Stratified spaces and the Novikov conjecture.

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lecture based on joint work with
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(and partially with Markus Banagl)

Kyoto, December 2013

The Novikov conjecture on closed orientable manifolds

- (you are all familiar with this, but let me set up the notation)
- (X, g) orientable closed compact Riemannian manifold; $\pi_1(X) := \Gamma$;
- X' Galois Γ-covering;
- $r : X \to B\Gamma$ a classifying map for X';
- $L_*(X) \in H_*(X, \mathbb{Q})$ is the homology L-class, i.e. the Poincaré dual of the Hirzebruch L-class in cohomology;
- Novikov conjecture: all the numbers

$$\{< \alpha, r_*(L_*(X)) >, \alpha \in H^*(B\Gamma, \mathbb{Q})\}$$

are oriented homotopy invariants.

The Mishchenko-Kasparov method

I call the following series of results the signature package for the pair $(X, r : X \to B\Gamma)$:

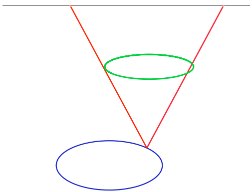
1. the signature operator with values in the Mischenko bundle defines a (signature) index class $\mathrm{Ind}(\widetilde{\eth}_{\mathrm{sign}}) \in K_*(C^*_r\Gamma)$,
2. the index class is a rational bordism invariant: i.e. it defines a group homomorphism $\Omega^{\mathrm{SO}}_*(B\Gamma) \to K_*(C^*_r\Gamma) \otimes \mathbb{Q}$;
3. The index class $\mathrm{Ind}(\widetilde{\eth}_{\mathrm{sign}}) \in K_*(C^*_r\Gamma)$ is a homotopy invariant;
4. there is an analytic K-homology class $[\eth_{\mathrm{sign}}] \in K_*(X)$ and $\mathrm{Ch}_*[\eth_{\mathrm{sign}}] = L_*(X)$ (rationally);
5. assembly $\beta : K_*(B\Gamma) \to K_*(C^*_r\Gamma)$ sends $r_*[\eth_{\mathrm{sign}}]$ into $\mathrm{Ind}(\widetilde{\eth}_{\mathrm{sign}})$;
6. if β is rationally injective then 1)$\to$ 5) $\Rightarrow$ homotopy invariance of all Novikov higher signatures.

Remark

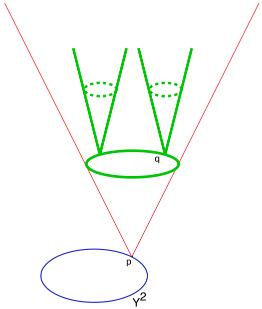
Item 3 (homotopy invariance of the index class) is sometimes replaced or decorated by the following two items:

- there is a (C^*-algebraic) symmetric signature $\sigma_{C^*}(X,r) \in K_*(C^*_r\Gamma)$, due to Mishchenko, which is
  - topologically defined
  - a bordism invariant
  - a *homotopy invariant*;
- the index class = the C^*-algebraic symmetric signature ($\Rightarrow$ the index class is a homotopy invariant)

Stratified pseudomanifolds



Above is an example of depth 1;below is an example of depth 2:



Very general questions

Let $\widehat{X}$ be a stratified pseudomanifold:

1. Is there a homology L-class $L_*(\widehat{X}) \in H_*(\widehat{X}, \mathbb{Q})$?
2. Can one state a Novikov conjecture for $\widehat{X}$?
3. What part of the signature package can be extended to $\widehat{X}$?

Some preliminary problems:

- Hirzebruch L-class (and its Poincaré dual) employs heavily the manifold structure;
- Mishchenko signature employs the structure of Algebraic Poincaré Complex of the singular chain complex; but for stratified spaces Poincaré duality does **not** hold .
- $\Omega_*^{\mathrm{pseudo}}(B\Gamma)$ is not interesting, in fact the coefficient groups $\Omega_*^{\mathrm{pseudo}}(\mathrm{point})$ are trivial
- the signature operator on Ω_c^* (of the regular part of a pseudomanifold) has many self-adjoint extensions (so, which one will be "connected to topology" ?!) .

