# Lie groupoids, cyclic homology and index theory

(Based on joint work with M. Pflaum and X. Tang)

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# Goal

- Goal: a *cohomological* index theorem for the pairing

$$K_0(A_{\mathsf{G}}) \times HC^{ev}(A_{\mathsf{G}}) \to \mathbb{C}.$$

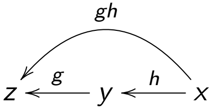
  $A_{\mathsf{G}} =$ (smooth) convolution algebra of a Lie groupoid G.
- $HC^\bullet(A_{\mathsf{G}})$ is unknown, so we use cocycles that we can construct from groupoid cohomology.
- On the level of $K$-theory, we have the *Baum–Connes* map

$$\mu : K_\bullet(\underline{BG}) \to K_\bullet(A_G),$$

  with $\underline{BG}$ the classifying map for proper actions.
- We therefore focus on proper actions of Lie groupoids.

# Lie groupoids

$G \rightrightarrows M$ Lie groupoid.




Examples:

1. $M \times M \rightrightarrows M$ pair groupoid,
2. $G \rightrightarrows \{e\}$ Lie group,
3. $G \times M \rightrightarrows M$ action of a Lie group,
4. $\mathcal{F} \subset TM$ foliations $\rightsquigarrow$ holonomy groupoid.

# What should we know about Lie groupoids?

Lie groupoids behave just like Lie groups:

- They have a **Lie algebroid** capturing the infinitesimal data,
- we can consider **representations** of Lie groupoids,
- we can define smooth groupoid cohomology $H^\bullet_{\mathrm{diff}}(\mathsf{G}; E)$,

but:

- Lie III is not valid: not every Lie algebroid integrates to a Lie groupoid (cf. Crainic–Fernandes)
- The adjoint representation does not exist.

# Lie algebroids

The infinitesimal data of a Lie groupoid is given by a Lie algebroid:

$$(A := \ker(dt), \rho := ds, [\ ,\ ]).$$

## Definition

A **Lie algebroid** is a vector bundle $A \to M$, whose space of sections carries a Lie bracket, equipped with a bundle map $\rho : A \to TM$ ("the anchor") satisfying

$$\begin{aligned} \rho([X, Y]) &= [\rho(X), \rho(Y)], \\ [X, fY] &= f[X, Y] + \rho(X)(f) \cdot Y. \end{aligned}$$

## Warning

Lie III is not true for Lie groupoids!

# Representations and Cohomology

## Definition

A representation of G is a vector bundle $E \to M$ equipped with

$$\lambda \in \Gamma^\infty(\mathsf{G}, \mathrm{Hom}(s^*E, t^*E))$$

satisfying

$$\lambda_{g_1} \circ \lambda_{g_2} = \lambda_{g_1 g_2}.$$

Smooth groupoid cohomology $H^\bullet_{\mathrm{diff}}(\mathsf{G}; E)$: $\Gamma^\infty(\mathsf{G}_k; s^*E)$

$$\begin{aligned}\delta\varphi(g_1, \ldots, g_k) = &\lambda_{g_1}\varphi(g_2, \ldots, g_k) \\ &+ \sum_{i=1}^{k-1}(-1)^i\varphi(g_1, \ldots, g_i g_{i+1}, \ldots, g_k) \\ &+ (-1)^k\varphi(g_1, \ldots, g_{k-1}).\end{aligned}$$

# Transversal densities

## Definition (Bundle of "transversal densities")

$$L := \bigwedge^{top} A \otimes \bigwedge^{top} T^*M$$

Evens–Lu–Weinstein: $L$ carries a canonical representation of G.

## Definition (Unimodularity)

A Lie groupoid G is *unimodular* if there exists an invariant nonvanishing section $\Omega$ of $L$.

