

BLANCHFIELD PAIRINGS AND GORDIAN DISTANCE

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ABSTRACT. A lower bound of the Gordian distance is presented in terms of the Blanchfield pairing. Our approach, in particular, allows us to show at least for 195 pairs of unoriented nontrivial prime knots with up to 10 crossings that their Gordian distance is equal to 3, most of which are difficult to treat otherwise.

1. INTRODUCTION

In this paper we present a lower bound of the Gordian distance of knots in terms of their Blanchfield pairings, extending the work [BF14b, BF15] by Borodzik and the first author on the unknotting number of a knot. We also describe our computational results on the lower bound for all pairs of nontrivial knots (possibly non-prime) with up to 10 crossings.

Blanchfield pairings and Gordian distance. Let J and K be knots in S^3 . Throughout this paper all knots are understood to be oriented. The *Gordian distance between J and K* , which we denote by $d(J, K)$, is defined to be the minimal number of crossing changes necessary to turn J into K . This induces a metric on the set of isotopy classes of knots. The *unknotting number* $u(J)$ of J is defined to be the Gordian distance between J and the unknot. We refer the reader to [BFP16, C19, Liv20] for recent studies on lower bounds on the Gordian distance of knots.

We set $\Lambda = \mathbb{Z}[t, t^{-1}]$, which is equipped with the involution given by $\overline{p(t)} = p(t^{-1})$ for $p(t) \in \Lambda$. Let R be a Noetherian unique factorization domain with (possibly trivial) involution $\bar{\cdot}: R \rightarrow R$. We call a homomorphism $\varphi: \Lambda \rightarrow R$ *admissible* if it satisfies the following:

- (1) $\varphi(t) \neq 1$ and $\varphi(t^{-1}) = \overline{\varphi(t)}$,
- (2) For every finitely generated R -module L the module $\text{Hom}_R(L, R)$ is free.

Every principle ideal domain satisfies (2), and so do Λ and its localization. (See [BF14b, Lemma 2.1] for the proof for Λ . The same proof works also for its localization.)

Let $\varphi: \Lambda \rightarrow R$ be an admissible homomorphism. We denote by $\Delta_J(t)$ the Alexander polynomial of a knot J . If $\varphi(\Delta_J(t)) \neq 0$, then the Blanchfield pairing of J associated with φ is defined:

$$\text{Bl}_{J,\varphi}: H_1^\varphi(X_J; R) \times H_1^\varphi(X_J; R) \rightarrow Q(R)/R,$$

where X_J is the complement of an open tubular neighborhood of J in S^3 and $Q(R)$ is the quotient ring of R . (See Section 2.2.) When φ is the identity map on Λ , $\text{Bl}_{J,\varphi}$

Date: September 27, 2022.

2020 Mathematics Subject Classification. Primary 57K10, Secondary 57K31.

Key words and phrases. Gordian distance, Blanchfield pairing, knot.

is the ordinary Blanchfield pairing of J . For a hermitian matrix A of size n over R with $\det A \neq 0$ we denote by $\lambda(A)$ the pairing

$$\lambda(A): R^n/AR^n \times R^n/AR^n \rightarrow Q(R)/R, (v, w) \mapsto \bar{v}^T A^{-1}w,$$

where we view v and w as represented by column vectors in R^n . We write A_0 for the image of A under the quotient map $R \rightarrow R/(\varphi(t) - 1)R$. We define $n_\varphi(J)$ to be the minimal size of a hermitian matrix A over R such that

- (1) $\lambda(A)$ is isometric to $\text{Bl}_{J, \varphi}$,
- (2) the matrix A_0 is congruent over $R/(\varphi(t) - 1)R$ to a diagonal matrix which has ± 1 on the diagonal.

We will show that such a hermitian matrix A exists for every admissible homomorphism φ and every knot J . (See Theorem 4.1.) When $\varphi: \Lambda \rightarrow S[t^{\pm 1}]$ is the embedding induced by a subring S of \mathbb{C} , we write $n_S(J)$ for $n_\varphi(J)$. In particular, $n_{\mathbb{Z}}(J)$ coincides with the invariant $n(J)$ first introduced in [BF15].

The Main theorem. The following is the main theorem of this paper. We write $-J$ for the mirror image of a knot J with opposite orientation, $J\sharp K$ for the connected sum of two knots J and K , and $a \cdot R$ for the principal ideal generated by $a \in R$.

Theorem 1.1. *Let J and K be knots in S^3 , and $\varphi: \Lambda \rightarrow R$ an admissible homomorphism. If $\varphi(\Delta_J(t)\Delta_K(t)) \neq 0$ and if $\varphi(\Delta_J(t)) \cdot R + \varphi(\Delta_K(t)) \cdot R = R$, then $d(J, K) \geq n_\varphi(-J\sharp K)$.*

In fact, in Section 4 we will state and prove a slightly stronger theorem as Theorem 4.1 which takes positive and negative crossing changes into account. In the case that $\Delta_J(t) = 1$ and φ is the identity map on Λ , this theorem recovers [BF15, Theorem 1.1] by Borodzik and the first author. In particular, they gave new obstructions for $u(J) = 2$ and $u(J) = 3$ as explained below. It is known that lower bounds of $n(J)$ are given by the Nakanishi index [N81], the Levine-Tristram signatures [BF14a, Le69, Mus65, Ta69, Tr69], the Lickorish obstruction [CoLi86, Lic85], the Murakami obstruction [Muk90] and the Jabuka obstruction [J09]. See [BF, BF15] for the details.

Corollary 1.2. *Let J and K be knots in S^3 . If $\Delta_J(t) \cdot \Lambda + \Delta_K(t) \cdot \Lambda = \Lambda$, then $d(J, K) \geq n(-J\sharp K)$.*

Corollary 1.3. *Let J and K be knots in S^3 . If the greatest common divisor of $\Delta_J(t)$ and $\Delta_K(t)$ is equal to ± 1 , then $d(J, K) \geq n_{\mathbb{Z}[\frac{1}{m}]}(-J\sharp K)$, where m is the minimal positive integer in $\Delta_J(t) \cdot \Lambda + \Delta_K(t) \cdot \Lambda$.*

Corollary 1.4. *Let J and K be knots in S^3 . If the greatest common divisor of $\Delta_J(t)$ and $\Delta_K(t)$ is equal to ± 1 , then $d(J, K) \geq n_{\mathbb{R}}(-J\sharp K)$.*

Note that $n_{\mathbb{Z}}(J) \geq n_{\mathbb{Z}[\frac{1}{m}]}(J) \geq n_{\mathbb{R}}(J)$ for every positive integer m . Borodzik and the first author [BF14a] proved that $n_{\mathbb{R}}(J)$ is completely determined by the Levine-Tristram signatures and the nullities.

We now consider the admissible homomorphism $\varphi_{-1}: \Lambda \rightarrow \mathbb{Z}$ sending t to -1 . Then it is well-known that $H_1^\varphi(X_J; \mathbb{Z})$ is isomorphic to $H_1(\Sigma(J); \mathbb{Z})$, where $\Sigma(J)$ is the double branched cover of S^3 along J . Since $\Sigma(J)$ is a rational homology 3-sphere, we have its linking pairing

$$\text{lk}_J: H_1(\Sigma(J); \mathbb{Z}) \times H_1(\Sigma(J); \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

It follows from [BF15, Lemma 3.3] that $\text{Bl}_{J, \varphi_{-1}}$ is isometric to 2lk_J , where 2lk_J is defined by $(2\text{lk}_J)(v, w) = 2 \cdot \text{lk}_J(v, w)$ for $v, w \in H_1(\Sigma(J); \mathbb{Z})$. Hence $n_{\varphi_{-1}}(J)$ coincides with the minimal size of an integral symmetric matrix representing 2lk_J and being congruent to the identity matrix modulo 2. Note also that $\varphi_{-1}(\Delta_J(t)) = \Delta_J(-1) = \pm \det(J)$. Thus we have the following corollary.

Corollary 1.5. *Let J and K be knots in S^3 . If $\det(J)$ and $\det(K)$ are coprime, then $d(J, K)$ is greater than or equal to the minimal size of an integral symmetric matrix representing $2\text{lk}_{-J\sharp K}$ and being congruent to the identity matrix modulo 2.*

When one of the knots J or K is the unknot, Corollary 1.5 gives a lower bound of the unknotting number and, in particular, the Lickorish obstruction for $u(J) = 1$ [CoLi86, Lic85]. Borodzik and the first author [BF15, Section 6] found examples of knots J where the lower bound gives new obstructions for $u(J) = 2$ and $u(J) = 3$. For example, they first determined $u(11_{a123}) = 3$ and $u(11_{n148}) = 3$. See also [BF] for the details.

Applications. Corollary 1.5 provides a new computable obstruction for $d(J, K) = 2$ for knots J and K in S^3 . The idea behind our approach is the following, which is the same as in [BF15, Section 5]. First it is well-known that the isometric type of the linking pairing $\text{lk}_{-J\sharp K}$ for J and K can be calculated in terms of Seifert matrices of J and K . Suppose that $\text{lk}_{-J\sharp K}$ is represented by an integral $n \times n$ matrix. Up to congruence there exists finitely many such matrices, which furthermore in many cases can be listed explicitly. It is then straightforward to verify whether or not $\text{lk}_{-J\sharp K}$ can be represented by any of these matrices.

