# On $PGL_3(\mathbb{C})$-torsions

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In this note, we explain how to obtain a $PGL_3(\mathbb{C})$-torsion for a mapping torus of a surface with punctures, using a concrete description of the action of a mapping class on Fock-Goncharov parameters.

Step 1. First, we fix an ideal triangulation of a surface S with punctures. In the case of a torus with one puncture, we choose the following triangulation:

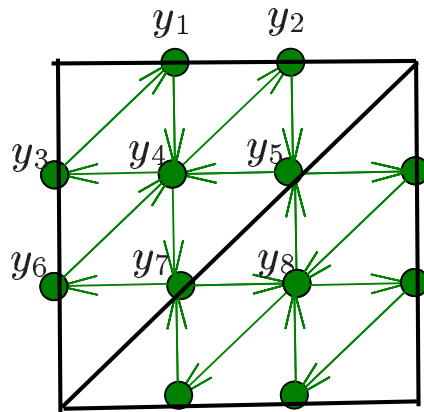torus with one puncture   triangulation
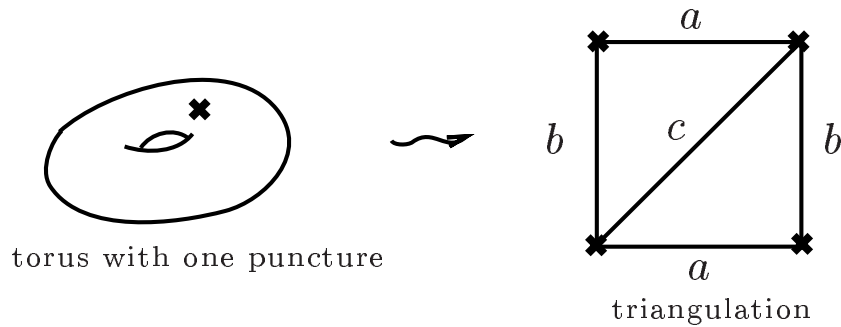
Step 2. For each triangle in the ideal triangulation, we put the following quiver:
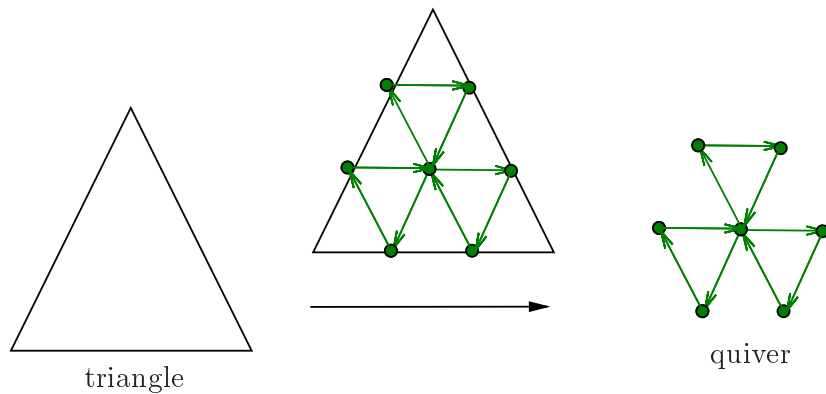
triangle   quiver

In the case of the triangulation of a torus with one puncture in Step 1, we have the following quiver:
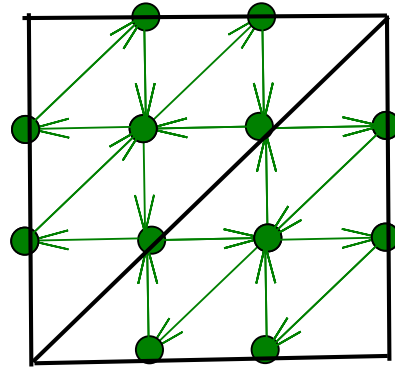
Step 3. For each vertex v in the quiver, we put a variable y_v. These variable $\{y_v\}$ are called Fock-Goncharov parameters [FG]. In the case of the quiver for a torus with one puncture in Step 2, we have variables $y_1, y_2, \ldots, y_8$:

Step 4. For a Dehn twist φ on a surface S, we write down the action $\varphi^{\#}$ on Fock-Goncharov parameters. This step can be done concretely according to the appendex in [TY] (see also [NTY]). In the case of the Dehn twists L, R on a torus with one puncture, we have the following

action:

$$L^{\#}(y_1) = \frac{y_1 y_3^2 (1+y_6) y_8}{(1+y_3)(1+y_3+y_3 y_8+y_3 y_6 y_8)}$$

$$L^{\#}(y_2) = \frac{y_2(1+y_3)y_4 y_6^2}{(1+y_6)(1+y_6+y_4 y_6+y_3 y_4 y_6)}$$

$$L^{\#}(y_3) = \frac{(1+y_3)y_5(1+y_6+y_4 y_6+y_3 y_4 y_6)}{1+y_6}$$

$$L^{\#}(y_4) = \frac{(1+y_6+y_4 y_6+y_3 y_4 y_6)y_8}{1+y_3+y_3 y_8+y_3 y_6 y_8}$$

$$L^{\#}(y_5) = \frac{1+y_6}{(1+y_3)y_4 y_6}$$

$$L^{\#}(y_6) = \frac{(1+y_6)y_7(1+y_3+y_3 y_8+y_3 y_6 y_8)}{1+y_3}$$

$$L^{\#}(y_7) = \frac{1+y_3}{y_3(1+y_6)y_8}$$

$$L^{\#}(y_8) = \frac{y_4(1+y_3+y_3 y_8+y_3 y_6 y_8)}{1+y_6+y_4 y_6+y_3 y_4 y_6}$$

$$R^{\#}(y_1) = \frac{y_1^2(1+y_2)y_4 y_7}{(1+y_1)(1+y_1+y_1 y_4+y_1 y_2 y_4)}$$

$$R^{\#}(y_2) = \frac{(1+y_1)y_2^2 y_5 y_8}{(1+y_2)(1+y_2+y_2 y_8+y_1 y_2 y_8)}$$

$$R^{\#}(y_3) = \frac{(1+y_1)y_3(1+y_2+y_2 y_8+y_1 y_2 y_8)}{1+y_2}$$

$$R^{\#}(y_4) = \frac{y_4(1+y_2+y_2 y_8+y_1 y_2 y_8)}{1+y_1+y_1 y_4+y_1 y_2 y_4}$$

$$R^{\#}(y_5) = \frac{1+y_2}{(1+y_1)y_2 y_8}$$

$$R^{\#}(y_6) = \frac{(1+y_2)(1+y_1+y_1 y_4+y_1 y_2 y_4)y_6}{1+y_1}$$

$$R^{\#}(y_7) = \frac{1+y_1}{y_1(1+y_2)y_4}$$

$$R^{\#}(y_8) = \frac{(1+y_1+y_1 y_4+y_1 y_2 y_4)y_8}{1+y_2+y_2 y_8+y_1 y_2 y_8}$$

Step 5. For a mapping class φ of a surface S, we consider the equation of fixed points:

$$y_i = \varphi^{\#}(y_i)$$

This equation should be the equation of a "geometric part" of the moduli space of $PGL(3;\mathbb{C})$-representations of the fundamental group of the mapping torus M_φ with φ. In the case of the

mapping class *LR* of a torus with one puncture, we have the following equation:

$$y_1 = (y_1^2 y_3^3 (1 + 2y_6 + y_4 y_6 + y_3 y_4 y_6 + y_6^2 + y_4 y_6^2 + y_2 y_4 y_6^2 + y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2) y_8^2) /$$
$$((1 + 2y_3 + y_3^2 + y_3 y_8 + y_3^2 y_8 + y_1 y_3^2 y_8 + y_3 y_6 y_8 + y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8)$$
$$(1 + 2y_3 + y_3^2 + 2y_3 y_8 + 2y_3^2 y_8 + y_1 y_3^2 y_8 + 2y_3 y_6 y_8 + 2y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8 +$$
$$y_3^2 y_8^2 + y_1 y_3^2 y_8^2 + 2y_3^2 y_6 y_8^2 + 2y_1 y_3^2 y_6 y_8^2 + y_1 y_3^2 y_4 y_6 y_8^2 +$$
$$y_3^2 y_6^2 y_8^2 + y_1 y_3^2 y_6^2 y_8^2 + y_1 y_3^2 y_4 y_6^2 y_8^2 + y_1 y_2 y_3^2 y_4 y_6^2 y_8^2))$$

$$y_2 = (y_2^2 y_4^2 y_6^3 (1 + 2y_3 + y_3^2 + y_3 y_8 + y_3^2 y_8 + y_1 y_3^2 y_8 + y_3 y_6 y_8 + y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8)) /$$
$$((1 + 2y_6 + y_4 y_6 + y_3 y_4 y_6 + y_6^2 + y_4 y_6^2 + y_2 y_4 y_6^2 + y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2)$$
$$(1 + 2y_6 + 2y_4 y_6 + 2y_3 y_4 y_6 + y_6^2 + 2y_4 y_6^2 + y_2 y_4 y_6^2 + 2y_3 y_4 y_6^2 +$$
$$y_2 y_3 y_4 y_6^2 + y_4^2 y_6^2 + y_2 y_4^2 y_6^2 + 2y_3 y_4^2 y_6^2 + 2y_2 y_3 y_4^2 y_6^2 + y_3^2 y_4^2 y_6^2 +$$
$$y_2 y_3^2 y_4^2 y_6^2 + y_2 y_3 y_4^2 y_6^2 y_8 + y_2 y_3^2 y_4^2 y_6^2 y_8 + y_1 y_2 y_3^2 y_4^2 y_6^2 y_8))$$

