On the L^p Novikov and Baum-Connes conjectures

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The right side of the Novikov and Baum-Connes conjectures is the K-theory of the reduced C^*-algebra $C^*_{red}(G)$ of the group G. This algebra is the completion of the algebra $L^1(G)$ acting on $L^2(G)$ by convolution. If we complete the algebra $L^1(G)$ in the norm of the convolution algebra on $L^p(G)$ we will get the Banach algebra which I will denote $C_r^{\times,p}(G)$. The K-theory of this algebra serves as the right side of the L^p-version of the Novikov and Baum-Connes conjectures. I will discuss the results on these conjectures for a certain class of groups.

This is a joint work with Guoliang Yu. My student Fan Fei Chong also participated in the Novikov conjecture part.

The construction of the assembly map requires a technique of asymptotic morphisms for Banach algebras. The definition of asymptotic morphisms for Banach algebras goes exactly in the same way as for C^*-algebras. Also the composition of asymptotic morphisms is defined as for C^*-algebras. The first non-trivial point is the construction of an asymptotic morphism corresponding to an extension of Banach algebras: $0 \to J \to A \to B \to 0$. Here we need a quasicentral continuous approximate unit for the ideal J. At this point, we will make a certain assumption concerning our Banach algebras.

Definition. A Banach algebra will be called σ-unital if it possesses a countable bounded approximate unit contained in a closed subalgebra isomorphic to $C_0(Y)$ for some locally compact space Y. In the case with an action of a locally compact group G, we assume that the subalgebra $C_0(Y)$ is G-invariant and the G-action on $C_0(Y)$ is induced by the G-action on Y.

(For the category of C^*-algebras, this coincides with the usual definition of σ-unitality.)

Lemma. Let J be a closed two-sided ideal in a Banach algebra B. Assume that J is σ-unital and the quotient B/J is separable. Then there is a quasicentral bounded approximate unit in J with respect to B. In the case of a G-action on B which satisfies the assumption of the above definition, the approximate unit can be chosen quasicentral with respect to G as well.

A Banach algebra B will be called an L^p-algebra if it isometrically embeds as a closed subalgebra into the algebra $\mathcal{L}(L^p(Z))$ of all bounded linear operators on some space $L^p(Z)$, where Z is a measure space. For such algebra, the opposite algebra B^{op} embeds into the dual space $L^q(Z)$ where $1/p + 1/q = 1$.

Note that when $W = l^p(Z)$ for a discrete space Z with an atomic measure, the algebra of compact operators $\mathcal{K}(W)$ is clearly σ-unital. This is also true in the equivariant case of a G-action induced by a G-action on Z (we assume G discrete in this case).

A possibility to construct an asymptotic morphism out of an exact sequence of Banach algebras using quasicentral approximate units as in the C^*-algebra case (the Connes-Higson construction) allows to define E-theory for σ-unital L^p Banach algebras and establish its main properties (product, Bott periodicity, exact sequences). Note that in the L^p case we use the algebra of compact operators on an L^p space in the definition of the E-theory groups.

We will also need E-theory on the category of $C_0(X)$-algebras. Recall that a Banach algebra A is a $C_0(X)$-Banach algebra if $C_0(X)$ acts on it as an algebra of central multipliers and $C_0(X) \cdot A = A$. On the category of $C_0(X)$-Banach algebras, one defines $\mathcal{R}E$-theory. Recall also that a G-Banach algebra is called *proper* if it is a $C_0(X)$-algebra with the G-action on X being proper. Note that the group $RE(X; A, B) = \mathcal{R}E(X; A \otimes C_0(X), B \otimes C_0(X))$ can also be defined as the group of continuous families of E-theory elements parametrized by the space X, like for KK-theory.

For any locally compact group G and an L^p-algebra $B \subset \mathcal{L}(L^p(Z))$ we define the norm on the crossed product algebra $C_c(G, B)$ as the operator norm for the action of this algebra on $L^p(G \times Z)$ by the formula:

$$(b \cdot l)(t, z) =$$

$$\int_G t^{-1}(b(s)) \cdot l(s^{-1}t, z) ds$$

where b is a compactly supported function on G with values in B and $l \in L^p(G \times Z)$. We define the reduced crossed product algebra $C_r^{\times,p}(G, B)$ as the completion of $C_c(G, B)$ in this norm.

In the special case when $B = \mathbf{C}$, the algebra $C_r^{\times,p}(G)$ is just the image of the algebra $L^1(G)$ in $\mathcal{L}(L^p(G))$ acting on $L^p(G)$ by convolution. If G is unimodular, the dual to the operator of convolution by an element $f(g) \in L^1(G)$ on $L^p(G)$ is the convolution operator by $\bar{f}^*(g)$ on $L^q(G)$, where $f \mapsto f^*$ is the usual involution on $L^1(G)$. Therefore, $C_r^{\times,p}(G) \simeq C_r^{\times,q}(G)$. By the interpolation theorem, since $C_r^{\times,p}(G)$ acts on both $L^p(G)$ and $L^q(G)$, it also acts on $L^2(G)$, so there is a natural homomorphism $C_r^{\times,p}(G) \to C_r^*(G)$. It is not clear however how the K-theory groups of these algebras are related.

We will use the descent homomorphism:

$$j^G : E^G(A, B) \to$$

$$E(C_c(G, A), C_r^{\times, p}(G, B)).$$

Let $\mathcal{E}G$ be the classifying space for proper actions of G. The left-hand side of the Baum-Connes conjecture (with coefficients in an L^p-algebra B) is the inductive limit $RE_*^G(\mathcal{E}G, B) = \varinjlim E_*^G(C_0(X), B)$ over all G-proper, G-compact, locally compact spaces X.

