at x if and only if $\tilde{\sigma}_{\gamma,i} := \sigma_{\gamma,i} \circ \mathrm{pr}_n|_{\widetilde{D}}$ is contained in O_y, the valuation ring of $\mathbb{C}(D)$ with respect to the valuation $w_y = \mathrm{ord}_y$. Indeed we may easily check, taking the natural injection $\mathbb{C}(C) \hookrightarrow \mathbb{C}(D)\,; f \mapsto f \circ \mathrm{pr}_n|_{\widetilde{D}}$ into accounts, that $\sigma_{\gamma,i}$ is holomorphic at $x = \mathrm{pr}_n|_{\widetilde{D}}(y)$ if and only if $\tilde{\sigma}_{\gamma,i}$ is holomorphic at y, or in other words, the inequality $w_y(\tilde{\sigma}_{\gamma,i}) \geq 0$ holds. Note that, by construction, $(-1)^i\tilde{\sigma}_{\gamma,i}$'s coincide with the coefficients of the characteristic polynomial of $\rho_{\widetilde{D}}(\gamma)$;

$$\det(tE_n - \rho_{\widetilde{D}}(\gamma)) = t^n - \tilde{\sigma}_{\gamma,1}(\chi_\rho)t^{n-1} + \cdots + (-1)^{n-1}\tilde{\sigma}_{\gamma,n-1}(\chi_\rho)t + (-1)^n.$$

Next we claim that the isotropy subgroup of $\mathrm{SL}_n(\mathbb{C}(D))$ at the vertex v_0 corresponding to the standard lattice $\sum_{j=1}^n O_y e_j$ is $\mathrm{SL}_n(O_y)$. Indeed, the isotropy subgroup at v_0 is a priori $Z(\mathrm{SL}_n(\mathbb{C}(D)))\mathrm{SL}_n(O_y)$, where $Z(\mathrm{SL}_n(\mathbb{C}(D)))$ denotes the centre of $\mathrm{SL}_n(\mathbb{C}(D))$ consisting of scalar matrices aI with $a \in \mathbb{C}(D)$, $a^n = 1$. But since O_y is integrally closed in $\mathbb{C}(D)$, all n-th roots of unity in $\mathbb{C}(D)$ are elements of O_y. This implies that $Z(\mathrm{SL}_n(\mathbb{C}(D)))$ is a subgroup of $\mathrm{SL}_n(O_y)$.

Now let us prove the sufficiency of the claim, and assume that γ fixes a vertex v of $\mathcal{B}_{n,\widetilde{D},y}$. Then there exists an element g of $\mathrm{GL}_n(\mathbb{C}(D))$ satisfying $gv_0 = v$. The isotropy subgroup of $\mathrm{SL}_n(\mathbb{C}(D))$ at v then coincides with $g\mathrm{SL}_n(O_y)g^{-1}$, which contains $\rho_{\widetilde{D}}(\gamma)$ by assumption. We may thus conclude that all coefficients $(-1)^i\tilde{\sigma}_{\gamma,i}(\gamma)$ of $\det(tE_n - \rho_{\widetilde{D}}(\gamma))$ are contained in O_y, as desired.

Next, to prove the necessity of the claim, assume that the functions $\sigma_{\widetilde{D},i}(\gamma)$ are contained in O_y for all i. We consider the rational canonical form of the matrix $\rho_{\widetilde{D}}(\gamma)$ over $\mathbb{C}(D)$; see for instance [DF04, Section 12.2]. Namely, we can find an element g of $\mathrm{GL}_n(\mathbb{C}(D))$ such that $g^{-1}\rho_{\widetilde{D}}(\gamma)g$ is a block sum of companion matrices $C_1 \oplus \cdots \oplus C_l$ over $\mathbb{C}(D)$ for some l. The product of the characteristic polynomials of C_i coincides with the characteristic polynomial of $\rho_{\widetilde{D}}(\gamma)$, which is, by the assumption, a polynomial with coefficients in O_y. It follows from Gauss's lemma (for primitive polynomials) that the characteristic polynomials of C_i are also polynomials with coefficients in O_y for all i, which implies that the companion matrices C_i are defined over O_y for all i. Thus $\rho_{\widetilde{D}}(\gamma)$ is contained in the conjugate $g\mathrm{SL}_n(O_y)g^{-1}$ of $\mathrm{SL}_n(O_y)$, and thus γ fixes the vertex gv_0 as desired. □

As a direct consequence of Theorem 4.4, we may verify that the action of π associated to an *ideal point* of $X_n(\pi)$ is nontrivial. Recall that an action of a group G on a simplicial complex Δ is said to be *nontrivial* if, for every vertex v of Δ, the isotropy subgroup G_v of G at v is a proper subgroup of G.

Corollary 4.5. *Let $\tilde{x}$ be an ideal point of an affine curve C contained in $X_n(\pi)$ and $\tilde{y}$ a lift of $\tilde{x}$ (namely, an ideal point of a lift D of C satisfying* $\mathrm{pr}_n|_{\widetilde{D}}(\tilde{y}) = \tilde{x}$*). Then the associated action of π on $\mathcal{B}_{n,\widetilde{D},\tilde{y}}$ is nontrivial.*

Proof. Let D be a lift of C in $R_n(\pi)$. Striving for a contradiction, suppose that the action of π induced on $\mathcal{B}_{n,\widetilde{D},\tilde{y}}$ is trivial, or in other words, suppose that there exists a vertex v of $\mathcal{B}_{n,\widetilde{D},\tilde{y}}$ at which the isotropy subgroup of π coincides with the whole group π. Theorem 4.4 then implies that the trace function I_γ does not have a pole at $\tilde{x}$ for *every* element γ of π. In particular every affine coordinate function of C is holomorphic at $\tilde{x}$ due to Theorem 4.1. The last assertion contradicts the fact that at least one coordinate function must have a pole at $\tilde{x}$ (recall that we have chosen $\tilde{x}$ from *ideal* points of C). □

4.3. **Ideal points of character varieties and tribranched surfaces.** Now we show that an essential tribranched surface in a 3-manifold is constructed from a nontrivial type-preserving action of its fundamental group on a Euclidean building. Such an action is obtained from an ideal point of an affine curve in the character variety as in Section 4.2, and consequently, an essential tribranched surface is detected by an ideal point under certain conditions.

