TORSION VOLUME FORMS ON CHARACTER VARIETIES

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1. Introduction

For a knot complement of S^3, Heusener [H03] showed that a 1-dimensional smooth part of the space of the conjugacy classes of irreducible SU_2-representations of the knot group carries a canonical orientation, and Dubois [D05, D06] constructed a canonical volume form on the part which induces Heusener's orientation via Reidemeister torsion. Later the author [K09] slightly generalized the volume form for a knot complement of a general rational homology 3-sphere, and studied compatibility with the actions by the 1st cohomology group with $\mathbb{Z}/2$ coefficients and the outer automorphism group of the fundamental group. The idea of regarding non-acyclic Reidemeister torsion as a volume form on a moduli space of group representations was first considered by Witten [Wi91]. He obtained an explicit formula to compute the symplectic volumes of moduli spaces of representations of surface groups in terms of the Reidemeister torsion volume forms.

The aim of the article is to construct a canonical complex volume form on a smooth part of lowest dimension of the $SL_2(\mathbb{C})$-character variety of a 3-manifold group analogously via Reidemeister torsion.

2. Reidemeister torsion

We begin with reviewing the definition of sigh-refined Reidemeister torsion, following Turaev [T01, T02]. See also [M66, P97]. Note that we consider the torsion of a twisted cochain complex instead of that of a twisted chain complex for the construction of a volume form on a moduli space of group representations.

Let $C_* = (C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_0)$ be a chain complex of finite dimensional vector spaces over a field $\mathbb{F}$. For given bases b_i of $\operatorname{Im} \partial_{i+1}$ and h_i of $H_i(C_*)$, we choose a basis $b_i h_i b_{i-1}$ of C_i as follows. Taking a lift of h_i in $\operatorname{Ker} \partial_i$ and combining it with b_i, we have a basis $b_i h_i$ of $\operatorname{Ker} \partial_i$. Then taking a lift of b_{i-1} in C_i and combining it with $b_i h_i$, we have a basis $b_i h_i b_{i-1}$ of C_i.

Definition 2.1. For given bases $c = \{c_i\}$ of C_* and $h = \{h_i\}$ of $H_*(C_*)$, we choose a basis $\{b_i\}$ of $\operatorname{Im} \partial_*$ and define

$$\tau(C_*, c, h) := (-1)^{|C_*|} \prod_{i=0}^{n} [b_i h_i b_{i-1}/c_i]^{(-1)^{i+1}} \in \mathbb{F}^*,$$

where

$$|C_*| := \sum_{j=0}^{n} (\sum_{i=0}^{j} \dim C_i)(\sum_{i=0}^{j} \dim H_i(C_*)),$$

and $[b_i h_i b_{i-1}/c_i]$ is the determinant of the base change matrix from c_i to $b_i h_i b_{i-1}$.

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It can be easily checked that $\tau(C_*, c, h)$ does not depend on the choices of b_i and $b_i h_i b_{i-1}$.

Let Y be a connected finite CW-complex and $\rho\colon \pi_1 Y \to GL_n(\mathbb{F})$ a linear representation. We regard the vector space $\mathbb{F}^n$ as a left $\mathbb{Z}[\pi_1 Y]$-module by

$$\gamma \cdot v := \rho(\gamma) v,$$

for $\gamma \in \pi_1 Y$ and $v \in \mathbb{F}^n$. Then we define the twisted cohomology group associated to ρ as

$$H^i_\rho(Y;\mathbb{F}^n) := H^i(\mathrm{Hom}_{\mathbb{Z}[\pi_1 Y]}(C_*(\widetilde{Y}), \mathbb{F}^n)),$$

where $\widetilde{Y}$ is the universal cover of Y. By a *cohomology orientation* of Y we mean an orientation ω of the cohomology group $H^*(Y;\mathbb{R}) = \bigoplus_i H^i(Y;\mathbb{R})$ as a vector space.