As a concrete example we show in Example 5.8 that the Gordian distance from the trefoil 3_1 to the connected sum $4_1\sharp 4_1$ of two copies of the figure 8 knot equals 3, in other words, $d(3_1, 4_1\sharp 4_1) = 3$. It seems like most other known lower bounds on the Gordian distance struggle to prove this result.

Next note that Corollary 1.5 provides the same lower bounds of $d(J, K)$, $d(rJ, K)$, $d(J, rK)$ and $d(rJ, rK)$, where we write rJ for a knot J with opposite orientation. This follows from the fact that the linking pairing $\text{lk}_{-J\sharp K}$ is invariant under the changes of the orientations of the knots, as it will be discussed in Section 5.1. Also, by definition, $d(J, K) = d(K, J) = d(mJ, mK) = d(mK, mJ)$, where we write mJ for the mirror image of a knot J . Therefore in describing numbers of pairs in the following computational results we do not distinguish the pairs (J, K) , (rJ, K) , (J, rK) and (rJ, rK) , and neither the pairs (J, K) , (K, J) , (mJ, mK) and (mK, mJ) .

Among all pairs (J, K) of nontrivial prime knots with up to 10 crossings there are 886 pairs for which Corollary 1.5 shows that $d(J, K) \geq 3$, but for which any of the lower bounds of $d(J, K)$ by the signature, the Rasmussen s -invariant [Ra10], the Ozsváth-Szabó τ -invariant [OS03] and the maximal rank of $H_1(\Sigma(J); \mathbb{F}_p)$ for all odd primes p is less than 3. For 195 pairs (J, K) of knots among the 886 ones, it is known that $u(J) + u(K) \leq 3$, and since $d(J, K) \leq u(J) + u(K)$, we can conclude that $d(J, K) = u(J) + u(K) = 3$. Also, if we count the numbers of such pairs of knots, instead, for all pairs (J, K) of nontrivial knots (possibly non-prime) with up to 10 crossings, then the first one 886 is replaced by 1696, and the second one 195 is by 360.

Details are given in Section 5 and on the website [FKS] by the authors. At the time of writing we have not yet implemented the obstruction to $d(J, K) = n$ for higher values of n .

Organization. Section 2 provides a brief review of Blanchfield pairings of a knot and a homology $S^2 \times S^1$ and intersection pairings of a 4-manifold with coefficients in R . In Section 3 we study a certain relation among the pairings of homology $S^1 \times S^2$'s and of a cobordism between them, which is a key ingredient in the proof of the main theorem. In Section 4 we state and prove a slightly stronger theorem deducing the main theorem. Section 5 is devoted to applications, where we describe more details of our approach and computational results with examples.

Convention and notation. All knots are understood to be oriented, and all manifolds are understood to be compact, connected and oriented, unless we say specifically otherwise. We do not assume that manifolds are smooth. Throughout this paper R is a Noetherian unique factorization domain with (possibly trivial) involution $\bar{\cdot} : R \rightarrow R$. The quotient field of R is denoted by $Q(R)$.

Acknowledgments. The website Knotinfo: Table of knot invariants [LM], maintained by Chuck Livingston and Allison H. Moore, has been an invaluable tool for finding examples and testing our algorithm. The authors would like to thank them for helpful information on data provided on the website. The research started and was carried out while the second and third authors were visiting the University of Regensburg. They were very grateful for the warm hospitality.

The first author was supported by the SFB 1085 “higher invariants” funded by the DFG. The second author was supported by JSPS KAKENHI Grant Numbers JP18K13404 and JP18KK0380. The third author was supported by JSPS KAKENHI Grant Numbers JP19H01785 and JP20K03596.

2. BLANCHFIELD PAIRINGS AND TWISTED INTERSECTION PAIRINGS

We begin with the definitions of the twisted Blanchfield pairings of a knot and a homology $S^1 \times S^2$ and the twisted intersection pairing of a 4-manifold associated with an admissible homomorphism $\varphi : \Lambda \rightarrow R$. We refer the reader to [H12, Chapter 2] for a thorough treatment of the classical Blanchfield pairing and to [FKLMN20, Appendix] for more details on (commutative) twisted Blanchfield pairings.

2.1. Twisted homology and cohomology groups. Let X be a topological space equipped with an epimorphism $H_1(X; \mathbb{Z}) \rightarrow \mathbb{Z}$ admitting the corresponding infinite cyclic covering \tilde{X} of X . Let Y be a subspace of X , and we write \tilde{Y} for the preimage of Y by the covering map $\tilde{X} \rightarrow X$. The singular chain complex $C_*(\tilde{X}, \tilde{Y})$ has the structure of a Λ -module, where the action by t corresponds to the deck transformation on \tilde{X} by $1 \in \mathbb{Z}$.

Recall that a homomorphism $\varphi : \Lambda \rightarrow R$ is called *admissible* if it satisfies the following:

- (1) $\varphi(t) \neq 1$ and $\varphi(t^{-1}) = \overline{\varphi(t)}$,
- (2) For every finitely generated R -module L the module $\text{Hom}_R(L, R)$ is free.

Let $\varphi : \Lambda \rightarrow R$ be an admissible homomorphism and L an R -module, which has the structure of a Λ -module induced by φ . For each nonnegative integer i we define

the i -th twisted homology group $H_i^\varphi(X, Y; L)$ and the i -th twisted cohomology group $H_\varphi^i(X, Y; L)$ of (X, Y) associated with φ as:

$$\begin{aligned} H_i^\varphi(X, Y; L) &= H_i(\overline{C_*(\tilde{X}, \tilde{Y})} \otimes_\Lambda L), \\ H_\varphi^i(X, Y; L) &= H^i(\text{Hom}_\Lambda(C_*(\tilde{X}, \tilde{Y}), L)). \end{aligned}$$

Here we denote by $\overline{C_*(\tilde{X}, \tilde{Y})}$ the module with the involuted Λ -structure, that is, $\overline{C_*(\tilde{X}, \tilde{Y})} = C_*(\tilde{X}, \tilde{Y})$ as abelian groups but multiplication by $a \in \Lambda$ corresponds to multiplication by \bar{a} . Similarly, for an R -module H we denote by \overline{H} the module with the involuted R -structure. When Y is empty, we write $H_i^\varphi(X; L)$ and $H_\varphi^i(X; L)$ respectively. Also, when φ is the identity map on Λ and $L = \Lambda$, then we write $H_i(X, Y; \Lambda)$ and $H^i(X, Y; \Lambda)$ respectively.

The *Kronecker pairing*

$$\kappa: H_\varphi^i(X, Y; L) \times H_i^\varphi(X, Y; R) \rightarrow L$$

is defined as the induced one by the sesquilinear pairing

$$\text{Hom}_\Lambda(C_i(\tilde{X}, \tilde{Y}), L) \times \overline{C_i(\tilde{X}, \tilde{Y})} \otimes_\Lambda R \rightarrow L, (f, c \otimes a) \mapsto \bar{a}f(c)$$

for $f \in \text{Hom}_\Lambda(C_i(\tilde{X}, \tilde{Y}), L)$, $c \in \overline{C_i(\tilde{X}, \tilde{Y})}$ and $a \in R$. Thus we have the evaluation map

$$H_\varphi^i(X, Y; L) \rightarrow \overline{\text{Hom}_R(H_i^\varphi(X, Y; R), L)}, f \mapsto \kappa(f, \cdot).$$

We set $R_0 = R/(\varphi(t) - 1)R$. If X is path-connected, then the 0-th twisted homology and cohomology groups are computed as follows (see for instance [HS71, Proposition 3.1]):

$$H_0^\varphi(X; L) = L \otimes_R R_0, \quad H_\varphi^0(X; L) = \{v \in L; \varphi(t)v = v\}.$$

Since $\varphi(t) \neq 1$ and R is an integral domain, we have

$$\text{Tor}_1^\Lambda(H_0(X; \Lambda), R) = \text{Tor}_1^\Lambda(\Lambda/(t-1)\Lambda, R) = \{x \in R; (\varphi(t) - 1)x = 0\} = 0.$$

It thus follows from the universal coefficient spectral sequence [Ro09, Theorem 10.90] that the homomorphism $H_1(X; \Lambda) \otimes_\Lambda R \rightarrow H_1^\varphi(X; R)$ is an isomorphism.

2.2. Blanchfield pairings. Recall that for a finitely generated R -module H with an exact sequence

$$R^l \rightarrow R^m \xrightarrow{r} H \rightarrow 0,$$

where $l \geq m$, the order of H is defined to be the greatest common divisor of the m -minors of a representation matrix of r , and is well-defined up to multiplication by a unit in R . (See for instance [H12, Lic97].)

Let J be a knot in S^3 . The complement X_J of an open tubular neighborhood of J in S^3 admits an isomorphism $H_1(X_J; \mathbb{Z}) \rightarrow \mathbb{Z}$ induced by the orientation of J . Recall that the Alexander polynomial $\Delta_J(t) \in \Lambda$ of J is defined to be the order of the Alexander module $H_1(X_J; \Lambda)$. Let $\varphi: \Lambda \rightarrow R$ be an admissible homomorphism, and suppose that $\varphi(\Delta_J(t)) \neq 0$. Since $H_1^\varphi(X_J; R)$ is isomorphic to $H_1(X_J; \Lambda) \otimes_\Lambda R$, $H_1^\varphi(X_J; R)$ is a torsion R -module and its order is equal to $\varphi(\Delta_J(t))$.