Let M be a compact, connected, irreducible and orientable 3-manifold. In the following argument, a "triangulation" of M should be understood to be *a piecewise-linear triangulation*, that is, the link of every i-simplex in the triangulation is piecewise-linearly homeomorphic to a $(2-i)$-simplex or the boundary of a $(3-i)$-simplex for $i = 0, 1, 2$, according as the i-simplex lies in ∂M or not; see for instance [He76, Chapter 1]. Now let K be a (possibly locally infinite) 2-dimensional Δ-complex. We call a map $f\colon M \to K$ *piecewise-linear* if, for some triangulation of M, the images of the vertices of every simplex in M span a simplex in K, and the restriction of f to each simplex of M is a linear map. We define $Y(K)$ to be the 1-dimensional subcomplex of the first barycentric subdivision of K consisting of all the barycentres of 1- and 2-simplices and all the edges connecting them.

Lemma 4.6. *Let $f\colon M \to K$ be a piecewise-linear map. Then the inverse image of $Y(K)$ under f is a tribranched surface in M.*

Proof. Consider a triangulation of M with respect to which f is a piecewise-linear map, and set Σ to be the inverse image of $Y(K)$ under f. Note that Σ is a compact subset of M since it is a closed subset of the compact manifold M. Let us use the notation introduced in Section 2.1.

We first show that (M, Σ) is locally homeomorphic to $(\overline{\mathbb{H}}, Y \times [0, \infty))$; recall that we define the topological space Y as

$$Y = \{\, re^{\sqrt{-1}\theta} \in \mathbb{C} \mid r \in \mathbb{R}_{\geq 0} \text{ and } \theta = 0, \pm 2\pi/3 \,\}.$$

The piecewise-linear map f maps each 3-simplex τ in M onto either a vertex, an edge or a 2-simplex in K. Corresponding to the image of τ in K, the restriction of Σ to τ is either the empty set, a normal disk (more precisely a triangle or a quadrilateral), or a 2-dimensional Δ-complex consisting of one triangle and two quadrilaterals sharing one common edge; see Figure 5. The inverse image Σ is the union of these subspaces glued up along 2-simplices in M.

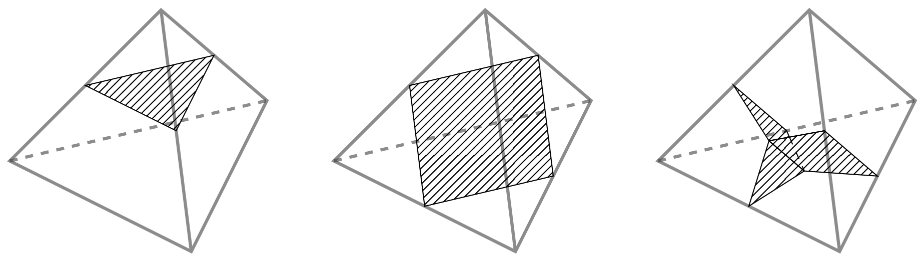

FIGURE 5. The inverse image Σ in a single 3-simplex τ of M

Now take an arbitrary point x of Σ, and let us study the topological structure around x. Firstly we know from the construction of Σ that x cannot be any vertex in M. If x is in the interior of a 3-simplex τ in M, then the above classification of types of Σ restricted to τ shows at once that (M, Σ) is locally homeomorphic to $(\mathbb{H}, Y \times (0, \infty))$ around x. Next suppose that x is in the interior of a 2-simplex in M. Then the above classification again shows that, for each 3-simplex τ containing the 2-simplex under consideration, a sufficiently small open neighborhood of x in $\Sigma \cap \tau$ is homeomorphic to $\mathbb{R} \times [0, \infty)$ or $Y \times [0, \infty)$. Since every 2-simplex is adjacent to at most two 3-simplices in M, a sufficiently small open neighborhood of x in Σ is homeomorphic to $\mathbb{R} \times [0, \infty)$ or $Y \times [0, \infty)$ if $x \in \partial M$, and to $\mathbb{R}^2$ or $Y \times \mathbb{R}$ otherwise. It thus follows that (M, Σ) is locally homeomorphic to $(\overline{\mathbb{H}}, Y \times [0, \infty))$ around x in this case.

As the final case suppose that x is the midpoint of an edge in M. Note that, for each 3-simplex τ containing the edge under consideration, a sufficiently small open neighborhood of x in $\Sigma \cap \tau$ is a sector in any cases of the above classification of $\Sigma \cap \tau$. Since we consider a piecewise-linear triangulation of M, a finite number of 3-simplices are glued along 2-simplices around every edge in M so that its link in M is homeomorphic to a closed interval or a circle according as the edge lies in ∂M or not. If we take a sufficiently small open neighborhood of x in Σ, we now see that its boundary is homeomorphic to the link of the edge under consideration since the neighbourhood is just the union of sectors glued up along 2-simplices; see Figure 6. This implies that, aroud x, Σ is homeomorphic to $\mathbb{R} \times [0, \infty)$ or $\mathbb{R}^2$ according as x is contained in ∂M or not, and thus (M, Σ) is locally homeomorphic to $(\overline{\mathbb{H}}, \mathbb{R} \times [0, \infty))$ in this case. In summary, we see that (M, Σ) is locally homeomorphic to $(\overline{\mathbb{H}}, Y \times [0, \infty))$ around each x in Σ.

