Group actions and metric embeddings

Date: September 8–11, 2015.

Place: Room 110, Graduate School of Science Bldg. No.3, Department of Mathematics, Kyoto University.

Lecture series

Damian Osajda (Wrocław)

Embedding infinite graphs into groups and applications

Abstract: The aim of this course is to explain the recent construction [Osa14] of finitely generated groups whose Cayley graphs contain isometrically some infinite sequences of finite graphs. For expanding sequences the resulting groups are not coarsely embeddable into Hilbert spaces and are counterex- amples to the Baum-Connes conjecture with coefficients. (Recall that a map $f\colon (X, d_X) \to (Y, d_Y)$ between metric spaces is a coarse embedding when $d_Y(f(x_n), f(y_n)) \to \infty$ iff $d_X(x_n, y_n) \to \infty$ for all sequences (x_n), (y_n).) The only other groups with those features are Gromov monsters [Gro03] — groups into which expanders embed weakly. For other families of graphs the technique allows to construct first examples of finitely generated groups with some exotic properties, e.g. a-T-menable groups without Yu's property A. The course will cover (hopefully, roughly, and up to some shifts) the following subjects:

Lectures 1 & 2. Formulation of the main goal, some motivations, and generalities on the approach. Basics of graphical small cancellation theory: definitions, examples, main properties (e.g. isometric embedding of relators). Some history, including a short discussion of Gromov's method [Gro03].

Lectures 3 & 4. A fairly detailed presentation of the proof of the main theorem from [Osa14] — embedding isometrically an infinite sequence of graphs into a finitely generated group. Applications to specific families of graphs. Finitely generated groups containing expanders. Further applications and remarks.

Lectures 5 & 6. Yu's property A and coarse embeddability into a Hilbert space. The construction of groups without property A acting properly on CAT(0) cubical complexes: Small cancellation labellings of graphs with walls, and defining a proper lacunary walling for such groups. Final remarks.

References

[Gro03] M. Gromov, *Random walk in random groups*, Geom. Funct. Anal. **13** (2003), no. 1, 73–146.

[Osa14] D. Osajda, *Small cancellation labellings of some infinite graphs and applications* (2014), preprint, available at `arXiv:1406.5015`.

Florent Baudier (Texas A&M)

Embeddability of metric spaces into Banach spaces and applications

Abstract: In this series of lectures we will present selected topics of an emerging and fast-developing field, often refer to as *quantitative metric geometry.* We will touch upon fundamental questions that originate from, or are related to, problems in theoretical computer science, group theory, topology, or theoretical physics. The common feature of all these problems is that they can be expressed in geometric terms. Building a geometric intuition will be central in our exposition. The series of lectures will be divided into three

relatively independent parts. In all three lectures we will be dealing with the general problem of "embedding faithfully" a certain type of metric space into a "nice" Banach space.

1. First Lecture

In the first lecture we will study the Lipschitz geometry of (infinite) locally finite metric spaces. In particular, we will describe the barycentric gluing technic introduced by the lecturer in [2] and explain its use in a striking theorem of M. Ostrovskii [12] which says that a locally finite metric space that is finitely representable in an infinite-dimensional Banach space is actually bi-Lipschitzly embeddable into it. The results presented in this lecture already have applications in geometric group theory regarding the non-equivariant compression of finitely generated groups [4].

2. Second Lecture

In the second lecture we will explain the philosophy of the Ribe Program (c.f. the survey of A. Naor [11]) which suggests that local properties of Banach spaces have purely metric characterizations. We will present various metric characterizations of the super-reflexivity property in terms of different families of graphs (binary trees [8], diamond graphs, Laakso graphs [9]). We will also explain a metric characterization of an asymptotic property, namely the (β)-renormability ([5], [7]), which can be seen as an asymptotic version of Bourgain's metric characterization of super-reflexivity.