$$y_3 = (y_5 (1 + 2y_3 + y_3^2 + y_3 y_8 + y_3^2 y_8 + y_1 y_3^2 y_8 + y_3 y_6 y_8 + y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8)$$
$$(1 + 2y_6 + 2y_4 y_6 + 2y_3 y_4 y_6 + y_6^2 + 2y_4 y_6^2 + y_2 y_4 y_6^2 + 2y_3 y_4 y_6^2 +$$
$$y_2 y_3 y_4 y_6^2 + y_4^2 y_6^2 + y_2 y_4^2 y_6^2 + 2y_3 y_4^2 y_6^2 + 2y_2 y_3 y_4^2 y_6^2 + y_3^2 y_4^2 y_6^2 +$$
$$y_2 y_3^2 y_4^2 y_6^2 + y_2 y_3 y_4^2 y_6^2 y_8 + y_2 y_3^2 y_4^2 y_6^2 y_8 + y_1 y_2 y_3^2 y_4^2 y_6^2 y_8)) /$$
$$((1 + 2y_6 + y_4 y_6 + y_3 y_4 y_6 + y_6^2 + y_4 y_6^2 + y_2 y_4 y_6^2 + y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2)$$
$$(1 + y_3 + y_3 y_8 + y_3 y_6 y_8))$$

$$y_4 = (y_8 (1 + y_3 + y_3 y_8 + y_3 y_6 y_8)(1 + 2y_6 + 2y_4 y_6 + 2y_3 y_4 y_6 + y_6^2 + 2y_4 y_6^2 + y_2 y_4 y_6^2 +$$
$$2y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2 + y_4^2 y_6^2 + y_2 y_4^2 y_6^2 + 2y_3 y_4^2 y_6^2 + 2y_2 y_3 y_4^2 y_6^2 +$$
$$y_3^2 y_4^2 y_6^2 + y_2 y_3^2 y_4^2 y_6^2 + y_2 y_3 y_4^2 y_6^2 y_8 + y_2 y_3^2 y_4^2 y_6^2 y_8 + y_1 y_2 y_3^2 y_4^2 y_6^2 y_8)) /$$
$$((1 + y_6 + y_4 y_6 + y_3 y_4 y_6)(1 + 2y_3 + y_3^2 + 2y_3 y_8 + 2y_3^2 y_8 + y_1 y_3^2 y_8 + 2y_3 y_6 y_8 +$$
$$2y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8 + y_3^2 y_8^2 + y_1 y_3^2 y_8^2 + 2y_3^2 y_6 y_8^2 + 2y_1 y_3^2 y_6 y_8^2 +$$
$$y_1 y_3^2 y_4 y_6 y_8^2 + y_3^2 y_6^2 y_8^2 + y_1 y_3^2 y_6^2 y_8^2 + y_1 y_3^2 y_4 y_6^2 y_8^2 + y_1 y_2 y_3^2 y_4 y_6^2 y_8^2))$$

$$y_5 = ((1 + y_6 + y_4 y_6 + y_3 y_4 y_6)(1 + 2y_6 + y_4 y_6 + y_3 y_4 y_6 + y_6^2 + y_4 y_6^2 +$$
$$y_2 y_4 y_6^2 + y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2)) /$$
$$(y_2 y_4^2 y_6^2 (1 + 2y_3 + y_3^2 + y_3 y_8 + y_3^2 y_8 + y_1 y_3^2 y_8 + y_3 y_6 y_8 + y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8))$$

$$y_6 = ((1 + 2y_6 + y_4 y_6 + y_3 y_4 y_6 + y_6^2 + y_4 y_6^2 + y_2 y_4 y_6^2 + y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2) y_7$$
$$(1 + 2y_3 + y_3^2 + 2y_3 y_8 + 2y_3^2 y_8 + y_1 y_3^2 y_8 + 2y_3 y_6 y_8 + 2y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8 +$$
$$y_3^2 y_8^2 + y_1 y_3^2 y_8^2 + 2y_3^2 y_6 y_8^2 + 2y_1 y_3^2 y_6 y_8^2 + y_1 y_3^2 y_4 y_6 y_8^2 + y_3^2 y_6^2 y_8^2 +$$
$$y_1 y_3^2 y_6^2 y_8^2 + y_1 y_3^2 y_4 y_6^2 y_8^2 + y_1 y_2 y_3^2 y_4 y_6^2 y_8^2)) / ((1 + y_6 + y_4 y_6 + y_3 y_4 y_6)$$
$$(1 + 2y_3 + y_3^2 + y_3 y_8 + y_3^2 y_8 + y_1 y_3^2 y_8 + y_3 y_6 y_8 + y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8))$$