We consider the following sequence of homomorphisms

$$\begin{aligned} \Phi_{J, \varphi}: H_1^\varphi(X_J; R) &\rightarrow H_1^\varphi(X_J, \partial X_J; R) && \rightarrow H_\varphi^2(X_J; R) \\ &\xleftarrow{\cong} H_\varphi^1(X_J; Q(R)/R) && \rightarrow \overline{\text{Hom}_R(H_1^\varphi(X_J; R), Q(R)/R)}. \end{aligned}$$

Here the homomorphisms are as follows:

- (1) the first one is the inclusion induced homomorphism;
- (2) the second one comes from Poincaré duality;
- (3) the third one is the Bockstein homomorphism

$$H_\varphi^1(X_J; Q(R)/R) \rightarrow H_\varphi^2(X_J; R)$$

corresponding to the short exact sequence of Λ -modules

$$0 \rightarrow R \rightarrow Q(R) \rightarrow Q(R)/R \rightarrow 0;$$

- (4) the last one is the evaluation map induced by the Kronecker pairing κ .

All of these can be checked to be isomorphisms. See [FKLMN20, Appendix] for the details. Thus $\Phi_{J,\varphi}$ induces a non-singular sesquilinear pairing

$$\text{Bl}_{J,\varphi}: H_1^\varphi(X_J; R) \times H_1^\varphi(X_J; R) \rightarrow Q(R)/R, (v, w) \mapsto \Phi_{J,\varphi}(v)(w),$$

which we call the *Blanchfield pairing of J associated with φ* . It is well-known that the pairing is hermitian. When φ is the identity map on Λ , the pairing is the ordinary Blanchfield pairing Bl_J of J .

We call a closed 3-manifold M equipped with an isomorphism $H_1(M; \mathbb{Z}) \rightarrow \mathbb{Z}$ a *homology $S^1 \times S^2$* . Let M be a homology $S^1 \times S^2$. We define the *Alexander polynomial* $\Delta_M(t) \in \Lambda$ of M to be the order of $H_1(M; \Lambda)$. We will show in Lemma 3.2 that $H_1^\varphi(M; R)$ is a torsion R -module. Its order is equal to $\varphi(\Delta_M(t))$. This can be checked for instance by using Reidemeister torsion [Tu01, Proposition 3.6 and Corollary 11.9].

Suppose that $\varphi(\Delta_M(t)) \neq 0$. Similarly, the following sequence of isomorphisms

$$\Phi_{M,\varphi}: H_1^\varphi(M; R) \rightarrow H_\varphi^2(M; R) \xleftarrow{\cong} H_1^\varphi(M; Q(R)/R) \rightarrow \overline{\text{Hom}_R(H_1^\varphi(M; R), Q(R)/R)}$$

induces a non-singular hermitian sesquilinear pairing

$$\text{Bl}_{M,\varphi}: H_1^\varphi(M; R) \times H_1^\varphi(M; R) \rightarrow Q(R)/R, (v, w) \mapsto \Phi_{M,\varphi}(v)(w),$$

which we call the *Blanchfield pairing of M associated with φ* .

We denote by M_J the result of 0-framed surgery of S^3 along J , which is a homology $S^1 \times S^2$. The equipped isomorphism $H_1(M_J; \mathbb{Z}) \rightarrow \mathbb{Z}$ sends the meridional element to 1. It can be checked that the inclusion induced homomorphism $H_1^\varphi(X_J; R) \rightarrow H_1^\varphi(M_J; R)$ is an isometry with respect to $\text{Bl}_{J,\varphi}$ and $\text{Bl}_{M_J,\varphi}$.

2.3. Intersection pairings. Let W be a topological 4-manifold equipped with an epimorphism $H_1(W; \mathbb{Z}) \rightarrow \mathbb{Z}$, and $\varphi: \Lambda \rightarrow R$ an admissible homomorphism. Recall that we set $R_0 = R/(\varphi(t) - 1)R$. We denote by $G^\varphi(W; R)$ and $G(W; R_0)$ the cokernels of the inclusion induced homomorphisms $H_2^\varphi(\partial W; R) \rightarrow H_2^\varphi(W; R)$ and $H_2(\partial W; R_0) \rightarrow H_2(W; R_0)$ respectively. It follows from the homology long exact sequence for $(W, \partial W)$ that $G^\varphi(W; R)$ and $G(W; R_0)$ are isomorphic to the kernels of the homomorphisms $H_2^\varphi(W, \partial W; R) \rightarrow H_1^\varphi(\partial W; R)$ and $H_2(W, \partial W; R_0) \rightarrow H_1(\partial W; R_0)$ respectively.

We consider the following sequence of homomorphisms

$$\Psi_{W,\varphi}: H_2^\varphi(W; \Lambda) \rightarrow H_2^\varphi(W, \partial W; \Lambda) \rightarrow H_\varphi^2(W; \Lambda) \rightarrow \overline{\text{Hom}_R(H_2^\varphi(W; R), R)}.$$

Here the homomorphisms are as follows:

- (1) the first one is the inclusion induced homomorphism;
- (2) the second one comes from Poincaré duality;
- (3) the third one is the evaluation map induced by the Kronecker pairing κ .

The map $\Psi_{W,\varphi}$ induces a hermitian sesquilinear pairing:

$$H_2^\varphi(W; R) \times H_2(W; R) \rightarrow R, (v, w) \mapsto \tilde{\Psi}_{W,\varphi}(v)(w),$$

which we call the *intersection pairing of W associated with φ* . Since $\Psi_{W,\varphi}$ factors through $H_2^\varphi(W, \partial W; \Lambda)$, the intersection pairing induces a pairing

$$G^\varphi(W; R) \times G^\varphi(W; R) \rightarrow R.$$

Similarly, considering the composition of φ and the quotient map $R \rightarrow R_0$, we have the intersection pairings of W over R_0 :

$$H_2(W; R_0) \times H_2(W; R_0) \rightarrow R_0, G(W; R_0) \times G(W; R_0) \rightarrow R_0.$$

When φ is the identity map on Λ , $R_0 = \mathbb{Z}$ and the first pairing is the ordinary intersection pairing of W .

3. COBORDISMS BETWEEN HOMOLOGY $S^1 \times S^2$ 'S

We describe a relation of twisted Blanchfield pairings of homology $S^1 \times S^2$'s and twisted intersection pairings of a certain cobordism between them.

3.1. Tame cobordisms. Let M and N be homology $S^1 \times S^2$'s, and $\varphi: \Lambda \rightarrow R$ an admissible homomorphism with $\varphi(\Delta_M(t)\Delta_N(t)) \neq 0$. We call a topological 4-manifold W equipped with an isomorphism $H_1(W; \mathbb{Z}) \rightarrow \mathbb{Z}$ a *φ -tame cobordism from M to N* if it satisfies the following:

- (1) ∂W is a disjoint union of $-M$ and N .
- (2) The inclusion induced homomorphisms $H_1(M; \mathbb{Z}) \rightarrow H_1(W; \mathbb{Z})$ and $H_1(N; \mathbb{Z}) \rightarrow H_1(W; \mathbb{Z})$ are isomorphisms, and compatible with the equipped isomorphisms, that is, the following diagram commutes:

$$\begin{array}{ccccc} H_1(M; \mathbb{Z}) & \longrightarrow & H_1(W; \mathbb{Z}) & \longleftarrow & H_1(N; \mathbb{Z}) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{Z} & & \end{array}$$

- (3) $H_1^\varphi(W; R) = 0$.

Note that $H_1^\varphi(\partial W; R) = H_1^\varphi(M; R) \oplus H_1^\varphi(N; R)$, and it is equipped with the Blanchfield pairing associated with φ

$$\text{Bl}_{-M,\varphi} \oplus \text{Bl}_{N,\varphi}: H_1^\varphi(\partial W; R) \times H_1^\varphi(\partial W; R) \rightarrow Q(R)/R.$$

The following proposition will be the key ingredient in the proof of the main theorem in Section 4, and proved in the following subsection. For a matrix A over R we write A_0 for its image under the quotient map $R \rightarrow R_0$.

Proposition 3.1. *Let M and N be homology $S^1 \times S^2$'s, $\varphi: \Lambda \rightarrow R$ an admissible homomorphism with $\varphi(\Delta_M(t)\Delta_N(t)) \neq 0$, and W a φ -tame cobordism from M to N . Then $G^\varphi(W; R)$ is a free R -module. Furthermore, every hermitian matrix A over R representing the twisted intersection pairing associated with φ on $G^\varphi(W; R)$ satisfies the following:*

- (1) $\lambda(A)$ is isometric to $\text{Bl}_{-M,\varphi} \oplus \text{Bl}_{N,\varphi}$.
- (2) A_0 represents the intersection pairing on $G(W; R_0)$.

3.2. Proof of Proposition 3.1. Let M be a homology $S^1 \times S^2$. The following lemma is standard.

Lemma 3.2. *The following hold:*

- (1) $H_1^\varphi(M; R)$ is a torsion R -module, and multiplication by $(\varphi(t) - 1)$ defines an automorphism on $H_1^\varphi(M; R)$.
- (2) $H_2^\varphi(M; R)$ is isomorphic to R_0 , and the homomorphism $H_2^\varphi(M; R) \otimes_R R_0 \rightarrow H_2(M; R_0)$ is an isomorphism.