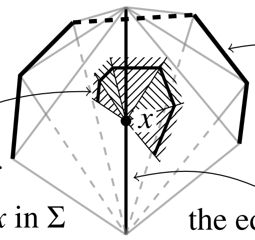

Figure 6. Around the midpoint x of an edge

Next we show that Σ satisfies (TBS1). Let C be an arbitrary component of the set $C(\Sigma)$ of branched points, and consider a sufficiently small tubular neighbourhood $\nu(C)$ of C in M. The intersection $\nu(C) \cap \Sigma$ naturally admits the structure of a fibre bundle over C whose fibre is homeomorphic to Y. We may identify $f(\nu(C) \cap \Sigma)$ with Y so that $f(C)$ corresponds to $\{0\}$. Then since the inverse image of $\{0\}$ under f is C, the topological space $f((\nu(C) \cap \Sigma) \setminus C)$ has 3 components, and so does $(\nu(C) \cap \Sigma) \setminus C$ by continuity of f. Therefore the fibre bundle $\nu(C) \cap \Sigma \to C$ above must be trivial, which implies that Σ satisfies (TBS1).

Finally we show that Σ satisfies (TBS2). We denote by M_0 the complement of a small open tubular neighbourhood of $C(\Sigma)$ in M. Let S be an arbitrary component of the subsurface $S(\Sigma)$, which can be regarded as a properly embedded subsurface in the orientable 3-manifold M_0. The image $f(S)$ is contained in a component Γ of the complement in $Y(K)$ of the subset consisting of all the barycentres of 2-simplices. Since Γ is bicollared in K, S is two-sided, and so orientable. Hence Σ satisfies (TBS2), and the lemma follows. $\square$

We now consider a type-preserving action of $\pi_1(M)$ on a Euclidean building $\mathcal{B}$. The simplicial complex structure of $\mathcal{B}^{(2)}$ naturally induces the Δ-complex structure of $\mathcal{B}^{(2)}/\pi_1(M)$, where, for each non-negative integer i, we denote by $\mathcal{B}^{(i)}$ the i-skeleton of $\mathcal{B}$. We say that a tribranched surface Σ is *dual to* a type-preserving action of $\pi_1(M)$ on a Euclidean building $\mathcal{B}$ if there exists a $\pi_1(M)$-equivariant piecewise-linear map $\tilde{f} \colon \widetilde{M} \to \mathcal{B}^{(2)}$ such that the inverse image of Σ in $\widetilde{M}$ coincides with the inverse image of $Y(\mathcal{B}^{(2)})$ under $\tilde{f}$.

Theorem 4.7. *Let n be a natural number greater than or equal to* 3*, and assume that the boundary ∂M of M is non-empty when n is strictly greater than* 3*. Then, for any nontrivial type-preserving action of $\pi_1(M)$ on a Euclidean building $\mathcal{B}$ of dimension $n - 1$, there exists an essential tribranched surface in M dual to the action.*

Proof. The proof is divided into two parts. In the first part we show that there exists a non-empty tribranched surface, which is not necessarily essential, dual to the action of $\pi_1(M)$ on $\mathcal{B}$, and in the second part we modify such a tribranched surface dual to the action to be essential by local surgeries.

Let us take a triangulation of M and consider the triangulation on $\widetilde{M}$ induced from it. We construct a $\pi_1(M)$-equivariant simplicial map $\tilde{f} \colon \widetilde{M} \to \mathcal{B}^{(2)}$ as follows. First consider the case of $n = 3$ (in the case the 2-skeleton of $\mathcal{B}$ coincides with $\mathcal{B}$ itself since it is of dimension 2). For each vertex v of M, we choose a lift $\tilde{v}$ of v in $\widetilde{M}$ and a vertex $\tilde{f}(\tilde{v})$ of $\mathcal{B}$. Then we define $\tilde{f}|_{\widetilde{M}^{(0)}}$ as

$$\tilde{f}|_{\widetilde{M}^{(0)}}(\gamma \cdot \tilde{v}) = \gamma \tilde{f}(\tilde{v})$$