Why can we hope ?

- Thom: the homology L-class is obtained by taking signatures of submanifolds;
- Goreski-MacPherson intersection (co)homology does satisfy a generalized Poincaré duality:if $\overline{p}$ and $\overline{q}$ are complementary perversities then there is a non-degenerate pairing: if $n = \dim \widehat{X}$

$$IH^{\ell}_{\overline{p}}(\widehat{X};\mathbb{Q}) \otimes IH^{n-\ell}_{\overline{q}}(\widehat{X};\mathbb{Q}) \to \mathbb{Q}$$

Recall that a perversity $\overline{p}$ is a function $\mathbb{Z}^{\geq 2} \to \mathbb{Z}$ such that

$$\overline{p}(2) = 0\,; \qquad \overline{p}(k) \leq \overline{p}(k+1) \leq \overline{p}(k) + 1.$$

- The lower middle perversity is $\overline{m}$, defined by $\overline{m}(k) = [(k-2)/2]$.
- The upper middle perversity is $\overline{n}$, defined by $\overline{n}(k) = [(k-1)/2]$.
- The complementary perversity $\overline{q}$ of a perversity $\overline{p}$ is the one with $\overline{p}(k) + \overline{q}(k) = k - 2$. Note: $\overline{m}$ and $\overline{n}$ are complementary. Finally: intersection cohomology is stratified homotopy invariant.

First possible solution

We restrict the class of pseudomanifolds. We consider Witt spaces:
$\widehat{X}$ is a Witt space if any even-dimensional link L has $IH_{\overline{m}}^{\dim L/2}(L;\mathbb{Q}) = 0$.
Then:

- lower and upper middle perversity intersection cohomology groups are isomorphic: $IH_{\overline{m}}^{*}(\widehat{X};\mathbb{Q}) \simeq IH_{\overline{n}}^{*}(\widehat{X};\mathbb{Q})$;
- consequently $IH_{\overline{m}}^{*}(\widehat{X};\mathbb{Q})$ does satisfy Poincaré duality
- there is a well defined signature
- $\Omega_{*}^{\mathrm{Witt}}(Y)$, Y a topological space, is rich and encodes the Goreski-MacPherson signature when $* = 4k$ and $Y =$point
- one can now define a homology L-class $L_{*}(\widehat{X}) \in H_{*}(\widehat{X};\mathbb{Q})$ à la Thom;
- Cheeger has proved that for suitable metrics the signature operator on a Witt space is **essentially self-adjoint** and **Fredholm**.

Cheeger results on Witt spaces

- Cheeger constructs the heat kernel for (the unique s.a. closure of) $\eth^2_{\mathrm{sign}}$ and proves that it is **trace class**: this proves the Fredholm property
- there is also a **general** de Rham theorem: if X is the regular part of a pseudomanifold $\widehat{X}$ (not necessarily Witt) then for the exterior differential $d : \Omega^*_c(X) \to \Omega^{*+1}_c(X)$ we can consider the minimal and the maximal extension and get two Hilbert complexes: $(L^2\Omega^*_{\max}, d_{\max})$ and $(L^2\Omega^*_{\min}, d_{\min})$. Cheeger (with some contribution by Nagase) proves that

$$H^*_{L^2,\max}(X) \simeq IH^*_{\overline{m}}(\widehat{X}) \; ; \qquad H^*_{L^2,\min}(X) \simeq IH^*_{\overline{n}}(\widehat{X})$$

- if $\widehat{X}$ is Witt then $H^*_{L^2,\max}(X) = H^*_{L^2,\min}(X)$ (in fact, $d_{\max} = d_{\min}$) and there is a Hodge decomposition theorem. Thus, in the Witt case, the Goresky-MacPherson signature is an index.