Proof. We first show (1). We consider the universal coefficient spectral sequence $E_{p,q}^2 = \text{Tor}_p^R(H_q^\varphi(M; R), R_0)$ with the differential map $d^2: E_{p,q}^2 \rightarrow E_{p-2,q+1}^2$ converging to $H_{p+q}(M; R_0)$ [Ro09, Theorem 10.90]. Since R_0 admits a free resolution of length 1, we have $E_{2,q}^2 = 0$ for all q . In particular, the differential map $d^2: E_{2,0}^2 \rightarrow E_{0,1}^2$ is the 0-map. Hence $H_1(M; R_0)$, which is isomorphic to R_0 , contains $H_1^\varphi(M; R) \otimes_R R_0$ and $E_2^{1,0}$ as direct summands. Here since $H_0^\varphi(M; R)$ is isomorphic to R_0 , so is $E_2^{1,0}$. It thus follows that $H_1^\varphi(M; R) \otimes_R R_0 = 0$, which shows that multiplication by $(\varphi(t) - 1)$ defines an automorphism on $H_1^\varphi(M; R)$. Let x_1, \dots, x_m be generators of $H_1^\varphi(M; R)$. Then we may write $x_i = (\varphi(t) - 1) \sum_{j=1}^m \lambda_{ij} x_j$ for some $\lambda_{ij} \in R$. Rearranging this we may write $\sum_{j=1}^m \mu_{ij} x_j = 0$ for $1 \leq i \leq m$, where $\mu_{ij} = (\varphi(t) - 1)\lambda_{ij} - \delta_{ij}$. We set $\Delta = \det(\mu_{ij}) \in R$. Then $\Delta x_i = 0$ for $1 \leq i \leq m$, and $\Delta H_1^\varphi(M; \mathbb{R}) = 0$. Since Δ is equal to $\det(-\delta_{ij}) = (-1)^m$ modulo $(\varphi(t) - 1)$, $\Delta \neq 0$, which shows that $H_1^\varphi(M; R)$ is a torsion R -module.

We next show (2). Poincaré duality implies that $H_2^\varphi(M; R)$ is isomorphic to $H_\varphi^1(M; R)$. We consider the universal coefficient spectral sequence $E_2^{p,q} = \overline{\text{Ext}_R^q(H_p^\varphi(M; R), R)}$ with the differential map $d_2: E_2^{p,q} \rightarrow E_2^{p-1,q+2}$ converging to $H_\varphi^{p+q}(M; R)$ [Le77, Theorem 2.3]. Note that here we need the involuted R -structure of $\overline{\text{Ext}_R^q(H_p^\varphi(M; R), R)}$ since we apply the universal coefficient spectral sequence to the chain complex $C_*(\widetilde{M}) \otimes_\Lambda R$ and then the cochain complex $\text{Hom}_\Lambda(C_*(\widetilde{M}), R)$ is identified with $\text{Hom}_R(C_*(\widetilde{M}) \otimes_\Lambda R, R)$ with the involuted R -structure by the Kronecker pairing. Since $H_0^\varphi(M; R)$ is isomorphic to R_0 , so is $E_2^{0,1}$. By (1) we have $E_2^{1,0} = 0$. Thus $H_2^\varphi(M; R)$ is isomorphic to R_0 . Now we again consider the universal coefficient spectral sequence $E_{p,q}^2 = \text{Tor}_p^R(H_q^\varphi(M; R), R_0)$ converging to $H_{p+q}(M; R_0)$. Recall that we have $E_{2,q}^2 = 0$ for all q . In particular, the differential map $d^2: E_{2,1}^2 \rightarrow E_{0,2}^2$ is the 0-map. Also, since $H_1^\varphi(M; \mathbb{R})$ is $(\varphi(t) - 1)$ -torsion free by (1), $E_{1,1}^2 = 0$. Thus the homomorphism $H_2^\varphi(M; R) \otimes_R R_0 \rightarrow H_2(M; R_0)$ is an isomorphism. \square

Let M and N be homology $S^1 \times S^2$'s, $\varphi: \Lambda \rightarrow R$ an admissible homomorphism with $\varphi(\Delta_M(t)\Delta_N(t)) \neq 0$, and W a φ -tame cobordism W from M to N .

Lemma 3.3. *The following hold:*

- (1) The evaluation map $H_\varphi^2(W; R) \rightarrow \overline{\text{Hom}_R(H_2^\varphi(W; R), R)}$ induced by the Kronecker pairing κ is an isomorphism.
- (2) The homomorphism $\text{Hom}_\Lambda(G^\varphi(W; R), R) \rightarrow \text{Hom}_R(H_2^\varphi(W; R), R)$ induced by the quotient map $H_2^\varphi(W; R) \rightarrow G^\varphi(W; R)$ is an isomorphism.

Proof. We first show (1). As in the proof of Lemma 3.2 we consider the universal coefficient spectral sequence $E_2^{p,q} = \overline{\text{Ext}_R^q(H_p^\varphi(W; R), R)}$ with the differential map $d_2: E_2^{p,q} \rightarrow E_2^{p-1,q+2}$ converging to $H_\varphi^{p+q}(W; R)$ [Le77, Theorem 2.3]. Since

$H_0^\varphi(W; R)$ is isomorphic to R_0 , which admits a free resolution of length 1, we have $E_2^{0,2} = 0$. Since $H_1(W; \Lambda) = 0$, we see at once that $E_2^{1,q} = 0$ for all q . In particular, we have $E_2^{1,1} = 0$, and see that the differential map $d_2: E_2^{2,0} \rightarrow E_2^{1,2}$ is the 0-map. Thus (1) follows.

Since $H_2^\varphi(\partial W; R) = H_2^\varphi(M; R) \oplus H_2^\varphi(N; R)$ is a torsion module by Lemma 3.2 (2), every homomorphism $H_2^\varphi(W; R) \rightarrow R$ induces a homomorphism $G^\varphi(W; R) \rightarrow R$, and (2) follows. \square

We now have the following sequence of the isomorphisms:

$$D_{W,\varphi}: H_2^\varphi(W, \partial W; R) \rightarrow H_2^\varphi(W; R) \rightarrow \overline{\text{Hom}_R(H_2^\varphi(W; R), R)} \xleftarrow{\cong} \overline{\text{Hom}_R(G^\varphi(W; R), R)},$$

Here the isomorphisms are as follows:

- (1) the first one comes from Poincaré duality;
- (2) the second one is the isomorphism as in Lemma 3.3 (1);
- (3) third ones is the isomorphism as in Lemma 3.3 (2).

Lemma 3.4. *The following hold:*

- (1) $H_2^\varphi(W, \partial W; R)$ is a free R -module.
- (2) $G^\varphi(W; R)$ is a free R -module.
- (3) The homomorphism $G^\varphi(W; R) \otimes_R R_0 \rightarrow G(W; R_0)$ is an isomorphism.

Proof. We have the isomorphism $D_{W,\varphi}: H_2^\varphi(W, \partial W; R) \rightarrow \overline{\text{Hom}_R(G^\varphi(W; R), R)}$. It follows from the second property of the admissible homomorphism φ that these R -modules are free, which shows (1).

We next show (2). Since $H_1^\varphi(W; R) = 0$, the homology long exact sequence for $(W, \partial W)$ descends to the following short exact sequence:

$$0 \rightarrow G^\varphi(W; R) \rightarrow H_2^\varphi(W, \partial W; R) \rightarrow H_1^\varphi(\partial W; R) \rightarrow 0.$$

By (1) we can find an isomorphism $f: R^r \rightarrow H_2^\varphi(W, \partial W; R)$, where $r = \text{rank}_R H_2^\varphi(W, \partial W; R)$. Let $g: R^s \rightarrow G^\varphi(W; R)$ be an epimorphism such that s is minimal among such epimorphisms. Thus we have a presentation matrix A of $H_1^\varphi(\partial W; R)$ so that the following diagram commutes:

$$\begin{array}{ccccccc} R^s & \xrightarrow{A} & R^r & \longrightarrow & H_1^\varphi(\partial W; R) & \longrightarrow & 0 \\ & & \downarrow f & & \parallel & & \\ 0 & \longrightarrow & G^\varphi(W; R) & \longrightarrow & H_2^\varphi(W, \partial W; R) & \longrightarrow & H_1^\varphi(\partial W; R) \longrightarrow 0 \end{array}$$

It is well-known that $H_1(M_J; \Lambda)$, isomorphic to $H_1(X_J; \Lambda)$, for a knot J in S^3 admits a nonsingular square presentation matrix, and so does $H_1^\varphi(M_J; R)$, isomorphic to $H_1(M_J; \Lambda) \otimes_\Lambda R$. Since M and N can be seen as the results of 0-framed surgery of a homology 3-sphere Σ along some knots in Σ , similarly, $H_1(M; \Lambda)$ and $H_1(N; \Lambda)$ admits a nonsingular square presentation matrix, and so do $H_1^\varphi(M; R)$ and $H_1^\varphi(N; R)$, isomorphic to $H_1(M; \Lambda) \otimes_\Lambda R$ and $H_1(N; \Lambda) \otimes_\Lambda R$ respectively. Let B and C be such presentation matrices of $H_1^\varphi(M; R)$ and $H_1^\varphi(N; R)$. Then $B \oplus C$ is another presentation matrix of $H_1^\varphi(\partial W; R) = H_1^\varphi(M; R) \oplus H_1^\varphi(N; R)$. Two presentation matrices of a same module can be transformed into each other by the following 4 operations and their inverses [Lic97, Theorem 6.1]:

- (1) Permutation of rows and columns;
- (2) Replacement of the matrix P by $P \oplus (1)$;

- (3) Addition of an extra column of zeros to the matrix;
- (4) Addition of a scalar multiple of a row (or column) to another row (or column).