Obstruction measured by the *modular class*

$$\log \delta \in H^1_{\mathrm{diff}}(\mathrm{G}; \mathbb{R}).$$

# Convolution algebra

$$\mathcal{A}_{\mathsf{G}} := \Gamma^{\infty}_{cpt}\left(\mathsf{G}; s^*\left|\bigwedge^{top} A^*\right|\right).$$

Product:

$$(f_1 * f_2)(g) := \int_{g_1 g_2 = g} f_1(g_1) f_2(g_2).$$

**Proposition**

*There exists a canonical map*

$$\chi : H^{\bullet}_{\mathrm{diff}}(\mathsf{G}; L) \to HC^{\bullet}(\mathcal{A}_{\mathsf{G}}).$$

In degree 0: given $\Omega \in \Gamma^{\infty}_{inv}(M; L)$,

$$\tau_{\Omega}(a) := \int_M \langle a, \Omega \rangle$$

is a trace.

# The index problem

We would like to evaluate the pairing

$$H^{\bullet}_{\mathrm{diff}}(\mathsf{G}; L) \times K_{\bullet}(\underline{BG}) \to \mathbb{C},$$

given by

$$\langle \nu, [Z, D] \rangle = \langle \chi(\nu), \mu([Z, D]) \rangle .$$

But $\mu([Z, D]) \notin K_0(\mathcal{A}_{\mathsf{G}})$.

Solution strategies:

- consider the extension problem for the cyclic cocycles $\chi(\nu) \in HC^{\bullet}(\mathcal{A}_{\mathsf{G}})$.
- Lift the pairing to $Z$.

**Definition**

An action of $\mathsf{G} \rightrightarrows M$ is given by a submersion $\mu : Z \to M$ together with

$$\mathsf{G}\,{}_t\!\times_\mu Z \to Z, \quad (g, z) \mapsto gz,$$

which is associative.

The action is *proper* if the map

$$\mathsf{G}\,{}_t\!\times_\mu Z \to Z \times Z, \quad (g, z) \mapsto (gz, z).$$

is proper.

# Invariant Pseudodifferential Calculus

Let G act on $Z$ with moment map $\mu : Z \to M$

## Definition

$P \in \Psi^k_{\text{inv}}(Z; \mathrm{G})$: family $P = \{P_x\}_{x \in M}$ of $\Psi DO's$ on $\mu^{-1}(x)$ such that:

- $x \mapsto P_x$ is smooth,
- $P$ is G-invariant:

$$P_{s(g)} = L_g^* \circ P_{t(g)} \circ L_{g^{-1}}^*.$$

- $\mathrm{supp}(P)$ is G-compact.

Left multiplication:

$$L_g : \mu^{-1}(s(g)) \to \mu^{-1}(t(g)) \rightsquigarrow L_g^* : C^\infty(\mu^{-1}(t(g))) \to C^\infty(\mu^{-1}(s(g)))$$

# Cut-off functions

## Definition

A cut-off function on $Z$ is a section $c \in \Gamma_c^\infty(Z; \left|\bigwedge^{top}\right| A^*)$ with

$$\int_{s(g)=\mu(z)} c(g^{-1}z) = 1, \quad \text{for all } z \in Z.$$

Choose $\Omega \in \Gamma^\infty(M; L)$.

$$\tau_\Omega(K) = \int_Z k(z,z)\langle c(z), \pi^*\Omega\rangle, \; k \in \Psi^{-\infty}_{\mathrm{inv}}(Z; \mathsf{G}).$$

$\tau_\Omega$ is a trace if and only if $\Omega$ is invariant.

(cf. H. Wang for the group action case)

# The characteristic map

Characteristic map:

$$\chi : H^\bullet_{\mathrm{diff}}(\mathsf{G}; L) \to HC^\bullet(\Psi^{-\infty}_{\mathrm{inv}}(Z; \mathsf{G}))$$

Defined by:

$$\chi(\varphi)(k_0 \otimes \ldots k_p)$$
$$:= \int_{Z_\mu^{(p+1)}} c(z_0)\varphi(z_0, \ldots, z_{2n})k_1(z_0, z_1)\cdots k_{2n}(z_{2n}, z_0)\pi^*\Omega.$$

index pairing:

$$H^\bullet_{\mathrm{diff}}(\mathsf{G}; L) \times K_0(\Psi^{-\infty}_{\mathrm{inv}}(Z; \mathsf{G})) \to \mathbb{C}.$$

# The index theorem for proper actions

$D$ invariant family of differential operators on $Z$.