for arbitrary $\gamma \in \pi_1(M)$ so that $\tilde{f}|_{\widetilde{M}^{(0)}}$ is $\pi_1(M)$-equivariant. Now assume that we have already constructed a $\pi_1(M)$-equivariant simplicial map $\tilde{f}|_{\widetilde{M}^{(i-1)}} \colon \widetilde{M}^{(i-1)} \to \mathcal{B}$ on the $(i-1)$-skeleton of $\widetilde{M}$, and let us take an arbitrary i-simplex σ of $\widetilde{M}$. We may extend the restriction $\tilde{f}|_{\partial\sigma}$ of $\tilde{f}|_{\widetilde{M}^{(i-1)}}$ onto $\partial\sigma$ to a map $\tilde{f}|_{\sigma}$ on σ due to the contractibility of the Euclidean building $\mathcal{B}$. Moreover we can take $\tilde{f}|_{\sigma}$ to be a simplicial map by subdividing M (and $\widetilde{M}$) if necessary. By continuing this procedure, we can extend $\tilde{f}|_{\widetilde{M}^{(0)}}$ to simplicial maps on $\widetilde{M}^{(1)}$, $\widetilde{M}^{(2)}$, and $\widetilde{M}$ inductively, and obtain a desired simplicial map $\tilde{f} \colon \widetilde{M} \to \mathcal{B}$. Next consider the case of $n \geq 4$. Since ∂M is non-empty by assumption, we can take a 2-dimensional subcomplex V which is a deformation retract of M. Denote by $\widetilde{V}$ the preimage of V under the universal covering map $\widetilde{M} \to M$. We define $\tilde{f}|_{\widetilde{V}^{(0)}}$ on $\widetilde{V}^{(0)}$ and, by subdividing M if necessary, we extend it to a $\pi_1(M)$-equivariant simplicial map $\tilde{f}|_{\widetilde{V}} \colon \widetilde{V} \to \mathcal{B}^{(2)}$ similarly to the case of $n = 3$. Note that the image of the extended map $\tilde{f}|_{\widetilde{V}}$ is contained in the 2-skeleton $\mathcal{B}^{(2)}$ of $\mathcal{B}$ since $\widetilde{V}$ is of dimension 2. By composing $\tilde{f}|_{\widetilde{V}}$ with a deformation retraction $\widetilde{M} \to \widetilde{V}$, we obtain a desired map $\tilde{f} \colon \widetilde{M} \to \mathcal{B}^{(2)}$.

We can slightly modify the above construction so that the restriction of $\tilde{f} \colon \widetilde{M} \to \mathcal{B}^{(2)}$ to each simplex is a linear map. Then $\tilde{f}$ induces on the quotients by $\pi_1(M)$ a piecewise-linear map $f \colon M \to \mathcal{B}^{(2)}/\pi_1(M)$. Set Σ to be the inverse image of $Y(\mathcal{B}^{(2)}/\pi_1(M))$ under f. By Lemma 4.6 we see that Σ is a tribranched surface in M. Note that the above construction of Σ is far from being canonical since it depends on many choices, for instance, of a triangulation of M and a $\pi_1(M)$-equivariant simplicial map $\tilde{f}$.

Next we show that Σ satisfies (ETBS1), which, in particular, implies that Σ is non-empty. Striving for a contradiction, suppose that there exists a component N of $M(\Sigma)$ such that the homomorphism $\pi_1(N) \to \pi_1(M)$ induced by the natural inclusion $N \hookrightarrow M$ is surjective. Let N_0 be a component of the preimage of N under the universal covering map $\widetilde{M} \to M$. Since $\tilde{f}(N_0)$ does not intersect $Y(\mathcal{B}^{(2)})$ by construction, it is contained in the open star of a certain vertex v of $\mathcal{B}^{(2)}$ in its barycentric subdivision. Obviously N_0 is a covering space over N, and thus the fundamental group $\pi_1(N)$ stabilises N_0. The image of the homomorphism $\pi_1(N) \to \pi_1(M)$ then also stabilises the open star of v containing $\tilde{f}(N_0)$ due to the $\pi_1(M)$-equivariance of $\tilde{f}$, and it is, in particular, contained in the isotropy subgroup $\pi_1(M)_v$ of $\pi_1(M)$ at v. Hence we conclude that $\pi_1(M)_v$ coincides with the whole group $\pi_1(M)$, combining the arguments above with the assumption on the surjectivity of the homomorphism $\pi_1(N) \to \pi_1(M)$, which contradicts nontriviality of the action of $\pi_1(M)$ on $\mathcal{B}$.

As we have already mentioned at the beginning of the proof, the tribranched surface Σ itself might not be essential. From now on we modify Σ to be essential as the second part of the proof. For a tribranched surface Σ given by the nontrivial type-preserving action of $\pi_1(M)$ on $\mathcal{B}$, we set

$$\begin{aligned} l(\Sigma) &= \text{the number of components of } C(\Sigma), \\ m(\Sigma) &= \sum_S (2 - \chi(S))^2 \quad (\text{where } \chi(S) \text{ is the Euler characteristic of the surface } S), \\ n(\Sigma) &= \text{the number of components of } \Sigma, \end{aligned}$$

where the sum in the second equation runs over all components S of $S(\Sigma)$. We see at once that these integers are all non-negative. We consider the triple $(l(\Sigma), m(\Sigma), n(\Sigma)) \in \mathbb{Z}^3$ with respect to the lexicographical order of $\mathbb{Z}^3$ as a *complexity* of a non-empty tribranched surface Σ. In the following we show that if Σ is not essential, there are operations of replacing Σ by another tribranched surface with lower complexity, which is also dual to the action of $\pi_1(M)$ on $\mathcal{B}$. Consequently a tribranched surface of minimal complexity dual to the action of $\pi_1(M)$ on $\mathcal{B}$ must be essential.

Let us consider the case where Σ does not satisfy (ETBS2). First assume that there exists a pair of components C and S of $C(\Sigma)$ and $S(\Sigma)$ respectively such that the natural homomorphism $\pi_1(C) \to \pi_1(S)$ is defined but not injective. This implies that S is a disk. Let S_1 and S_2 be the other components of $S(\Sigma)$ whose boundary contain parallel copies of C as components (the surfaces S_1 and S_2 might coincide). Take a small neighbourhood B of S which is homeomorphic to a ball and intersects S_1 and S_2 in the collars of C. Figure 7 illustrates a local picture of the neighbourhood B. Choose properly embedded disks D_1 and D_2 in B bounding $S_1 \cap \partial B$ and