It thus follows that the difference between the minimal number of generators of the module generated by columns of A and the number of rows of A is equal to that for $B \oplus C$, which is 0. Hence we have $s = r$, and so A is a square matrix. Since $\det A = \det B \det C \neq 0$, we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^r & \xrightarrow{A} & R^r & \longrightarrow & H_1^\varphi(\partial W; R) & \longrightarrow & 0 \\
& & \downarrow g & & \downarrow f & & \parallel & & \\
0 & \longrightarrow & G^\varphi(W; R) & \longrightarrow & H_2^\varphi(W, \partial W; R) & \longrightarrow & H_1^\varphi(\partial W; R) & \longrightarrow & 0
\end{array}$$

It now follows from the five lemma that g is an isomorphism, and (2) follows.

Finally, we show (3). It suffices to see that the homomorphisms $H_2^\varphi(\partial W; R) \otimes_R R_0 \rightarrow H_2(\partial W; R_0)$ and $H_2^\varphi(W; R) \otimes_R R_0 \rightarrow H_2(W; R_0)$ are isomorphisms. By Lemma 3.2 (2) the first one is an isomorphism. We consider the universal coefficient spectral sequence $E_{p,q}^2 = \text{Tor}_p^R(H_q^\varphi(W; R), R_0)$ with the differential map $d^2: E_{p,q}^2 \rightarrow E_{p-2,q+1}^2$ converging to $H_{p+q}(W; R_0)$ [Ro09, Theorem 10.90]. Since R_0 admits a free resolution of length 1, we have $E_{2,0}^2 = E_{2,1}^2 = 0$. In particular, the differential map $d_2: E_{2,1}^2 \rightarrow E_{0,2}^2$ is the 0-map. Also, since $H_1^\varphi(W; R) = H_1^\varphi(M; R) \oplus H_1^\varphi(N; R)$ is $(\varphi(t) - 1)$ -torsion free by Lemma 3.2 (1), $E_{1,1}^2 = 0$. Thus the homomorphism $H_2^\varphi(W; R) \otimes_R R_0 \rightarrow H_2(W; R_0)$ is an isomorphism, and (3) follows. \square

Lemma 3.5. *The homomorphism $H_\varphi^2(W, \partial W; Q(R)) \rightarrow H_\varphi^2(W; Q(R))$ is an isomorphism.*

Proof. Since $H_1^\varphi(W; Q(R)) = H_2^\varphi(\partial W; Q(R)) = 0$ by Lemma 3.2 (1), it follows from the homology long exact sequence for $(W, \partial W)$ that the homomorphism $H_2^\varphi(W; Q(R)) \rightarrow H_2^\varphi(W, \partial W; Q(R))$ is an isomorphism. By Poincaré duality we have the desired isomorphism. \square

We consider the following sequence of homomorphisms

$$\begin{aligned}
\overline{\Psi}_{W,\varphi}: H_2^\varphi(W, \partial W; R) &\rightarrow H_2^\varphi(W, \partial W; Q(R)) &\rightarrow H_\varphi^2(W; Q(R)) \\
&\xleftarrow{\cong} H_\varphi^2(W, \partial W; Q(R)) &\rightarrow \overline{\text{Hom}}_R(H_2^\varphi(W, \partial W; R), Q(R)).
\end{aligned}$$

Here the homomorphisms are as follows:

- (1) the first one is a localization map;
- (2) the second one comes from Poincaré duality;
- (3) the third one is the isomorphism as in Lemma 3.5;
- (4) the last one is the evaluation map induced by the Kronecker pairing κ .

As the twisted intersection pairing of W the map $\overline{\Psi}_{W,\varphi}$ induces a hermitian sesquilinear pairing

$$H_2^\varphi(W, \partial W; R) \times H_2^\varphi(W, \partial W; R) \rightarrow Q(R), (v, w) \mapsto \overline{\Psi}_{W,\varphi}(v)(w),$$

which we call the *twisted intersection pairing* of $(W, \partial W)$.

The following lemma is proved totally parallel to one of the claims in the proof of [BF15, Lemma 2.8].

Lemma 3.6. *The twisted intersection pairings of W and $(W, \partial W)$ and the twisted Blanchfield pairings of M and N associated with φ fit into the following commutative diagram:*

$$\begin{array}{ccc}
H_2^\varphi(W; R) \times H_2^\varphi(W; R) & \longrightarrow & R \\
\downarrow & & \downarrow \\
H_2^\varphi(W, \partial W; R) \times H_2^\varphi(W, \partial W; R) & \longrightarrow & Q(R) \\
\downarrow & & \downarrow \\
H_1^\varphi(\partial W; R) \times H_1^\varphi(\partial W; R) & \xrightarrow{\text{Bl}_{-M, \varphi} \oplus \text{Bl}_{N, \varphi}} & Q(R)/R.
\end{array}$$

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. First, $H_2^\varphi(W, \partial W; R)$ and $G^\varphi(W; R)$ are free R -modules by Lemma 3.4 (1) and (2). We pick an arbitrary basis \mathcal{B} of $G^\varphi(W; R)$, and set A to be the hermitian matrix representing the twisted intersection pairing associated with φ on $G^\varphi(W; R)$ with respect to \mathcal{B} . By Lemma 3.4 (3) we see that A satisfies (2).

We write \mathcal{C} for the basis of $H_2^\varphi(W, \partial W; R)$ dual to \mathcal{B} by the isomorphism $D_{W, \varphi}: H_2^\varphi(W, \partial W; R) \rightarrow \overline{\text{Hom}_R(G^\varphi(W; R), R)}$. Then the homomorphism $G^\varphi(W; R) \rightarrow H_2^\varphi(W, \partial W; R)$ is given by the map $v \mapsto Av$ with respect to \mathcal{B} and \mathcal{C} . By Lemma 3.6 we see that A^{-1} represents the twisted intersection pairing of $(W, \partial W)$ associated with φ with respect to \mathcal{C} , and we have the following commutative diagram of the pairings:

$$\begin{array}{ccc}
R^s \times R^s & \xrightarrow{(v, w) \mapsto v^T A w} & R \\
\downarrow (v, w) \mapsto (Av, Aw) & & \downarrow \\
R^s \times R^s & \xrightarrow{(v, w) \mapsto v^T A^{-1} w} & Q(R) \\
\downarrow & & \downarrow \\
H_1^\varphi(\partial W; R) \times H_1^\varphi(\partial W; R) & \xrightarrow{\text{Bl}_{-M, \varphi} \oplus \text{Bl}_{N, \varphi}} & Q(R)/R,
\end{array}$$

where $s = \text{rank}_R G^\varphi(W; R) = \text{rank}_{R_0} G(W; R_0)$. Since the vertical maps form short exact sequences, we now see that A satisfies (1). \square

4. PROOF OF THE MAIN THEOREM

We now prove Theorem 1.1. In fact, we state and prove a slightly stronger theorem which takes positive and negative crossing changes into account. A crossing change for a knot in S^3 is called positive (resp. negative) if it turns a negative (resp. positive) crossing in a knot diagram into positive (resp. negative) one.

Theorem 1.1 is a direct corollary of the following theorem. Recall that for a matrix A over R we write A_0 for its image under the quotient map $R \rightarrow R_0$.

Theorem 4.1. *Let J and K be knots in S^3 such that J can be turned into K by n_+ positive crossing changes and n_- negative ones, and let $\varphi: \mathbb{Z}[t^\pm] \rightarrow R$ be an admissible homomorphism. If $\varphi(\Delta_J(t)\Delta_K(t)) \neq 0$ and if $\varphi(\Delta_J(t))R$ and $\varphi(\Delta_K(t))R$ are*

coprime, then there exists a hermitian matrix A of size $n_+ + n_-$ over R satisfying the following:

- (1) $\lambda(A)$ is isometric to $\text{Bl}_{-J\sharp K, \varphi}$.
- (2) A_0 is the diagonal matrix with n_+ entries of 1 and n_- ones of -1 .

Considering all knots J with $\Delta_J(t) = 1$ and the identity map on Λ as φ in Theorem 4.1, we recover [BF15, Theorem 1.1].

The following proposition, combined with Proposition 3.1, implies Theorem 4.1. Note that we have

$$\text{Bl}_{-M_J, \varphi} \oplus \text{Bl}_{M_K, \varphi} = \text{Bl}_{-J, \varphi} \oplus \text{Bl}_{K, \varphi} = \text{Bl}_{-J\sharp K, \varphi}.$$

Proposition 4.2. *Let J and K be knots in S^3 such that J can be turned into K by n_+ positive crossing changes and n_- negative ones, and let $\varphi: \Lambda \rightarrow R$ be an admissible homomorphism. If $\varphi(\Delta_J(t)\Delta_K(t)) \neq 0$ and if $\varphi(\Delta_J(t)) \cdot R + \varphi(\Delta_K(t)) \cdot R = R$, then there exists a smooth φ -tame cobordism W from M_J to M_K satisfying the following:*

- (1) $G(W; R_0)$ is a free R_0 -module of rank $n_+ + n_-$.
- (2) The intersection pairing on $G(W; R_0)$ is represented by the diagonal matrix with n_+ entries of 1 and n_- ones of -1 .