Definition 2.2. For a representation $\rho\colon \pi_1 Y \to GL_n(\mathbb{F})$, a basis h of $H^*_\rho(Y;\mathbb{F}^n)$ and a cohomology orientation ω, we define the *sign-refined Reidemeister torsion* $T_\rho(Y, h, \omega)$ as follows. We choose a lift $\tilde{\sigma}_i$ in $\widetilde{Y}$ of each cell σ_i in Y and bases h_0 of $H^*(Y;\mathbb{R})$ positively oriented with respect to ω and $\langle \xi_1, \dots, \xi_n \rangle$ of $\mathbb{C}^n$. Then we define

$$T_\rho(Y, h, \omega) := \tau_0^n \tau(\mathrm{Hom}_{\mathbb{Z}[\pi_1 Y]}(C_{-*}(\widetilde{Y}), \mathbb{F}^n), c) \ \in \mathbb{F}^* / \det \rho(\pi_1 Y)$$

where

$$\begin{aligned} \tau_0 &:= \mathrm{sgn}\, \tau(C^{-*}(Y;\mathbb{R}), c_0, h_0), \\ c_0 &:= \langle \sigma_1^*, \dots, \sigma^*_{\dim C_*(Y)} \rangle, \\ c &:= \langle \tilde{\sigma}_{1,1}, \dots, \tilde{\sigma}_{1,n}, \dots, \tilde{\sigma}_{\dim C_*(X),1}, \dots, \tilde{\sigma}_{\dim C_*(X),n} \rangle, \end{aligned}$$

and $\tilde{\sigma}_{i,j}$ is the cochain which maps $\tilde{\sigma}_i$ to ξ_j and $\tilde{\sigma}_k$ to 0 for $k \neq i$.

It is known that $T_\rho(Y, h, \omega)$ does not depend on the choices of $\tilde{\sigma}_i$, h_0 and $\langle \xi_1, \dots, \xi_n \rangle$, and that $T_\rho(Y, h, \omega)$ is a simple homotopy invariant.

3. Character varieties

Next we study a smooth subspace of the $SL_2(\mathbb{C})$-character variety of a 3-manifold group where a desired complex volume form is defined.

Let M be a compact connected oriented 3-manifold whose boundary consists of m tori T_i and let μ_i an oriented simple closed curve in T_i for each i. A typical example is a link complement of a rational homology 3-sphere equipped with the meridians of the link. We fix a tree embedded in M and connecting one point on μ_i for each i, and regard it as a base point. By abuse of notation we use the same letter μ_i for the element in $\pi_1 M$ represented by μ_i.

For a representation $\rho\colon \pi_1 M \to SL_2(\mathbb{C})$, its character $\chi_\rho\colon \pi_1 M \to \mathbb{C}$ is given by

$$\chi_\rho(\gamma) = \mathrm{tr}\, \rho(\gamma)$$

for $\gamma \in \pi_1 M$. The $SL_2(\mathbb{C})$-*character variety* X is the set of the characters χ_ρ of representations $\rho\colon \pi_1 M \to SL_2(\mathbb{C})$, which is the algebro-geometric quotient of the complex affine algebraic set $\mathrm{Hom}(\pi_1 M, SL_2(\mathbb{C}))$. We denote by $t\colon \mathrm{Hom}(\pi_1 M, SL_2(\mathbb{C})) \to X$ the quotient map which maps a representation to its character. It is known that the fiber of t at the character of an irreducible representation consists only of equivalent representations. See [CS83, LM85, S02] for more details.

Following Weil [We64], the Zariski tangent space $T_\rho Hom(\pi_1 M, SL_2(\mathbb{C}))$ can be identified with a subspace of the vector space $Z^1_{\mathrm{Ad}\circ\rho}(\pi_1 M; \mathfrak{sl}_2(\mathbb{C}))$ of group 1-cocycles by the inclusion

$$\left.\frac{d\rho_t}{dt}\right|_{t=0} \mapsto \left(\gamma \mapsto \left.\frac{d\rho_t(\gamma)\rho(\gamma^{-1})}{dt}\right|_{t=0}\right),$$

where $\rho_0 = \rho$ and $\gamma \in \pi_1 M$. It is easy to check that the tangent space to the orbit by the conjugation corresponds to the vector space $B^1_{\mathrm{Ad}\circ\rho}(\pi_1 M; \mathfrak{sl}_2(\mathbb{C}))$ of group 1-coboundaries.

We define R_0 to be the set of irreducible $SL_2(\mathbb{C})$-representations ρ of $\pi_1 M$ such that

$$\dim H^1_{\mathrm{Ad}\circ\rho}(M; \mathfrak{sl}_2(\mathbb{C})) = m \quad \text{and} \quad \rho(\mu_i) \neq \pm I,$$

where $\mathrm{Ad}\colon SL_2(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C}))$ is the adjoint representation and I the identity matrix. We set $X_0 = t(R_0)$.