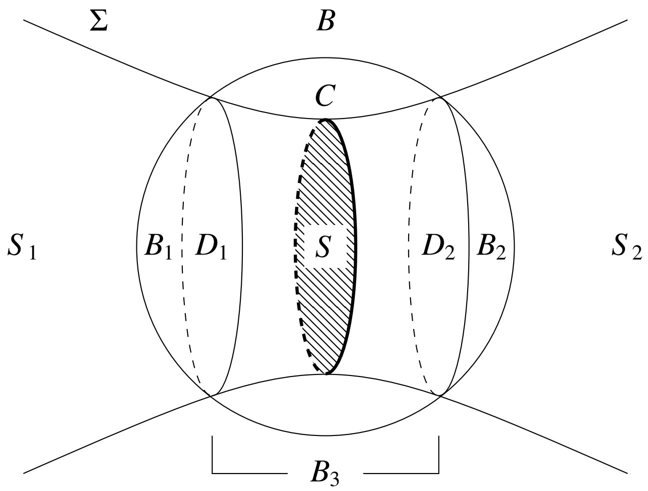

FIGURE 7. A neighbourhood B of the surface component S

$S_2 \cap \partial B$ respectively and not intersecting S. We construct a map $g\colon B \to \mathcal{B}^{(2)}/\pi_1(M)$ such that $g|_{\partial B} = f|_{\partial B}$ and that $g^{-1}(Y(\mathcal{B}^{(2)}/\pi_1(M))) = D_1 \cup D_2$ as follows. Since $f(\partial D_1)$ and $f(\partial D_2)$ are contained in open edges of $Y(\mathcal{B}^{(2)}/\pi_1(M))$ near the vertex $f(C)$, the maps $g|_{\partial D_1} = f|_{\partial D_1}$ and $g|_{\partial D_2} = f|_{\partial D_2}$ extend to D_1 and D_2 respectively so that $g(D_1)$ and $g(D_2)$ are contained in the same open edges. The ball B is divided into 3 balls B_1, B_2 and B_3 by D_1 and D_2, where ∂B_1 does not contain D_2 but contains D_1, ∂B_2 does not contain D_1 but contains D_2, and ∂B_3 contains both disks. There exists a unique 2-simplex of $\mathcal{B}^{(2)}$ which contains $f(C)$ as its barycentre, and the open star of each of its 3 vertices contains one of $g(\partial B_1 \setminus D_1)$, $g(\partial B_2 \setminus D_2)$ and $g(\partial B_3 \setminus (D_1 \cup D_2))$. We can thus extend $g|_{\partial B_1}$, $g|_{\partial B_2}$ and $g|_{\partial B_3}$ to B_1, B_2 and B_3 respectively so that all of $g(B_1 \setminus D_1)$, $g(B_2 \setminus D_2)$ and $g(B_3 \setminus (D_1 \cup D_2))$ do not intersect $Y(\mathcal{B}^{(2)}/\pi_1(M))$. Then we see at once that the inverse images of $Y(\mathcal{B}^{(2)}/\pi_1(M))$ under the maps $g|_{B_1}$, $g|_{B_2}$ and $g|_{B_3}$ are D_1, D_2 and $D_1 \cup D_2$ respectively. Figure 8 illustrates the image $g(B)$ of the neighbourhood B of S. We now define $f'\colon M \to \mathcal{B}^{(2)}/\pi_1(M)$ so that $f'|_{M \setminus B} = f|_{M \setminus B}$ and $f'|_B = g$. Then $f'^{-1}(Y(\mathcal{B}^{(2)}/\pi_1(M)))$ is another tribranched surface and has a lower complexity since $l(\Sigma)$ decreases. Since each component of

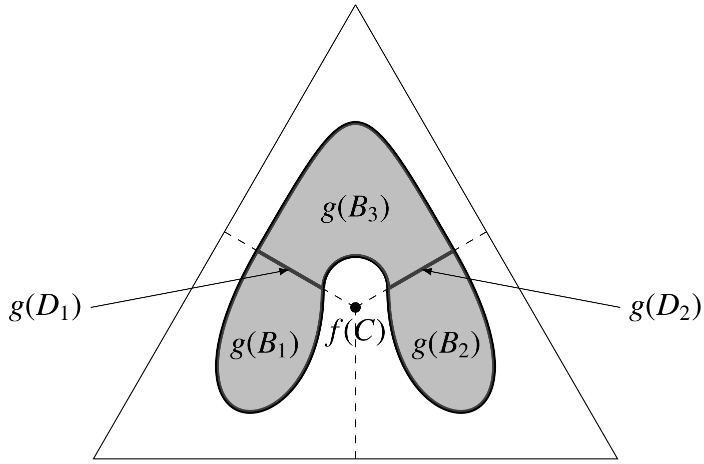

FIGURE 8. The image $g(B)$ for the pair (C, S)

the inverse image of B in $\widetilde{M}$ is homeomorphic to B, the map f' is induced by a $\pi_1 M$-equivariant map $\tilde{f}' \colon \widetilde{M} \to \mathcal{B}^{(2)}$ obtained from $\tilde{f}$ by the same modification in $\widetilde{M}$.