Proof. We first construct a smooth cobordism W from M_J to M_K . We write $s = n_+ + n_-$. We recall that a positive (resp. negative) crossing change of a knot is realized by performing 1-surgery (resp. (-1) -surgery) along the boundary of an embedded disk in S^3 intersecting the knot in precisely two points with opposite orientations. Such embedded disks in S^3 corresponding to crossing changes can be picked to be disjoint. Thus there exist simple closed curves c_1, \dots, c_{n_+} with 1-framings and c_{n_++1}, \dots, c_s with (-1) -framings in X_J satisfying the following:

- (1) c_1, \dots, c_s form an unlink in S^3 .
- (2) The linking number $\text{lk}(c_i, J)$ of c_i and J is 0 for each i .
- (3) The manifold obtained from X_J by Dehn surgery along c_1, \dots, c_s is diffeomorphic to X_K .

Viewing c_1, \dots, c_s also as lying in M_J , we define W to be the result of adding s 2-handles to $M_J \times [0, 1]$ along $c_1 \times \{1\}, \dots, c_s \times \{1\}$. It follows from the construction of W that ∂W is a disjoint union of $-M_J$ and M_K . Note that we can think of the cobordism $-W$ from $-M_K$ to M_J also as the result of adding s 2-handles to $M_K \times [0, 1]$ along such simple closed curves in $M_K \times \{1\}$.

We next show that W is a φ -tame cobordism from M_J to M_K . Since the curves c_1, \dots, c_s are nullhomologous in M_J , the inclusion induced homomorphism $H_1(M_J; \mathbb{Z}) \rightarrow H_1(W; \mathbb{Z})$ is an isomorphism, and, similarly, so is the other one $H_1(M_K; \mathbb{Z}) \rightarrow H_1(W; \mathbb{Z})$. Since the images in $H_1(W; \mathbb{Z})$ of the meridional elements of $H_1(M_J; \mathbb{Z})$ and $H_1(M_K; \mathbb{Z})$ coincide, W is equipped with an isomorphism $H_1(W; \mathbb{Z}) \rightarrow \mathbb{Z}$ such that the following diagram commutes:

$$\begin{array}{ccccc} H_1(M_J; \mathbb{Z}) & \longrightarrow & H_1(W; \mathbb{Z}) & \longleftarrow & H_1(M_K; \mathbb{Z}) \\ & & \downarrow & & \\ & & \mathbb{Z} & & \end{array}$$

It follows from the Mayer-Vietoris homology exact sequence that the inclusion induced homomorphism $H_1^\varphi(M_J; R) \rightarrow H_1^\varphi(W; R)$ is surjective, and, similarly, so is

the other one $H_1^\varphi(M_K; R) \rightarrow H_1^\varphi(W; R)$. Since $\varphi(\Delta_J(t))$ and $\varphi(\Delta_K(t))$ annihilate $H_1^\varphi(M_J; R)$ and $H_1^\varphi(M_K; R)$ respectively, they also do $H_1^\varphi(W; R)$. The assumption that $\varphi(\Delta_J(t))R$ and $\varphi(\Delta_K(t))R$ are coprime implies that there exist $a, b \in R$ such that $1 = a\varphi(\Delta_J(t)) + b\varphi(\Delta_K(t))$, which thus annihilates $H_1^\varphi(W; R)$. Hence we have $H_1^\varphi(W; R) = 0$.

Finally, we show that W satisfies (1) and (2) as in the statement. Recall that c_1, \dots, c_s form an unlink in S^3 , and $\text{lk}(c_i, J) = 0$ for each i . In particular, c_1, \dots, c_s are nullhomologous in M_J . Now we can find disjoint subsurfaces F'_1, \dots, F'_s in $M_J \times [0, 1]$ with $\partial F'_i = c_i \times \{1\}$ for each i . For example, such a subsurface F'_i can be constructed in $M_J \times (1 - \frac{i}{s}, 1]$ for each i as the union of $c_i \times [1 - \frac{i-1}{s}, 1]$, a punctured disk in $M_J \times \{1 - \frac{i-1}{s}\}$ whose boundary is $c_i \times \{1 - \frac{i-1}{s}\}$, and tubes in $M_J \times (1 - \frac{i}{s}, 1 - \frac{i-1}{s}]$ along arcs in J . We write F_i for the union of F'_i and the core of the 2-handle attached to c_i for each i . It follows from the Mayer-Vietoris homology exact sequence that (1) is satisfied, and the images of the integral homology classes of the closed subsurfaces F_1, \dots, F_s under the composition of maps $H_2(W; \mathbb{Z}) \rightarrow H_2(W; R_0) \rightarrow G(W; R_0)$ form a basis of $G(W; R_0)$. It is straightforward to see that

$$F_i \cdot F_j = 0 \text{ for } i \neq j, \quad F_i \cdot F_i = \begin{cases} 1 & \text{for } i = 1, \dots, n_+, \\ -1 & \text{for } i = n_+ + 1, \dots, s, \end{cases}$$

and (2) follows from the following commutative diagram of the intersection pairings of W :

$$\begin{array}{ccc} H_2(W; \mathbb{Z}) \times H_2(W; \mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ H_2(W; R_0) \times H_2(W; R_0) & \longrightarrow & R_0. \end{array}$$

□

5. APPLICATIONS AND EXAMPLES

Applying Theorem 4.1 and Corollary 1.5, we formulate obstructions for $d(J, K) \leq 2$ for knots J and K in S^3 , and describe our computational results obtained for all pairs of nontrivial knots (possibly non-prime) with up to 10 crossings. The full details of the results are available on the website [FKS].

5.1. Obstructions for $d(J, K) \leq 2$. First as an application of Theorem 4.1 we present obstructions for $d(J, K) = 1$ for knots J and K in S^3 .

The following theorem, which generalizes the unknotting number one obstruction by Murakami [Muk90], Fogel [F93] and John Rickard, is easily deduced from Theorem 4.1.

Proposition 5.1. *Let J and K be knots in S^3 such that J can be turned into K by a single crossing change, where we set $\epsilon = 1$ if the crossing change is positive and $\epsilon = -1$ otherwise. We pick representatives $\Delta_J(t), \Delta_K(t) \in \Lambda$ of the Alexander polynomials so that $\Delta_J(1) = \Delta_K(1) = 1$. Let $\varphi: \Lambda \rightarrow R$ be an admissible homomorphism. If $\varphi(\Delta_J(t)\Delta_K(t)) \neq 0$ and if $\varphi(\Delta_J(t)) \cdot R + \varphi(\Delta_K(t)) \cdot R = R$, then there exists a generator g of $H_1^\varphi(X_{-J\sharp K}; R)$ such that*

$$\text{Bl}_{-J\sharp K, \varphi}(g, g) = \frac{\epsilon}{\varphi(\Delta_J(t)\Delta_K(t))} \in Q(R)/R.$$

By considering the admissible homomorphism $\varphi_{-1}: \Lambda \rightarrow \mathbb{Z}$ sending t to -1 , we have the following corollary, which generalizes the unknotting number one obstruction by Lickorish [Lic85].

Corollary 5.2. *Let J and K be knots in S^3 such that J can be turned into K by a single crossing change, where we set $\epsilon = 1$ if the crossing change is positive and $\epsilon = -1$ otherwise. If $\det(J)$ and $\det(K)$ are coprime, then there exists a generator g of $H_1(\Sigma(-J\sharp K); \mathbb{Z})$ such that*

$$\text{lk}_{-J\sharp K, \varphi}(g, g) = \frac{2\epsilon}{\det(J)\det(K)} \in \mathbb{Z}/\det(J)\det(K)\mathbb{Z}.$$

Proof. It follows from [BF15, Lemma 3.3] that $\text{Bl}_{J, \varphi_{-1}}$ is isometric to 2lk_J . Note also that $\varphi_{-1}(\Delta_J(t)) = \Delta_J(-1) = \pm \det(J)$. It thus follows from Proposition 5.1 that there exists a generator h of $H_1(\Sigma(-J\sharp K); \mathbb{Z})$ such that

$$2\text{lk}_{-J\sharp K, \varphi}(h, h) = \frac{\epsilon}{\det(J)\det(K)} \in \mathbb{Z}/\det(J)\det(K)\mathbb{Z}.$$

We set $g = 2h$. Since $\det(J)\det(K)$ is an odd number, g is also a generator of $H_1(\Sigma(-J\sharp K); \mathbb{Z})$, and it has the required properties. \square

Next as an application of Corollary 1.5 we formulate an obstruction for $d(J, K) = 2$ for knots J and K in S^3 .

Before stating the obstruction we recall a classification of symmetric integral 2×2 -matrices up to congruence. The full classification has been already known to Gauß. The following is a slightly weaker result. We refer to Conway and Sloane [CS99, Section 15.3] for an excellent exposition and also to [BF15, Lemma 5.2] for a proof.

Lemma 5.3. *Let C be a nonsingular symmetric integral 2×2 -matrix. Then, either C is congruent to a matrix of the form*

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

such that

- (1) $0 < |a| \leq |b| \leq |\det(C)|$,
- (2) $c \in \{0, \dots, \lfloor \frac{|a|}{2} \rfloor\}$,

or C is congruent to a matrix of the form

$$\begin{pmatrix} a & c \\ c & 0 \end{pmatrix}$$

with $c^2 = \det(C)$, $c \geq 0$ and $a \in \{-c, \dots, c\}$.