Proposition 3.1. (i) *The space X_0 is a complex m-manifold.*
(ii) *The tangent space $T_{\chi_\rho} X_0$ is isomorphic to $H^1_{\mathrm{Ad}\circ\rho}(M; \mathfrak{sl}_2(\mathbb{C}))$.*

Proof. Let $\rho \in R_0$ and let V be an irreducible component of X containing χ_ρ. It follows from the result of Thurston [CS83, Proposition 3.1.2] that

$$\dim V \geq m.$$

On the other hand

$$\dim V \leq \dim T_{\chi_\rho} V \leq \dim T_{\chi_\rho} X \leq \dim H^1_{\mathrm{Ad}\circ\rho}(\pi_1 M; \mathfrak{sl}_2(\mathbb{C})) = m.$$

Therefore the above inequalities are all equalities, and so V is the unique component containing χ_ρ and χ_ρ is a smooth point of V. Furthermore $T_{\chi_\rho} V$ is isomorphic to $H^1_{\mathrm{Ad}\circ\rho}(M; \mathfrak{sl}_2(\mathbb{C}))$.

Now it suffices to show that X_0 is a Zariski open subspace. It follows from [CS83, Lemma 1.4.2] that the subspace of X consisting of the characters of irreducible representations is Zariski open. Since the subspace of X consisting of the characters of representations such that

$$\dim H^1_{\mathrm{Ad}\circ\rho}(M; \mathfrak{sl}_2(\mathbb{C})) \leq m \quad \text{and} \quad \rho(\mu_i) \neq \pm I$$

is also Zariski open, so is X_0. □

4. Torsion volume forms

Here we construct a Reidemeister torsion volume form on X_0.

Let $\overline{M}$ be the closed oriented manifold obtained by gluing solid tori Z_i to M along T_i for all i so that μ_i is identified with a meridian of Z_i. The manifold $\overline{M}$ has a natural cohomology orientation $\omega_{\overline{M}}$ represented by bases h^i of $H^i(\overline{M}; \mathbb{R})$ such that h^i and h^{3-i} are dual with respect to the cup product $H^i(\overline{M}; \mathbb{R}) \times H^{3-i}(\overline{M}; \mathbb{R}) \to \mathbb{R}$. Each solid torus Z_i also has a natural cohomology orientation ω_{Z_i} represented by

$$\langle [pt]^*, [Z_i, \partial Z_i]^* \rangle,$$

where $[pt]$ is the homology class represented by a point, and $[Z_i, \partial Z_i]^*$ is the Poincaré dual of the fundamental class. The natural cohomology orientation ω_{T_i} of each T_i is represented by

$$\langle [pt]^*, [\lambda_i]^*, [\mu_i]^*, [T_i]^* \rangle,$$

where λ is an oriented simple closed curve in T_i such that λ_i, μ_i is a longitude-meridian pair. We regard the Mayer-Vietoris long exact sequence as a cochain complex:

$$H^* = (H^0(\overline{M};\mathbb{R}) \to H^0(M;\mathbb{R}) \oplus (\oplus_i H^0(Z_i;\mathbb{R})) \to \oplus_i H^0(T_i;\mathbb{R}) \to H^1(\overline{M};\mathbb{R}) \to \dots).$$

We define a cohomology orientation ω_M so that

$$\operatorname{sgn} \tau(H^{-*}, h, \emptyset) = 1,$$

where h is the basis obtained by combining bases of $H^*(\overline{M};\mathbb{R}), H^*(M;\mathbb{R}), H^*(Z_i;\mathbb{R})$ and $H^*(T_i;\mathbb{R})$ representing $\omega_{\overline{M}}, \omega_M, \omega_{Z_i}$ and ω_{T_i} respectively.

The Killing form of $\mathfrak{sl}_2(\mathbb{C})$ induces non-degenerate cup products

$$\cup\colon H^i_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) \times H^{3-i}_{\mathrm{Ad}\circ\rho}(M,\partial M;\mathfrak{sl}_2(\mathbb{C})) \to H^3_{\mathrm{Ad}\circ\rho}(M,\partial M;\mathfrak{sl}_2(\mathbb{C})), \tag{1}$$

$$H^i_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})) \times H^{2-i}_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})) \to H^2_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})). \tag{2}$$

Lemma 4.1. *For $\rho \in R_0$, the following hold:*

(i) $\dim H^0_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) = \dim H^3_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) = 0$,

(ii) $\dim H^2_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) = m$,

(iii) $\dim H^0_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})) = \dim H^2_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})) = 1$ *for all i,*

(iv) $\dim H^1_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})) = 2$ *for all i.*

Proof. Since ρ is non-abelian, we observe that

$$H^0_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) = \mathfrak{sl}_2(\mathbb{C})^{\mathrm{Ad}\circ\rho(\pi_1 M)} = 0.$$

The boundary of M is non-empty, and so $H^3_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) = 0$, which shows (i).