Next assume that there exists a pair of components S and N of $S(\Sigma)$ and $M(\Sigma)$ respectively such that the natural homomorphism $\pi_1(S) \to \pi_1(N)$ is defined but not injective. By Dehn's lemma, there exits a compressing disk D of S in N. Take a small neighbourhood B of D which is homeomorphic to a ball and intersects an annulus in S. Figure 9 illustrates a local picture of the neighbourhood B. Choose properly embedded disks D_1 and D_2 in B bounding

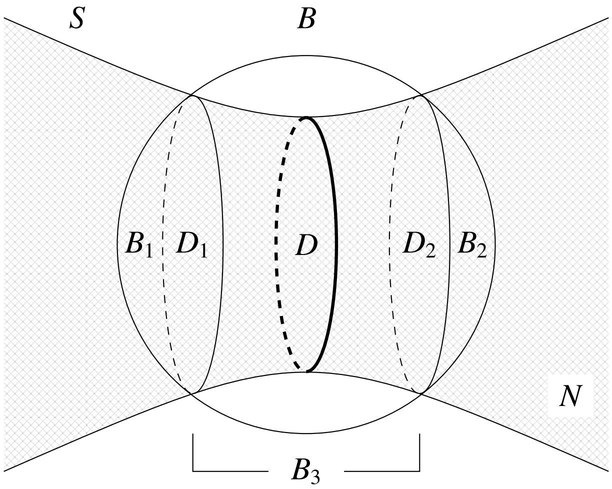

FIGURE 9. A neighbourhood B of the compression disk D

the components of the boundary of the annulus. We construct a map $g \colon B \to \mathcal{B}^{(2)}/\pi_1(M)$ such that $g|_{\partial B} = f|_{\partial B}$ and that $g^{-1}(Y(\mathcal{B}^{(2)}/\pi_1(M))) = D_1 \cup D_2$ as follows. Since $f(\partial D_1)$ and $f(\partial D_2)$ are contained in the open star of a vertex in $Y(\mathcal{B}^{(2)}/\pi_1(M))$ which is a barycentre of an edge of $\mathcal{B}$, the maps $g|_{\partial D_1} = f|_{\partial D_1}$ and $g|_{\partial D_2} = f|_{\partial D_2}$ extend to D_1 and D_2 respectively so that $g(D_1)$ and $g(D_2)$ are contained in the same star. The ball B is divided into 3 balls B_1, B_2 and B_3 by D_1 and D_2, where ∂B_1 does not contain D_2 but contains D_1, ∂B_2 does not contain D_1 but contains D_2, and ∂B_3 contains both disks. Since $g(\partial B_1 \setminus D_1)$ and $g(\partial B_2 \setminus D_2)$ are contained in the open star of a vertex of $\mathcal{B}^{(2)}$ in its barycentric subdivision (corresponding to N), and since $g(\partial B_3 \setminus (D_1 \cup D_2))$ is contained in that of another vertex of $\mathcal{B}^{(2)}$, we can extend $g|_{\partial B_1}$, $g|_{\partial B_2}$ and $g|_{\partial B_3}$ to B_1, B_2 and B_3 respectively so that $g(B_1 \setminus D_1)$, $g(B_2 \setminus D_2)$ and $g(B_3 \setminus (D_1 \cup D_2))$ do not intersect $Y(\mathcal{B}^{(2)}/\pi_1(M))$. Then we see at once that the inverse images of $Y(\mathcal{B}^{(2)}/\pi_1(M))$ under the maps $g|_{B_1}$, $g|_{B_2}$ and $g|_{B_3}$ are D_1, D_2 and $D_1 \cup D_2$ respectively. Figure 10 illustrates the image $g(B)$ of the neighbourhood B of the compression disk D. Now we define $f' \colon M \to \mathcal{B}^{(2)}/\pi_1(M)$ so that $f'|_{M\setminus B} = f|_{M\setminus B}$ and

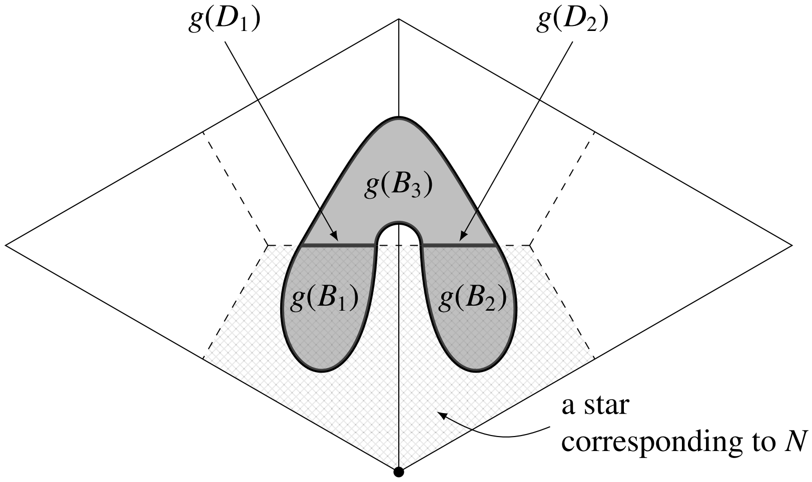

FIGURE 10. The image $g(B)$ for the pair (S, N)

$f'|_B = g$. Set $\Sigma' = f'^{-1}(Y(\mathcal{B}^{(2)}/\pi_1(M)))$, which is another tribranched surface with the same $l(\Sigma')$ as $l(\Sigma)$. We show in the followings that $m(\Sigma')$ is strictly less than $m(\Sigma)$, which implies that Σ' has a lower complexity than Σ. Set $S' = (S \setminus B) \cup D_1 \cup D_2$. First suppose that S' is connected. Then we can calculate as

$$\begin{aligned} m(\Sigma) - m(\Sigma') &= (2 - \chi(S))^2 - (2 - \chi(S'))^2 \\ &= 4 + 4(2 - \chi(S')) > 0 \end{aligned}$$

by using $\chi(S') = \chi(S) + 2 \leq 2$. Next suppose that S' has two components S'_1 and S'_2. Note that neither S'_1 nor S'_2 is a sphere. Then we can calculate as

$$\begin{aligned} m(\Sigma) - m(\Sigma') &= (2 - \chi(S))^2 - (2 - \chi(S'_1))^2 - (2 - \chi(S'_2))^2 \\ &= 2(2 - \chi(S'_1))(2 - \chi(S'_2)) > 0 \end{aligned}$$

by using $\chi(S'_1) + \chi(S'_2) = \chi(S) + 2$, $\chi(S'_1) < 2$ and $\chi(S'_2) < 2$. In both the cases $m(\Sigma')$ decreases from $m(\Sigma)$, as desired. Similarly to the previous case, f' is induced by a $\pi_1 M$-equivariant map $\tilde{f}' \colon \widetilde{M} \to \mathcal{B}^{(2)}$.