3. Third Lecture

In the third lecture we will leave the realm of Lipschitz geometry and focus on the coarse and uniform geometries of metric spaces. We will explain why metric spaces whose balls are relatively compact are "almost bi-Lipschitzly" embeddable into "nearly super-reflexive" Banach spaces [6]. We will describe a deep result of N. Kalton and L. Randrianarivony [10] and its applications to the compression theory of metric spaces [3]. If time permits we shall scratch the surface of the equivariant embedding theory of some finitely generated groups such as the discrete Heisenberg group [1].

References

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[2] F. Baudier, *Metrical characterization of super-reflexivity and linear type of Banach spaces*, Arch. Math. **89** (2007), 419–429.

[3] F. Baudier, *Quantitative nonlinear embeddings into Lebesgue sequence spaces*, J. Topol. Anal., to appear, arXiv: 1210.0588 (2013), 32 pages.

[4] F. Baudier, *On the metric geometry of stable metric spaces, submitted for publication*, arXiv:1409.7738 (2014), 22 pages (2014).

[5] F. Baudier, N. J. Kalton, and G. Lancien, *A new metric invariant for Banach spaces*, Studia Math. **199** (2010), 73–94.

[6] F. Baudier and G. Lancien, *Tight embeddability of proper and stable metric spaces*, Anal. Geom. Metr. Spaces **3** (2015), 140–156.

[7] F. Baudier and S. Zhang, *(β)-distortion of some infinite graphs*, submitted for publication.

[8] J. Bourgain, *The metrical interpretation of superreflexivity in Banach spaces*, Israel J. Math. **56** (1986), 222–230.

[9] W. B. Johnson and G. Schechtman, *Diamond graphs and super-reflexivity*, J. Topol. Anal. **1** (2009), no. 2, 177–189.

[10] N. J. Kalton and N. L. Randrianarivony, *The coarse Lipschitz structure of $\ell_p \oplus \ell_q$*, Math. Ann. **341** (2008), 223–237.

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[12] M. I. Ostrovskii, *Embeddability of locally finite metric spaces into Banach spaces is finitely determined*, Proc. Amer. Math. Soc. **140** (2012), no. 8, 2721–2730.

Mikael de la Salle (Lyon)

Actions of groups on Banach spaces and applications

ABSTRACT: The purpose of this series of lectures is to present some recent developpement on rigidity for group actions on Banach spaces. This includes Banach space property (T), the fixed point property for affine actions on Banach spaces, and applications to coarse embeddability of expanders in Banach spaces. We will mainly focus on Banach space valued spectral gaps, as in the work Lafforgue [6, 7] and some of its later extensions [8, 10, 3, 4, 5].

A tentative plan for the lecture would be :

Lecture 1: *Geometry of Banach spaces, Banach space spectral gap and property (T).* The first lecture will present general facts on some Banach space version of spectral gap, in particular [9]. This will require to present some basic facts on the geometry of Banach spaces (uniform convexity, type, cotype...) and variants of Kazhdan's property (T) [1, 2, 6, 9].

Lecture 2: *Working out one example.* The second lecture will focus on one spectific group : $G = \mathrm{SL}(3, \mathbf{R})$ or $G = \mathrm{SL}(3, \mathbf{Q}_p)$ depending on the audience's preference. We will present Lafforgue's proof of property (T) for G, and explain how this proof extends to Banach space property (T) [6, 7, 10].

Lecture 3: *Fixed point property and strong property (T).* The third lecture will explore the fixed point property for affine actions by isometries on Banach spaces. This will naturally lead to Lafforgue's notion of strong property (T) [1, 6, 10, 5].

REFERENCES

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[3] T. de Laat and M. de la Salle, *Strong property (T) for higher rank simple Lie groups*, to appear in Proc. London Math. Soc. (2015).

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Invited talks

Bruno Duchesne (Lorraine)

Amenable invariant random subgroups

Abstract: Let G be a locally compact second countable group. An *Invariant Random Subgroup* (IRS) is a probability measure on the set of closed subgroups of G, which is invariant under the action of G by conjugation.