For a nonzero integer d we denote by \mathcal{C}_d the set of matrices which are of such forms as in Lemma 5.3 and which are congruent to the identity matrix modulo 2. Now Corollary 1.5 and Lemma 5.3 immediately imply the following proposition. See [BF15, Proposition 5.3] for a corresponding proposition in the case of $\Delta_J(t) = 1$.

Proposition 5.4. *Let J and K be knots in S^3 with $d(J, K) \leq 2$. If $\det(J)$ and $\det(K)$ are coprime, then there exists $C \in \mathcal{C}_d \cup \mathcal{C}_{-d}$, where $d = \det(J)\det(K)$, such that $\lambda(C)$ is isometric to $2\text{lk}_{-J\sharp K}$.*

The following lemma gives an elementary way to check whether 2lk_J is isometric to $\lambda(C)$ for a given integral 2×2 -matrix C .

Lemma 5.5 ([BF15, Lemma 5.4]). *Let J and K be knots in S^3 and C a symmetric integral 2×2 -matrix with $\det(C) = \pm \det(J) \det(K)$. Then $\lambda(C)$ is isometric to $2\text{lk}_{-J\sharp K}$ if and only if there exist generators v_1, v_2 of $H_1(\Sigma(-J\sharp K))$ such that*

$$2\text{lk}_{-J\sharp K}(v_1, v_2) = (i, j)\text{-entry of } C^{-1} \in \mathbb{Q}/\mathbb{Z}$$

for any $i, j \in \{1, 2\}$.

Similarly, we can extend our approach using the linking pairing to give an obstruction for $d(J, K) = m$ for an arbitrary positive integer m . See [BF15, Section 5.3] for the obstruction for lk_J to be isometric to $\lambda(C)$ for a positive-definite symmetric $m \times m$ -matrix C .

Note that our approach using the linking pairing does not see orientations of knots. First, it is well-known that the linking pairing is additive under the connected sum of knots: $\text{lk}_{J\sharp K} = \text{lk}_J \oplus \text{lk}_K$. Second, the linking pairing is invariant under the change of the orientation of a knot: $\text{lk}_{rJ} = \text{lk}_J$, where we write rJ for a knot J with opposite orientation. This follows from the facts that lk_J is an invariant of the double branched cover $\Sigma(J)$ of S^3 along J as an oriented 3-manifold and that $\Sigma(J)$ and $\Sigma(rJ)$ are isomorphic as oriented 3-manifolds. Therefore $\text{lk}_{-J\sharp K}$, $\text{lk}_{-J\sharp rK}$, $\text{lk}_{-rJ\sharp K}$ and $\text{lk}_{-rJ\sharp rK}$ are all isometric, and consequently the obstructions by Corollary 5.2 and Proposition 5.4 provide the same lower bounds of $d(J, K)$, $d(J, rK)$, $d(rJ, K)$ and $d(rJ, rK)$.

5.2. Examples. We have written a computer program which given Seifert surfaces of knots J and K calculates the linking pairing $\text{lk}_{-J\sharp K}$ and determines whether the obstructions by Corollary 5.2 and Proposition 5.4 show that $d(J, K) \geq 2$ and $d(J, K) \geq 3$ respectively. It is well-known that the linking pairing lk_J of J is isometric to $\lambda(A+A^T)$ for a Seifert matrix A of J . We have computed the results for all pairs of nontrivial knots (possibly non-prime) together with their mirror images with up to 10 crossings, using the database of Seifert surfaces in Knotinfo [LM], and compared the results with the following lower bounds. The full details of the results are available on the website [FKS].

- (1) The signature $\sigma(J)$ gives the lower bound [Muk85, Mus65]:

$$d(J, K) \geq \frac{1}{2} |\sigma(J) - \sigma(K)|$$

- (2) The Rasmussen s -invariant $s(J)$ gives the lower bound [Ra10]:

$$d(J, K) \geq \frac{1}{2} |s(J) - s(K)|$$

- (3) The Ozsváth-Szabó τ -invariant $\tau(J)$ gives the lower bound [OS03]:

$$d(J, K) \geq |\tau(J) - \tau(K)|$$

- (4) For every odd prime p the rank of $H_1(\Sigma(J); \mathbb{F}_p)$ over a finite field \mathbb{F}_p gives the lower bound (see Corollary 5.12):

$$d(J, K) \geq |\text{rank } H_1(\Sigma(J); \mathbb{F}_p) - \text{rank } H_1(\Sigma(K); \mathbb{F}_p)|.$$

- (5) Torisu [To98] showed a necessary and sufficient condition for pairs of 2-bridge knots of $d(J, K) = 1$ in terms of their 2-bridge notations.

We denote by $g_4^{\text{smooth}}(J)$ the smooth 4-ball genus of J . The lower bounds (1), (2) and (3) follow from the inequality $d(J, K) \geq g_4^{\text{smooth}}(-J\sharp K)$, the lower bounds of $g_4^{\text{smooth}}(J)$ by the invariants and the additivity of them. Note that $-\sigma(J) =$

$s(J) = 2\tau(J)$ for an alternating knot J . The s -invariant and the τ -invariant have been computed for all prime knots with up to 12 crossings (see [LM] and [BG12]). The lower bound (4) is well-known, and see Section 5.3 for a proof.

An upper bound of the Gordian distance is given by the triangle inequality for a knot I :

$$d(J, K) \leq d(I, J) + d(I, K).$$

In particular, when I is the unknot, we have the following upper bound:

$$d(J, K) \leq u(J) + u(K).$$

The unknotting number has been computed for all prime knots with up to 10 crossings, except for 10_{11} , 10_{47} , 10_{51} , 10_{54} , 10_{61} , 10_{76} , 10_{77} , 10_{79} and 10_{100} . (It is known that $u(K) = 2$ or 3 for the above excepted knots K .) Literature on the unknotting number is extensive. See [LM] and the references given there. We do not have useful upper bounds on $d(J, K)$ using any other numerical invariants of knots.

As discussed in Section 5.1 the obstructions by Corollary 5.2 and Proposition 5.4 provide the same lower bounds of $d(J, K)$, $d(J, rK)$, $d(rJ, K)$ and $d(rJ, rK)$. Also, by definition, $d(J, K) = d(K, J) = d(mJ, mK) = d(mK, mJ)$, where we write mJ for the mirror image of a knot J . Therefore in describing numbers of pairs in our computational results we do not distinguish the pairs (J, K) , (rJ, K) , (J, rK) and (rJ, rK) , and neither the pairs (J, K) , (K, J) , (mJ, mK) and (mK, mJ) .

Applying Corollary 5.2, we have the following results. Among all pairs (J, K) of nontrivial prime knots with up to 10 crossings such that at least one of J and K is not a 2-bridge knot there are 10272 pairs for which Corollary 5.2 shows that $d(J, K) \geq 2$ but the value on the right hand side in any of the 4 lower bounds (1), (2), (3) and (4) is 0 or 1. For 1853 pairs (J, K) of knots among the 10272 ones, it is known that $u(J) + u(K) \leq 2$, and hence we can conclude that $d(J, K) = u(J) + u(K) = 2$. Also, if we count the numbers of such pairs of knots, instead, for all pairs (J, K) of nontrivial knots (possibly non-prime) with up to 10 crossings such that at least one of J and K is not a 2-bridge knot, then the first one 10272 is replaced by 12577, and the second one 1853 is unchanged.

Example 5.6. Let $(J, K) = (8_{17}, 8_{21})$, where the bridge indices of 8_{17} and 8_{21} are equal to 3, and $\det(8_{17}) = 37$ and $\det(8_{21}) = 15$. A calculation of $\text{lk}_{-8_{17}\#8_{21}}$ from their Seifert surfaces shows that there exists no generator g of $H_1(\Sigma(-8_{17}\#8_{21}))$ such that

$$\text{lk}_{-8_{17}\#8_{21}}(g, g) = \pm \frac{2}{37 \cdot 15} \in \mathbb{Z}/(37 \cdot 15)\mathbb{Z}.$$

It thus follows from Corollary 5.2 that $d(8_{17}, 8_{21}) \geq 2$. On the other hand, since $-\sigma(8_{17}) = s(8_{17}) = 2\tau(8_{17}) = 0$ and $-\sigma(8_{21}) = s(8_{21}) = 2\tau(8_{21}) = 2$, we have $\frac{1}{2}|\sigma(8_{17}) - \sigma(8_{21})| = \frac{1}{2}|s(8_{17}) - s(8_{21})| = |\tau(8_{17}) - \tau(8_{21})| = 1$. Also, since $\text{rank } H_1(\Sigma(8_{17}); \mathbb{F}_p)$ is equal to 1 for $p = 37$ and to 0 for other odd primes p , and since $\text{rank } H_1(\Sigma(8_{21}); \mathbb{F}_p)$ is equal to 1 for $p = 3, 5$ and to 0 for other p , it follows that $|\text{rank } H_1(\Sigma(8_{17}); \mathbb{F}_p) - \text{rank } H_1(\Sigma(8_{21}); \mathbb{F}_p)|$ is equal to 1 for $p = 3, 5, 37$ and to 0 for other p . Thus any of the lower bounds (1), (2), (3) and (4) only gives $d(8_{17}, 8_{21}) \geq 1$. Moreover, since it is known that $u(8_{17}) = u(8_{21}) = 1$, we have $d(8_{17}, 8_{21}) \leq u(8_{17}) + u(8_{21}) = 2$. Therefore we can conclude that $d(8_{17}, 8_{21}) = 2$.