" Old " results by Albin-Leichtnam-Mazzeo-P.:

In the following paper
The signature package on Witt spaces by P. Albin, E. Leichtnam, R. Mazzeo and P.P. *Ann. Sci. Ecole Normale Superieure*. 2012. pp 241-310
we achieve the following:

- we reprove Cheeger's result using techniques from microlocal analysis;
- we re-obtain the existence of a signature class $[\eth_{\rm sign}] \in K_*(\widehat{X})$ (Cheeger + Moscovici-Wu); ($\rightarrow$ "open" question)
- we extend the analysis to the $C_r^*\Gamma$-context;
- we thus define $\mathrm{Ind}(\widetilde{\eth}_{\rm sign}) \in K_*(C_r^*\Gamma)$;
- we use ideas of Hilsum-Skandalis and prove the stratified homotopy invariance of $\mathrm{Ind}(\widetilde{\eth}_{\rm sign})$

" Old " results: Part 2.

- If $\beta : K_*(B\Gamma) \rightarrow K_*(C^*_r\Gamma)$ is assembly then $\beta(r_*[\eth_{\mathrm{sign}}]) = \mathrm{Ind}(\widetilde{\eth}_{\mathrm{sign}})$ in $K_*(C^*_r\Gamma)$
- we prove that the index class is a Witt-bordism invariant, i.e. defines a homomorphism $\Omega^{\mathrm{Witt}}_*(B\Gamma) \rightarrow K_*(C^*_r\Gamma) \otimes \mathbb{Q}$;
- Moscovici-Wu proved that the Chern character of $[\eth_{\mathrm{sign}}]$ is, rationally, the Goresky-MacPherson-Thom L-class
- using all this we establish the stratified homotopy invariance of the higher signatures of a Witt space

$$\{< \alpha, r_* L_*(\widehat{X}) >, \quad \alpha \in H^*(B\Gamma, \mathbb{Q})\}$$

when the assembly map is rational injective.

Example 1. Complex projective varieties are Witt spaces (warning: we are considering them with an *ad hoc* conic-type metric).

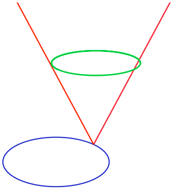
Example 2. Certain quotients by non-free actions of Lie groups are Witt spaces.
For example, a semifree S^1-action on a smooth manifold M produces a singular quotient with links $\mathbb{C}P^N$. If N is odd we get a Witt space. However, if N is even we do not get a Witt space.

Comment 1. Friedman and McClure have now a definition of Mishchenko signature $\sigma^{\rm Witt}(\widehat{X}, r)$ for Witt spaces. Since this is also a Witt-bordism invariant, $\sigma^{\rm Witt} : \Omega_*^{\rm Witt}(B\Gamma) \to K_*(C_r^*\Gamma)$, one can see that $\sigma^{\rm Witt}(\widehat{X}, r)$ is rationally equal to our index class. If Γ satisfies the Farrell-Jones L-theory conjecture then Friedman and McClure could try to prove the Novikov conjecture for $\widehat{X}$ Witt and such that $\pi_1(\widehat{X}) = \Gamma$.

Techniques

- we resolve the pseudomanifold $\widehat{X}$ to a manifold with corners $\widetilde{X}$ (Verona + Brasselet-Hector-Saralegi + ALMP). $\widetilde{X}$ has an additional structure: it has an an iterated fibration structure on the boundary (boundary hypersurfaces are fibrations + compatibility relations at the corners between these fibrations).

Example: if $\widehat{X}$ is a depth-one space



then $\widetilde{X}$ is a manifold with boundary and the boundary is a fibration with base equal to the singular stratum (the bottom circle) and fiber the links (the green circles).

Iterated conic metrics. Let us concentrate on the depth one case (the one of the figure).

So there is a decomposition of $\widehat{X}$ into two strata: Y, the singular set (the bottom circle) and X, the regular part (this is the union of the red cones (without the vertices)).
The link of a point $p \in Y$ is a smooth closed manifold Z (the green circle).
A neighborhood of p looks like $B \times C(Z)$, with B a ball in an euclidean space.
In fact a tubular neighborhood of Y is a bundle of cones $C(Z) \to T \xrightarrow{\pi} Y$, as in the figure.
A conic metric looks like $dx^2 + x^2 g_Z + \pi^* h_Y$ and the de Rham operator $\eth_{\mathrm{dR}}$ looks like (here $f = \dim Z$ and $\mathbf{N}$ is the number operator)