$$D \text{ elliptic} \rightsquigarrow \mathrm{Ind}(D) \in K_0(\Psi^{-\infty}_{\mathrm{inv}}(Z;\mathsf{G})).$$

**Theorem**

*For* $\nu \in H^{2k}_{\mathrm{diff}}(\mathsf{G};L)$, *we have*

$$\langle \nu, \mathsf{Ind}(D)\rangle := \frac{1}{(2\pi\sqrt{-1})^k}\int_{\mathcal{F}^*}\pi^*\langle c, \Phi_Z(\nu)\rangle\,\mathrm{Td}(\mathcal{F}^*\otimes\mathbb{C})\,\mathsf{ch}_{\mathcal{F}}(\sigma(D)).$$

# The van Est map

$\mathcal{F}$ = foliation by the fibers of $\mu : Z \to M$.

$$\text{van Est map}: \quad \Phi_Z : H^\bullet_{\text{diff}}(\mathsf{G}; E) \to H^\bullet_{\mathcal{F}}(Z; \mu^* E)^{\mathsf{G}}$$

**Theorem (Crainic)**

*The van Est map $\Phi_Z$ is an isomorphism in degree $\bullet \leq n$ and injective for $\bullet = n + 1$, when the* G*-action is proper and the fibers of the moment map $\mu : Z \to M$ are homologically n-connected.*

# G-invariant cohomology

**Lemma**

*Let* $\alpha \in \Omega^{top}_{\mathcal{F}}(Z;\mu^* L)^{\mathsf{G}}$. *The integral*

$$\int_Z \langle c, \alpha \rangle \,,$$

*vanishes on exact forms. The linear map*

$$\int_Z \langle c, - \rangle : H^{top}_{\mathcal{F}}(Z;\pi^* L)^{\mathsf{G}} \to \mathbb{C}$$

*is independent of* $c$.

# Ingredients of the proof

1. The support of the parametrix $k(z_1, z_2) \in \Psi^{-\infty}_{\mathrm{inv}}(Z; \mathsf{G})$ can be localized to the diagonal in $Z\,{}_\pi\!\times_\pi Z$. Near this diagonal, the groupoid cocycle defines an invariant differential form.
2. Introduce the *asymptotic* G-invariant pseudodifferential calculus. This induces a G-invariant $\star$-product on $T^*_\pi Z$, a regular Poisson manifold.
3. Compare the trace on the deformation quantization $\mathscr{A}^{\hbar}_{T^*_\pi Z}$ agrees with the trace obtained by the Fedosov construction.
4. Compute the higher index pairing by taking the limit $\hbar \to 0$, and use the algebraic index theorem for regular Poisson manifolds.

# Special cases

- By specializing G, $Z$ and the elements in $H^\bullet_{\mathrm{diff}}(\mathsf{G}; L)$ we can now recover various well-known index theorems.
- In all these cases, the most interesting aspect is to identify the van Est map in that situation.

# Special case: $Z = \mathsf{G}$

$E \to M$ vector bundle, $D \in \mathcal{U}(A) \otimes \mathrm{End}(E)$.

$$D \text{ elliptic} \rightsquigarrow \mathrm{Ind}(D) \in K_0(\mathcal{A}_{\mathsf{G}}).$$

## Theorem

*Let* $\nu \in H^{2k}_{\mathrm{diff}}(\mathsf{G}; L)$. *Then*

$$\mathrm{Ind}_\nu(D)) = \frac{1}{(2\pi\sqrt{-1})^k} \int_{A^*} \pi^* \Phi_{\mathsf{G}}(\nu) \mathrm{Td}^{\pi^! A}(\pi^! A \otimes \mathbb{C}) \rho^*_{\pi^! A} \mathrm{ch}(\sigma(D)).$$

- $f : N \to M$ a surjective submersion, $f^! A$ is the pull-back in the category of Lie algebroids.
- $\Phi_{\mathsf{G}} : H^\bullet_{\mathrm{diff}}(\mathsf{G}; L) \to H^\bullet_{\mathrm{Lie}}(A; L)$ van Est map.