The equation

$$\sum_{i=0}^{3} (-1)^i \dim H^i_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) = 3\chi(M) = 0$$

together with (i) gives

$$\dim H^2_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) = \dim H^1_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) = m,$$

which shows (ii).

Since $\rho|_{\pi_1 T_i}$ is non-trivial, we observe that

$$\dim H^0_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})) = \dim \mathfrak{sl}_2(\mathbb{C})^{\mathrm{Ad}\circ\rho(\pi_1 T_i)} = 1.$$

From (2) we have

$$\dim H^2_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})) = \dim H^0_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})) = 1,$$

which shows (iii).

The equation

$$\sum_{i=0}^{2} (-1)^i \dim H^i_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})) = 3\chi(T_i) = 0$$

together with (iii) gives

$$\dim H^1_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) = \dim H^0_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_1(\mathbb{C})) + \dim H^2_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_1(\mathbb{C})) = 2,$$

which shows (iv). □

We denote by $\theta\colon H^2_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) \to \oplus_{i=0}^m H^0_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C}))^*$ the composition of the homomorphisms

$$H^2_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) \to H^2_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})) \text{ and } H^2_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})) \to H^0_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C}))^*$$

induced by the natural inclusion and (1) respectively.

Lemma 4.2. *For $\rho \in R_0$, it follows that θ is an isomorphism.*

Proof. We need to show that the homomorphism $H^2_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) \to \oplus H^2_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C}))$ is an isomorphism. From (2) and Lemma 4.1 (i)

$$\dim H^3_{\mathrm{Ad}\circ\rho}(M,\partial M;\mathfrak{sl}_2(\mathbb{C})) = \dim H^0_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) = 0.$$

Therefore it follows from the long exact sequence for the pair $(M,\, \partial M)$ that the above homomorphism is surjective. From Lemma 4.1 (ii), (iii)

$$\dim H^2_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C})) = \sum_{i=0}^m \dim H^2_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})) = m,$$

which deduces the desired conclusion. □

It is easily seen that for $\rho \in R_0$, the complex vector space $H^0_{\mathrm{Ad}\circ\rho}(T_i;\mathfrak{sl}_2(\mathbb{C})) = \mathfrak{sl}_2(\mathbb{C})^{\mathrm{Ad}\circ\rho(\pi_1 T_i)}$ is generated by

$$P_{i,\rho} := \rho(\mu_i) - \frac{1}{2}\operatorname{tr}\rho(\mu_i) I$$

for all i. We define a basis h_ρ of $H^2_{\mathrm{Ad}\circ\rho}(M;\mathfrak{sl}_2(\mathbb{C}))^*$ to be

$$\langle \theta^{-1}(P^*_{1,\rho}), \dots, \theta^{-1}(P^*_{m,\rho}) \rangle$$

for each i.

Definition 4.3. We choose a triangulation of M. A linear form $\tau_{\chi_\rho}\colon \wedge_{i=1}^m T_{\chi_\rho} X_0 \to \mathbb{C}$ is defined by

$$\tau_{\chi_\rho}(v_1 \wedge \cdots \wedge v_m) := \begin{cases} T_\rho(M, \langle v_1, \dots, v_m, h_\rho \rangle, \omega_M) & \text{if } v_1 \wedge \cdots \wedge v_m \neq 0, \\ 0 & \text{if } v_1 \wedge \cdots \wedge v_m = 0. \end{cases}$$

We call the m-form τ the *Reidemeister torsion volume form.*

It follows from the simple homotopy invariance of Reidemeister torsion and the conjugation invariance that τ does not depend on the choices of a triangulation and a representative ρ. The following theorem is now straightforward to be checked from the definition of Reidemeister torsion.

Theorem 4.4. *The Reidemeister torsion volume form τ is a complex volume form on X_0.*

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