Applying Proposition 5.4, we have the following results. Among all pairs (J, K) of nontrivial prime knots with up to 10 crossings there are 886 pairs for which

Proposition 5.4 shows that $d(J, K) \geq 3$ but the value on the right hand side in any of the 4 lower bounds (1), (2), (3) and (4) is less than 3. For 195 pairs (J, K) of knots among the 886 ones, it is known that $u(J) + u(K) \leq 3$, and hence we can conclude that $d(J, K) = u(J) + u(K) = 3$. Also, if we count the numbers of such pairs of knots, instead, for all pairs (J, K) of nontrivial knots (possibly non-prime) with up to 10 crossings, then the first one 886 is replaced by 1696, and the second one 195 is by 360.

Example 5.7. Let $(J, K) = (8_7, 9_{40})$, where $\det(8_7) = 23$ and $\det(9_{40}) = 75$. A calculation of $\text{lk}_{-8_7\#9_{40}}$ from their Seifert surfaces show that there exists no matrix $C \in \mathcal{C}_d \cup \mathcal{C}_{-d}$, where $d = 23 \cdot 75$, such that $\lambda(C)$ is isometric to $2\text{lk}_{-8_7\#9_{40}}$. It thus follows from Proposition 5.4 that $d(8_7, 9_{40}) \geq 3$. On the other hand, since $-\sigma(8_7) = s(8_7) = 2\tau(8_7) = -2$ and $-\sigma(9_{40}) = s(9_{40}) = 2\tau(9_{40}) = 2$, we have $\frac{1}{2}|\sigma(5_2) - \sigma(9_{40})| = \frac{1}{2}|s(5_2) - s(9_{40})| = |\tau(5_2) - \tau(9_{40})| = 2$. Note that both of 8_7 and 9_{40} are alternating knots. Also, since $\text{rank } H_1(\Sigma(8_7); \mathbb{F}_p)$ is equal to 1 for $p = 23$ and to 0 for other odd primes p , and since $\text{rank } H_1(\Sigma(9_{40}); \mathbb{F}_p)$ is equal to 2 for $p = 5$, to 1 for $p = 3$ and to 0 for other p , it follows that $|\text{rank } H_1(\Sigma(8_7); \mathbb{F}_p) - \text{rank } H_1(\Sigma(9_{40}); \mathbb{F}_p)|$ is equal to 2 for $p = 5$, to 1 for $p = 3, 23$ and to 0 for other p . Thus any of the lower bounds (1), (2), (3) and (4) only gives $d(8_7, 9_{40}) \geq 2$. Moreover, since it is known that $u(8_7) = 1$ and $u(9_{40}) = 2$, we have $d(8_7, 9_{40}) \leq u(8_7) + u(9_{40}) = 3$. Therefore we can conclude that $d(8_7, 9_{40}) = 3$.

Example 5.8. Let $(J, K) = (3_1, 4_1\#4_1)$, where $\det(3_1) = 3$ and $\det(4_1\#4_1) = 25$. Similarly, a calculation of $\text{lk}_{-3_1\#4_1\#4_1}$ shows that we can apply Proposition 5.4 to have $d(3_1, 4_1\#4_1) \geq 3$. Since $-\sigma(3_1) = s(3_1) = 2\tau(3_1) = -2$ and $-\sigma(4_1\#4_1) = s(4_1\#4_1) = 2\tau(4_1\#4_1) = 0$, we have $\frac{1}{2}|\sigma(3_1) - \sigma(4_1\#4_1)| = \frac{1}{2}|s(3_1) - s(4_1\#4_1)| = |\tau(3_1) - \tau(4_1\#4_1)| = 1$. Note that both of 3_1 and $4_1\#4_1$ are alternating knots, and that $4_1\#4_1$ is a slice knot. Also, since $\text{rank } H_1(\Sigma(3_1); \mathbb{F}_p)$ is equal to 1 for $p = 3$ and to 0 for other odd primes p , and since $\text{rank } H_1(\Sigma(4_1\#4_1); \mathbb{F}_p)$ is equal to 2 for $p = 5$ and to 0 for other p , it follows that $|\text{rank } H_1(\Sigma(3_1); \mathbb{F}_p) - \text{rank } H_1(\Sigma(4_1\#4_1); \mathbb{F}_p)|$ is equal to 2 for $p = 5$, to 1 for $p = 3$ and to 0 for other p . Thus any of the lower bounds (1), (2) and (3) only gives $d(3_1, 4_1\#4_1) \geq 1$, and the lower bound (4) does $d(3_1, 4_1\#4_1) \geq 2$. Moreover, since it is known that $u(3_1) = u(4_1) = 1$, we have $d(3_1, 4_1\#4_1) \leq u(3_1) + u(4_1\#4_1) \leq u(3_1) + 2u(4_1) = 3$. Therefore we can conclude that $d(3_1, 4_1\#4_1) = 3$.

- Remark 5.9.*
- (1) Kawauchi [K12], Murakami [Muk85] and Nakanishi [N81] gave obstructions for $d(J, K) = 1$ in terms of the Alexander polynomial.
 - (2) Darcy and Summers [DS97, DS98], Motegi [Mot96] and Torisu [To98] gave obstructions for $d(J, K) = 1$ applicable for many pairs of 2-bridge knots or Montesinos knots.
 - (3) Darcy [D] and Moon [Moo10] gave tables of lower and upper bounds of the Gordian distance of many pairs of knots with 10 crossings or less. For example, their tables describe $1 \leq d(8_{17}, 8_{21}) \leq 2$ and $2 \leq d(3_1, 4_1\#4_1) \leq 3$, but our lower bound determines that $d(8_{17}, 8_{21}) = 2$ and $d(3_1, 4_1\#4_1) = 3$ as in Examples 5.6 and 5.8.
 - (4) Miyazawa [Mi11] showed lower bounds of the Gordian distance using the Jones, HOMFLY and Q -polynomials of knots and gave lists of the Gordian distance or its lower bounds for many pairs of knots with 10 crossings or less.

For example, Miyazawa [Mi11, Corollary 3.5] showed that $d(8_7, 9_{40}) \geq 2$, but our lower bound determines that $d(8_7, 9_{40}) = 3$ as in Example 5.7.

5.3. Minimal number of generators of the Alexander modules. For readers' convenience we prove the following well-known lower bounds of the Gordian distance. Wendt [W37] first gave lower bounds by the ranks of the first homology groups of the cyclic branched covers. For a knot J and a homomorphism $\varphi: \Lambda \rightarrow R$, which is not necessarily admissible, we denote by $m_\varphi(J)$ the minimal number of generators of $H_1^\varphi(X_J; R)$.

Proposition 5.10. *Let J and K be knots in S^3 , and let $\varphi: \Lambda \rightarrow R$ be a homomorphism over a principal ideal domain. Then $d(J, K) \geq |m_\varphi(J) - m_\varphi(K)|$.*

This proposition can be easily deduced from the following lemma.

Lemma 5.11. *Let J and K be knots in S^3 such that J can be turned into K by a single crossing change, and let $\varphi: \Lambda \rightarrow R$ be an admissible homomorphism over a principal ideal domain. Then $|m_\varphi(J) - m_\varphi(K)|$ is equal to 0 or 1.*

Proof. Let c be a simple closed curve in X_J corresponding to a crossing change turning J into K as in the proof of Proposition 4.2. Note that c is nullhomologous. Let Z be an open tubular neighborhood of c in X_J , and we denote by Y the complement of Z . Then we consider the Mayer-Vietoris homology exact sequence:

$$H_1(\partial Z; R) \rightarrow H_1^\varphi(Y; R) \oplus H_1(Z; R) \rightarrow H_1^\varphi(X_J; R) \rightarrow 0,$$

where $H_1(Z; R)$ and $H_1(\partial Z; R)$ are free R -modules of rank 1 and 2 respectively, and the inclusion induced homomorphism $H_1(\partial Z; R) \rightarrow H_1(Z; R)$ is surjective. Now by a standard argument on finitely generated modules over a principal ideal domain we have $m - 1 \leq m_\varphi(J) \leq m$, where m is the minimal number of generators of $H_1^\varphi(Y; R)$. Similarly, we also have $m - 1 \leq m_\varphi(K) \leq m$. Therefore we have $|m_\varphi(J) - m_\varphi(K)| \leq 1$. \square

In the case of the admissible homomorphism $\varphi_{-1}: \Lambda \rightarrow \mathbb{F}_p$ sending t to -1 for an odd prime p , the twisted homology group $H_1^{\varphi_{-1}}(X_J; \mathbb{F}_p)$ is isomorphic to $H_1(\Sigma(J); \mathbb{F}_p)$. (See for instance [BF15, Lemma 3.3].) Thus we have the following corollary.

Corollary 5.12. *Let J and K be knots in S^3 , and let p be an odd prime. Then*

$$d(J, K) \geq |\text{rank } H_1(\Sigma(J); \mathbb{F}_p) - \text{rank } H_1(\Sigma(K); \mathbb{F}_p)|.$$

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