# Example: the pair groupoid

- Pair groupoid $M \times M \rightrightarrows M \rightsquigarrow$ Lie algebroid $TM$.
- $M \times M$ is proper: $H^k_{\mathrm{diff}}(G; L) = 0 \ k > 0$.
- Only one index (trace) $\Rightarrow$ Atiyah–Singer.

Discrete group $\Gamma$ acting freely on $\tilde{M}$ with quotient $\tilde{M}/\Gamma = M$.

- $\mathsf{G}_{\Gamma,M} := \tilde{M} \times_\Gamma \tilde{M} \rightrightarrows M$, Lie algebroid $TM$.
- $H^\bullet_{\mathrm{diff}}(\mathsf{G}_{\Gamma,M}; \mathbb{C}) \cong H^\bullet_{\mathrm{grp}}(\Gamma; \mathbb{C}) \cong H^\bullet(B\Gamma; \mathbb{C})$
- Covering index theorem:

$$\left\langle \mathrm{Ind}_\Gamma(\tilde{D}), \alpha \right\rangle = \frac{1}{(2\pi\sqrt{-1})^k} \int_{T^*M} \psi^* \alpha \mathrm{Td}(T^*M \otimes \mathbb{C}) \mathrm{ch}(\sigma(D))$$

- $\psi : H^\bullet(B\Gamma) \to H^\bullet(M)$ van Est map.

# Homogeneous spaces of Lie groups

- $K \subset G$ compact subgroup of a Lie group. $X := G/K$.
- $V \in \mathrm{Rep}(K) \rightsquigarrow \tilde{V} := G \times_K V$.
- Elliptic $G$-equivariant differential operator

$$D : \Gamma_c^\infty(X; \tilde{V}) \to \Gamma_c^\infty(X; \tilde{V}).$$

**Theorem**

$$\mathrm{Ind}_\nu(D) = \frac{1}{(2\pi\sqrt{-1})^k} \left\langle \Phi_X(\nu), \hat{A}(\mathfrak{g}; K)\mathrm{ch}(\sigma(D)) \right\rangle.$$

- $\hat{A}(\mathfrak{g}; K)\mathrm{ch}(\sigma(D))$ characteristic classes in $H^\bullet(\mathfrak{g}; K)$. Remark

$$T_{[e]}X \cong \mathfrak{g}/\mathfrak{k} \quad \Longrightarrow \quad \Omega^\bullet(TX)^G \cong \bigwedge^\bullet(\mathfrak{g}/\mathfrak{k})^*.$$

- $\Phi_X : H_d^\bullet(G; \bigwedge \mathfrak{g}) \to H^\bullet(\mathfrak{g}, K; \bigwedge^{\mathrm{top}} \mathfrak{g})$ the “van Est map”.
- $\langle\ ,\ \rangle : H^\bullet(\mathfrak{g}, K; \bigwedge^{\mathrm{top}} \mathfrak{g}) \times H^\bullet(\mathfrak{g}, K) \to \mathbb{C}$ natural pairing.

$G$ is unimodular: $H^0_d(G;\bigwedge^{\mathrm{top}}\mathfrak{g})\cong\mathbb{C}\rightsquigarrow$ trace $\rightsquigarrow$ $L^2$-index theorem of Connes–Moscovici.

- $G$ semisimple, unimodular. $K\subset G$ maximal compact, $\mathrm{rank}(G)=\mathrm{rank}(K)\Longrightarrow \dim(G/K)=$ even.
- $V_\mu$ irrep $K$ with heights weight $\mu$ $\rightsquigarrow$ Dirac operator

$$\not{D}_V:\Gamma^\infty_c(X;\tilde{V}\otimes\tilde{S}^+)\to\Gamma^\infty_c(X;\tilde{V}\otimes\tilde{S}^-)$$

- Atiyah–Schmid

$$\mathrm{Ind}_\Omega(\not{D}_\mu)=\prod_{\alpha>0}\frac{(\mu+\rho_K,\alpha)}{(\rho_G,\alpha)}$$

- Atiyah and Schmidt showed that for $\mu$ nonsingular (i.e., $\langle\mu+\rho_K,\alpha\rangle\neq 0$) $\ker_{L^2}(\not{D}_\mu)$ is a discrete series representation with formal dimension given by the formula above
- $\mu$ singular corresponds to a "limit of discrete series".