Finally, we consider the case where Σ does not satisfy (ETBS3). Then we see as follows that, after eliminating a component of Σ contained in a ball in M or a collar of ∂M, the resultant tribranched surface is also dual to the action of $\pi_1(M)$ on $\mathcal{B}$. If there is a component of Σ contained in a ball B, we can construct a map $f' \colon M \to \mathcal{B}^{(2)}/\pi_1(M)$ so that $f'|_{M\setminus B} = f|_{M\setminus B}$ and that $f'(B)$ does not intersect $Y(\mathcal{B}^{(2)}/\pi_1(M))$, because $f(\partial B)$ is contained in a contractible component of the complement of $Y(\mathcal{B}^{(2)}/\pi_1(M))$ in $\mathcal{B}^{(2)}/\pi_1(M)$. If there is one contained in a collar of ∂M, we set $f' \colon M \to \mathcal{B}^{(2)}/\pi_1(M)$ to be the composition of a deformation retraction from M to the complement of the collar with the restriction of f to it. In both the cases the complexity of a new tribranched surface defined as $f'^{-1}(Y(\mathcal{B}^{(2)}/\pi_1(M)))$ is lower than the original one's, since $l(\Sigma)$ and $m(\Sigma)$ do not increase and $n(\Sigma)$ decreases. Also, it is straightforward to see that f' is induced by a $\pi_1 M$-equivariant map $\tilde{f}' \colon \widetilde{M} \to \mathcal{B}^{(2)}$ obtained from $\tilde{f}$ by the same modification in $\widetilde{M}$. The proof is now completed. □

Remark 4.8. (1) Since a tribranched surface of minimal complexity dual to the action of $\pi_1(M)$ on a Euclidean building is not necessarily unique, the construction of an essential tribranched surface in the proof is far from being canonical.

(2) The same argument in the proof shows that the theorem holds also for a nontrivial type-preserving action of $\pi_1(M)$ on a contractible colorable chamber complex of dimension $(n-1)$.

The following is the main theorem of this article, which is now a direct consequence of Corollary 4.5 and Theorem 4.7.

Theorem 4.9. *Let n be a natural number greater than or equal to* 3, *and assume that the boundary ∂M of M is non-empty when n is strictly greater than* 3. *Then, for each ideal point $\tilde{x}$ of an affine curve C in $X_n(M)$, there exists an essential tribranched surface Σ contained in M dual to the action $\pi_1(M)$ on the Bruhat–Tits building $\mathcal{B}_{n,\widetilde{D},\tilde{y}}$ associated to $\tilde{x}$, where D is a lift of C in $R_n(M)$ and $\tilde{y}$ is an ideal point of D satisfying* $\mathrm{pr}_n|_{\widetilde{D}}(\tilde{y}) = \tilde{x}$.

5. An application to small Seifert manifolds

One of great advantages of extending Culler–Shalen theory to higher dimensional representations is that we may apply our extended theory also to a *non-Haken* 3-manifold, that is, a 3-manifold which does not contain any essential surfaces. Here we describe an application of Theorem 4.9 to a class of 3-manifolds called *small Seifert manifolds*, which contain non-Haken 3-manifolds.

A *Seifert manifold* is a compact, orientable 3-manifold admitting the structure of a Seifert fibred space whose base orbifold is a compact surface with cone points. A *small Seifert manifold* is a Seifert manifold with at most 3 singular fibres. We refer the reader to [Ja80, Chapter IV] for details on Seifert manifolds.

Let p, q and r be natural numbers greater than or equal to 3. We denote by $S^2(p,q,r)$ the 2-sphere with three cone points whose cone angles are $2\pi/p$, $2\pi/q$ and $2\pi/r$ respectively, and consider a small Seifert manifold M with the base orbifold $S^2(p,q,r)$. Such a 3-manifold is known to be irreducible, and it is Haken if and only if its first homology group $H_1(M,\mathbb{Z})$ is infinite. The fundamental group $\pi_1(M)$ has a presentation of the form

$$\langle x, y, h \mid h\text{: central}, x^p = h^a, y^q = h^b, (xy)^r = h^c \rangle$$

for certain integers a, b, c satisfying $(a,p) = (b,q) = (c,r) = 1$. The orbifold fundamental group $\pi_1^{\mathrm{orb}}(S^2(p,q,r))$ of $S^2(p,q,r)$ is isomorphic to the *ordinary triangle group* (or the *von Dyck group*) $\Delta(p,q,r)$ defined as

$$\langle x, y \mid x^p = y^q = (xy)^r = 1 \rangle,$$

and by identifying $\pi_1^{\mathrm{orb}}(S^2(p,q,r))$ with $\Delta(p,q,r)$, we may regard the natural homomorphism $\pi_1(M) \to \pi_1^{\mathrm{orb}}(S^2(p,q,r))$ induced by the projection $M \to S^2(p,q,r)$ as the group homomorphism which maps x and y identically and sends h to the unit (in particular it is a surjection). It is easy to see that the first homology group $H_1(M, \mathbb{Z})$ is infinite if and only if the equality

$$\frac{a}{p} + \frac{b}{q} = \frac{c}{r}$$

holds. From this observation we thus find that M tends to be non-Haken in most cases.