$$\eth_{\mathrm{dR}} \sim \begin{pmatrix} \frac{1}{x}\eth_{\mathrm{dR}}^{Z} + \eth_{\mathrm{dR}}^{Y} & -\partial_x - \frac{1}{x}(f - \mathbf{N}) \\ \partial_x + \frac{1}{x}\mathbf{N} & -\frac{1}{x}\eth_{\mathrm{dR}}^{Z} - \eth_{\mathrm{dR}}^{Y} \end{pmatrix}$$

with $\eth_{\mathrm{dR}}^{Y}$ and $\eth_{\mathrm{dR}}^{Z}$ the de Rham operators of h_Y and g_Z, respectively.
Main idea: consider $x\eth_{\mathrm{dR}}$ instead ! It can be considered on $\widetilde{X}$.
This is an elliptic differential operator in the edge-calculus of Mazzeo.
This brings us to microlocal techniques à la Melrose.

What happens in the non-Witt case ?

- We loose essential-self-adjointness. We must **reduce the maximal domain** and hope to get something self-adjoint and Fredholm.
- Topologically there are results by Markus Banagl.
- Banagl, inspired by work of Cheeger in the isolated-singularity-case, defined a new class of stratified psuedomanifold that are non-Witt but that admit a signature nevertheless. There is also a homology L-class.
- These are stratified pseudomanifold with a Lagrangian structure along the strata.
- Banagl techniques are purely sheaf-theoretic (derived functors, Verdier duality etc etc).
- We want to give a de Rham-Hodge treatment of Banagl theory and extend to these manifolds index theory and analytic Novikov theory.

How do we proceed ?

Let us concentrate again on the depth-one case and consider first $\eth_{\mathrm{dR}}$.Recall that a tubular neighborhood T of the singular set looks like

$$C(Z) \to T \to Y$$

The level set $x = 1$ defines a fibration $Z \to H \to Y$. If Z is even-dimensional and has cohomology in middle degree then we are NOT in the Witt case. We have the following

Fundamental Lemma *Any $u \in \mathcal{D}_{\max}(\eth_{\mathrm{dR}})$ has an asymptotic expansion at Y,*

$$u \sim x^{-f/2}(\alpha_1(u) + dx \wedge \beta_1(u)) + \widetilde{u}$$

with the terms in this expansion distributional:

$$\alpha_1(u), \beta_1(u) \in H^{-1/2}(Y; \Lambda^* T^* Y \otimes \mathcal{H}^{f/2}(H/Y)), \quad \widetilde{u} \in xH^{-1}(X, \Lambda^* X)$$

Here $\mathcal{H}^{f/2}(H/Y)$ is the flat Hodge bundle over Y (with typical fiber $\mathcal{H}^{f/2}(Z)$).

Cheeger boundary condition

The distributional differential forms $\alpha(u)$, $\beta(u)$ serve as 'Cauchy data' at Y which we use to define *Cheeger ideal boundary conditions*. Here is what we do: for any subbundle

$$\begin{array}{ccc} W & \longrightarrow & \mathcal{H}^{f/2}(H/Y) \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

that is parallel with respect to the flat connection, we define

$$\mathcal{D}_{\max,W}(\eth_{\mathrm{dR}}) = \{u \in \mathcal{D}_{\max}(\eth_{\mathrm{dR}}) :$$
$$\alpha_1(u) \in H^{-1/2}(Y; \Lambda^* T^* Y \otimes W), \quad \beta_1(u) \in H^{-1/2}(Y; \Lambda^* T^* Y \otimes (W)^{\perp})\}.$$

Canonical choices are to take W equal to the zero section (minimal domain) or W equal to $\mathcal{H}^{f/2}(H/Y)$ (maximal domain)

Inductive procedure

Let $\widehat{X}$ be a general pseudomanifold with an iterated conic metric g and singular strata $\{Y^1, \dots, Y^T\}$. A collection of bundles

$$\mathcal{W} = \{W^1 \to Y^1, W^2 \to Y^2, \dots, W^T \to Y^T\}$$

is a **(Hodge) mezzoperversity** adapted to g if, inductively, each W^i is a flat subbundle of the Hodge bundle with the "previous" boundary conditions.