This is a quite recent point of view on probability preserving actions, focused on the stabilizers of points. I will try give some flavour on that subject and eventually focus on IRS supported on amenable subgroups. In particular, I will show that such IRS lives in the amenable radical of G answering a question of Abert, Glasner and Virag.

Yosuke Kubota (Tokyo)

Topological phases via coarse geometry

Abstract: In condensed matter physics, topological properties of the Hamiltonian of a quantum system (a self-adjoint operator together with a spectral gap) are studied. A classification of topological phases in 10 types is given by Kitaev in relation to the Bott periodicity, which is formulated by Freed-Moore in terms of twisted equivariant K-theory. In this talk, we introduce a variant of the notion of topological phases reflecting metric structure of the real space. This framework enables us to deal with quantum systems which is not periodic (such as materials with disorders or quasi-crystals). Moreover, the twisted equivariant K-groups of Roe algebras gives generalizations of existing invariants such as the Hall conductance for the integer quantum Hall effect or the Kane-Mele $\mathbb{Z}_2$-invariant for topological insulators. As a consequence, we obtain a mathematical proof of bulk-edge correspondence for possibly non-periodic topological phases of arbitrary types by using the coarse Mayer-Vietoris exact sequence.

Hiraku Nozawa (Ritsumeikan)

Property (T) and local rigidity of Riemannian foliations

Abstract: The main result of this talk is that, if the fundamental group of a closed manifold M has property (T) of Kazhdan, then any minimal Riemannian foliation on M is locally rigid among foliations on M in the sense of deformation theory. It implies that certain Lie foliations on double coset spaces of semisimple Lie groups are locally rigid. We will explain how to prove the main result based on a variant of a fixed point theorem for almost actions of groups with property (T) due to Fisher-Margulis, which was used by them to prove the local rigidity of isometric actions of any discrete group with property (T) on any closed Riemannian manifold.

Tatsuki Seto (Nagoya)

Toeplitz operators and the Roe-Higson type index theorem

Abstract: In 1988, Roe defined a C^*-algebra, which is called the Roe algebra, and proved an index theorem on a complete Riemannian manifold M which is partitioned by a closed hypersurface N. His theorem states that Connes' pairing of Roe's cyclic one cocycle with the Roe's odd index of a Dirac operator on M is calculated by the Fredholm index of the Dirac operator D_N on N. Higson gave a simple proof for Roe's index theorem in 1991.

If M is even dimensional, Roe-Higson's index theorem is trivial since the Fredholm index of D_N is always zero. In this talk, I will talk about another non-trivial index formula with Toeplitz operators as an even dimensional analogue of Roe-Higson's theorem.

Yoshinori Yamasaki (Ehime)

Ramanujan Cayley graphs and the Hardy-Littlewood conjecture

Abstract: Ramanujan graph is a sparse graph with good expansion (or high connectivity) properties. In this talk, the following problem will be discussed: Whether can we determine the bound of the valency of graphs which guarantees to be Ramanujan for each fixed number of vertices? We will explain that, when the graphs are given as Cayley graphs on some specific groups such as cyclic groups, the explicit determination of the bound is deeply related to the distribution of primes represented by quadratic polynomials and hence to the Hardy-Littlewood conjecture. This is the joint work with Miki Hirano and Kohei Katata.

Takamitsu Yamauchi (Ehime)

Hereditarily infinite-dimensional spaces concerning asymptotic dimension

Abstract: Asymptotic dimension was introduced by Gromov (1993) as a large scale analogue of covering dimension. On covering dimension, Walsh (1979) constructed a compact metric space such that every finite dimensional nonempty subspace has dimension zero. Such spaces are said to be hereditarily infinite-dimensional. In this talk, we consider an analogous property for asymptotic dimension.