In the case where M is Haken, we may readily construct an affine curve in $X_3(M)$ consisting of abelian characters because the first homology group $H_1(M, \mathbb{Z})$ is infinite. In the following we verify that $X_3(M)$ contains an affine curve also in the case where M is non-Haken. It thus follows from Theorem 4.9 that an essential tribranched surface Σ contained in M is detected by an ideal point of the curve. One can never obtain such an essential tribranched surface by utilising classical Culler–Shalen theory (since the $\mathrm{SL}_2(\mathbb{C})$-character variety $X_2(M)$ is of dimension 0 in the case).

We may regard the group $\Delta(p, q, r)$ as a subgroup of index 2 of the *Schwartzian triangle group* $\Gamma(p, q, r)$ defined as

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1\rangle,$$

identifying x with ab and y with bc respectively. It follows from the argument in [Go88, Section 6] that there exists a family of $\mathrm{SL}_3(\mathbb{C})$-representations $\rho_s \colon \Gamma(p, q, r) \to \mathrm{SL}_3(\mathbb{C})$ with complex parameter s defined by

$$\rho_s(a) = \begin{pmatrix} 1 & 0 & 0 \\ -2s\cos\frac{\pi}{p} & -1 & 0 \\ -2\cos\frac{\pi}{r} & 0 & -1 \end{pmatrix},$$

$$\rho_s(b) = \begin{pmatrix} -1 & -2s^{-1}\cos\frac{\pi}{p} & 0 \\ 0 & 1 & 0 \\ 0 & -2\cos\frac{\pi}{q} & -1 \end{pmatrix},$$

$$\rho_s(c) = \begin{pmatrix} -1 & 0 & -2\cos\frac{\pi}{r} \\ 0 & -1 & -2\cos\frac{\pi}{q} \\ 0 & 0 & 1 \end{pmatrix}$$

(the representations above are minor modifications of the ones introduced in [Go88], where $\cos\frac{\pi}{p}$, $\cos\frac{\pi}{q}$ and $\cos\frac{\pi}{r}$ are replaced by $\cos\frac{2\pi}{p}$, $\cos\frac{2\pi}{q}$ and $\cos\frac{2\pi}{r}$ respectively in the matrices). A simple computation enables us to obtain the equation

$$\operatorname{tr}\rho_s(abac) = 8(s + s^{-1})\cos\frac{\pi}{p}\cos\frac{\pi}{q}\cos\frac{\pi}{r} + 16\cos^2\frac{\pi}{p}\cos^2\frac{\pi}{r} + 4\cos^2\frac{\pi}{q} - 1,$$

which shows that the restrictions of $\operatorname{tr}\rho_s$ to $\Delta(p, q, r)$ define a nontrivial affine curve contained in $X_3(\Delta(p, q, r))$. Since the natural homomorphism $\pi_1(M) \to \pi_1^{\mathrm{orb}}(S^2(p, q, r))$ is surjective, the morphism $X_3(\pi_1^{\mathrm{orb}}(S^2(p, q, r))) \to X_3(M)$ induced on the character varieties is an embedding. Therefore one readily sees that, by identifying $X_3(\Delta(p, q, r))$ with $X_3(\pi_1^{\mathrm{orb}}(S^2(p, q, r)))$, the character variety $X_3(M)$ also contains a nontrivial curve.

6. Questions

We conclude with a list of questions. Let M be a compact, connected, irreducible and orientable 3-manifold. It is known by Boyer and Zhang [BZ98], Motegi [Mo88], and Schanuel and Zhang [SZ01] that there exists an essential surface not detected by any ideal points of any affine curves in $X_2(M)$ for a certain 3-manifold M. We may now propose the following important question:

Question 6.1. *Does there exist an essential surface* (*without branched points*) *not detected by any ideal points of any affine curves in $X_2(M)$ but detected by an ideal point of an affine curve in $X_n(M)$ for $n \geq 3$ as in Theorem 4.9?*

Here we remark that an essential surface (without any branched points) is also an essential tribranched surface in our terminology. [Note: as we have mentioned at the end of Section 0, (a much stronger form of) Question 6.1 has been already solved affirmatively in [FKN18].]

Now recall that we have imposed a little too strong assumption on the boundary of the 3-manifold M under consideration in the proof of Theorem 4.9; namely we have assumed there that the boundary of M is not empty when n is strictly greater than 3.

Question 6.2. *Does the same conclusion as Theorem* 4.9 *hold without the assumption that the boundary ∂M is non-empty when n is strictly greater than* 3*?*

Next let M be a small Seifert manifold whose base orbifold is $S^2(p, q, r)$ with $p, q, r \geq 3$. Recall that we saw in Section 5 that $X_3(M)$ contains a nontrivial curve.

Question 6.3. *What essential tribranched surfaces are detected as in Theorem 4.9 by an ideal point of the nontrivial curve considered in Section* 5*?*

The final question is concerning the characterisation of the class of 3-manifolds containing essential tribranched surfaces.

Question 6.4. *Does every aspherical* 3*-manifold contain an essential tribranched surface?*

[Note: Friedl, Nagel and the second-named author [FKN17, Theorem 1.2] proved that if M is closed and $\operatorname{rank} \pi_1(M) \geq 4$, then M contains an essential tribranched surface. They constructed such essential tribranched surfaces using open boook decompositions.]

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