$$y_7 = ((1 + y_3 + y_3 y_8 + y_3 y_6 y_8)(1 + 2y_3 + y_3^2 + y_3 y_8 + y_3^2 y_8 + y_1 y_3^2 y_8 +$$
$$y_3 y_6 y_8 + y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8)) /$$
$$(y_1 y_3^2 (1 + 2y_6 + y_4 y_6 + y_3 y_4 y_6 + y_6^2 + y_4 y_6^2 + y_2 y_4 y_6^2 + y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2) y_8^2)$$
$$y_8 = (y_4 (1 + y_6 + y_4 y_6 + y_3 y_4 y_6)(1 + 2y_3 + y_3^2 + 2y_3 y_8 + 2y_3^2 y_8 + y_1 y_3^2 y_8 + 2y_3 y_6 y_8 +$$
$$2y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8 + y_3^2 y_8^2 + y_1 y_3^2 y_8^2 + 2y_3^2 y_6 y_8^2 + 2y_1 y_3^2 y_6 y_8^2 +$$
$$y_1 y_3^2 y_4 y_6 y_8^2 + y_3^2 y_6^2 y_8^2 + y_1 y_3^2 y_6^2 y_8^2 + y_1 y_3^2 y_4 y_6^2 y_8^2 + y_1 y_2 y_3^2 y_4 y_6^2 y_8^2)) /$$
$$((1 + y_3 + y_3 y_8 + y_3 y_6 y_8)(1 + 2y_6 + 2y_4 y_6 + 2y_3 y_4 y_6 + y_6^2 + 2y_4 y_6^2 + y_2 y_4 y_6^2 +$$
$$2y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2 + y_4^2 y_6^2 + y_2 y_4^2 y_6^2 + 2y_3 y_4^2 y_6^2 + 2y_2 y_3 y_4^2 y_6^2 +$$
$$y_3^2 y_4^2 y_6^2 + y_2 y_3^2 y_4^2 y_6^2 + y_2 y_3 y_4^2 y_6^2 y_8 + y_2 y_3^2 y_4^2 y_6^2 y_8 + y_1 y_2 y_3^2 y_4^2 y_6^2 y_8))$$

Remark that the mapping torus in this example is the complement of the figure-eight knot.

Step 6. For the equation of fixed points, we choose a solution. We can show that, for a pesudo-Anosov mapping class φ, we have always a solution of the equation of fixed points [KT].

In the case of LR, we have a solution:

$$y_1 = 1$$
$$y_2 = 1$$
$$y_3 = \frac{1}{2}\left(-1 + \sqrt{-3}\right)$$
$$y_4 = 1$$
$$y_5 = \frac{1}{2}\left(-1 - \sqrt{-3}\right)$$
$$y_6 = \frac{1}{2}\left(-1 + \sqrt{-3}\right)$$
$$y_7 = \frac{1}{2}\left(-1 - \sqrt{-3}\right)$$
$$y_8 = 1$$

Step 7. For a mapping class φ of a surface S, we evaluate at a solution $y = y^{\text{sol}}$ the characteristic polynomial $\Delta(t)$ of the Jacobi matrix of $\varphi^{\#}$:

$$\Delta(t) = \det\left(t - \left(\frac{\partial \varphi^{\#}(y_i)}{\partial y_j}\right)\right)\bigg|_{y=y^{\text{sol}}}$$

In the case of the mapping class LR and the solution $y = y^{\text{sol}}$ in step 6, we have the following polynomial.

$$\Delta(t) = (-1 + t)^2 (1 - 5t + t^2)(1 - 9t + 44t^2 - 9t^3 + t^4)$$

Step 8. Last, we consider the following limit:

$$\tau := \lim_{t \to 1} \frac{\Delta(t)}{(t-1)^{2p}},$$

where p is the number of punctures on S. One of our results is that this number is identified with $PGL_3(\mathbb{C})$-torsion with the adjoint representation of the representation which corresponds the solution, except sign ambiguity [KT]. A $PGL_3(\mathbb{C})$-torsion is a generalization of a Porti's torsion for a $PSL_2(\mathbb{C})$-representation [P]. In the case of LR, we have:

$$\begin{aligned}
\tau &= \lim_{t \to 1} \frac{\Delta(t)}{(t-1)^2} \\
&= -3 \times 28 \\
&= -84
\end{aligned}$$

References

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