New results of Albin-Leichtnam-Mazzeo-P

- Every mezzoperversity induces a closed self-adjoint domain $\mathcal{D}_{\mathcal{W}}(\eth_{\mathrm{dR}})$;
- $(\eth_{\mathrm{dR}}, \mathcal{D}_{\mathcal{W}}(\eth_{\mathrm{dR}}))$ is Fredholm with discrete spectrum;
- We can define domain for the exterior derivative as an unbounded operator on L^2 differential forms: $\mathcal{D}_{\mathcal{W}}(d)$;
- Similarly we define $\mathcal{D}_{\mathcal{W}}(\delta)$ and show that these domains are mutually adjoint;
- they satisfy $\mathcal{D}_{\mathcal{W}}(\eth_{\mathrm{dR}}) = \mathcal{D}_{\mathcal{W}}(d) \cap \mathcal{D}_{\mathcal{W}}(\delta)$;
- if $\mathcal{D}_{\mathcal{W}}(d)^{[k]} = \mathcal{D}_{\mathcal{W}}(d) \cap L^2\Omega^k(X)$ then $d : \mathcal{D}_{\mathcal{W}}(d)^{[k]} \to \mathcal{D}_{\mathcal{W}}(d)^{[k+1]}$ so that $\mathcal{D}_{\mathcal{W}}(d)^{[*]}$ forms a complex;
- the corresponding de Rham cohomology groups, $H^*_{\mathcal{W}}(\widehat{X})$, are finite dimensional and metric independent;
- there is a Hodge theorem.

More results of Albin-Leichtnam-Mazzeo-P

- if $F : \widehat{Y} \to \widehat{X}$ is a stratified map and $\mathcal{W}$ is a mezzoperversity for X, then there is an induced mezzoperversity $F^\sharp \mathcal{W}$ on Y;
- if F is a stratified homotopy equivalence then $H^*_{\mathcal{W}}(\widehat{X}) \simeq H^*_{F^\sharp \mathcal{W}}(\widehat{Y})$;
- given a mezzoperversity $\mathcal{W}$ there is a dual mezzoperversity $\mathcal{DW}$ defined in terms of Hodge-$\star$
- there is a natural non-degenerate pairing $H^\ell_{\mathcal{W}}(\widehat{X}) \times H^{n-\ell}_{\mathcal{DW}}(\widehat{X}) \to \mathbb{R}$
- this sharpens $H^\ell_{\min}(\widehat{X}) \times H^{n-\ell}_{\max}(\widehat{X}) \to \mathbb{R}$
- if $\mathcal{W} = \mathcal{DW}$ then we say that $\mathcal{W}$ is self-dual (it might not exists)
- a space admitting a self-dual mezzoperversity is a **Cheeger space**;
- on a Cheeeger space we have a non-degenerate pairing $H^\ell_{\mathcal{W}}(\widehat{X}) \times H^{n-\ell}_{\mathcal{W}}(\widehat{X}) \to \mathbb{R}$ and thus a signature;
- a self-dual mezzoperversity defines a Fredhom signature operator $(\eth_{\rm sign}, \mathcal{D}_{\mathcal{W}}(\eth_{\rm sign}))$; the index is equal to the signature
- the signature is independent of $\mathcal{W}$!!!
- there is a homology L-class
- $\Rightarrow$ on a Cheeger space $\exists$ Novikov higher signatures and we show they are stratified homotopy invariants if assembly is rationally injective.

References

All of the above can be found in the following two preprints:

- P. Albin, E. Leichtnam, R. Mazzeo, P. Piazza.
  "Hodge theory on Cheeger spaces. "
  arXiv: 1307.5473.
- P. Albin, E. Leichtnam, R. Mazzeo, P. Piazza.
  "The Novikov conjecture on Cheeger spaces."
  arXiv: 1308.2844

- in a further paper
  P. Albin, M. Banagl, E. Leichtnam, R. Mazzeo, P. Piazza.
  "Refined intersection homology on non-Witt spaces."
  arXiv:1308.3725
  we show that the topologically defined groups of Banagl and our de Rham-type cohomology groups are isomorphic.
- this implies that the two definitions of signature, and therefore of homology L-class, are compatible.
- Thanks for your attention !