In particular, the left-hand side for the L^p Baum-Connes conjecture with $B = \mathbf{C}$ is the same as for the conventional Baum-Connes conjecture.

The assembly map

$$\mu_p : RE_*^G(\mathcal{E}G, B) \to$$

$$K_*(C_r^{\times,p}(G, B))$$

is defined as follows. We take a non-negative function $c \in C_c(X)$ satisfying $\int_G c(g^{-1}x)dg = 1$ and define the idempotent element $[c] \in C_c(G, C_0(X))$ by

$$[c](g, x) = c(x)^{1/2} c(g^{-1}x)^{1/2}.$$

For any $a \in E_*^G(C_0(X), B)$, we first apply the homomorphism j^G to get an element $j^G(a) \in E(C_c(G, C_0(X)), C_r^{\times,p}(G, B))$, and then take the E-theory product of $j^G(a)$ with the idempotent $[c]$.

We call the injectivity of μ_p the L^p Novikov conjecture, and the bijectivity of μ_p the Baum-Connes conjecture (with coefficients in B).

Let us first discuss the Novikov conjecture.

Suppose that Γ is discrete and that there exists a $C_0(X)$-Banach Γ-algebra $\mathcal{A}$, such that actually $C_0(X) \subset \mathcal{A}$, and X is Γ-proper. Assume that $\mathcal{A}$ is E-equivalent to $\mathcal{S} = C_0(\mathbf{R})$, graded by even-odd functions, equivariantly with respect to any finite subgroup of Γ. More precisely, we assume that there exists a "Bott element" in the group $E^\Gamma(\mathcal{S}, \mathcal{A})$ which is invertible in E-theory when restricted to any finite subgroup of Γ.

Then the Novikov conjecture can be proved using the following diagram:

$$
\begin{array}{ccc}
RE_*^{\Gamma}(\mathcal{E}\Gamma, B\hat{\otimes}\mathcal{S}) & \rightarrow & K_*(C_r^{\times,p}(\Gamma, B\hat{\otimes}\mathcal{S})) \\
\downarrow & & \downarrow \\
RE_*^{\Gamma}(\mathcal{E}\Gamma, B\hat{\otimes}\mathcal{A}) & \rightarrow & K_*(C_r^{\times,p}(\Gamma, B\hat{\otimes}\mathcal{A})).
\end{array}
$$

Here the horizontal arrows are the assembly maps, the vertical arrows are the Bott maps.

We prove that the bottom horizontal arrow and the left vertical arrow are isomorphisms. The injectivity of the upper horizontal arrow follows.

The main tool in the proof is the Mayer-Vietoris exact sequence in E-theory.

Example when such algebra $\mathcal{A}$ exists: Γ *admits a coarse uniform embedding into a discrete* l^p*-space.* This is certainly satisfied in the assumptions of the next theorem.

Theorem. The L^p-version of the Baum-Connes conjecture with coefficients is true for any countable discrete group Γ which admits an affine-isometric proper action on some space $l^p(Z)$ so that the linear part of this action is induced by the action of Γ on Z. (Z is a discrete countable set equipped with the atomic measure.)

For any finite subset $F \subset Z$ and $V_F = l^p(F)$, we consider $\Lambda^*(V_F)$ as the l^p-space of all finite subsets of F. We construct the inductive limit of the spaces $L^p(\Lambda^*(V_F))$. Namely, if $F_1 \subset F_2$ are finite subsets of Z and $\alpha > 0$, let $f_{F_2-F_1}$ be the scalar function on $V_{F_2-F_1}$ defined by:

$$f_{F_2-F_1}(v) = \exp(-\alpha \sum_{i=1}^{n} |x_i|^p/p),$$

where $F_2 - F_1 = \{z_i | \ i = 1, \cdots, n\}$, $v = \sum_{i=1}^n x_i \delta_{z_i}$.

The isometric embedding:

$$L^p(\Lambda^*(V_{F_1})) \to L^p(\Lambda^*(V_{F_2}))$$

is given by: $\xi \to \xi \otimes (c_n f_{F_2 - F_1})$, where the constant $c_n(p, \alpha)$ is chosen in such a way that $c_n f_{F_2 - F_1}$ has L^p-norm 1.

For any $\alpha > 0$, we define the Banach space $\mathcal{B}_\alpha$ as the inductive limit of $L^p(\Lambda^*(V_F))$. We define the Banach algebra $\mathcal{K}_\alpha$ as the algebra of all compact operators on the Banach space $\mathcal{B}_\alpha$. To simplify notation we will omit the coefficient algebra in the statement of the Baum-Connes conjecture with coefficients.

Lemma. The right-hand side of the Baum-Connes conjecture $K_*(C_r^{\times,p}(\Gamma))$ is isomorphic to $K_*(C_r^{\times,p}(\Gamma,\mathcal{K}_\alpha))$.

The image of the Baum-Connes map can be described using localization algebras and finite propagation techniques (developed by G. Yu in the 90's). The sketch of the argument below also uses quasi-projections, i.e. elements which satisfy the property $||P^2 - P|| < \delta$ for some small δ and $||P|| < c$. The numbers δ and c depend on the context.

Propagation of operators on the space $L^p(\Lambda^*(V_F))$ is defined using their support on $L^p(V_F)$. The propagation of an element $a \in C_c(\Gamma, \mathcal{K}(V_F))$ is defined as the supremum of propagations of all operators $a(g),\ g \in \Gamma$.

The main part of the proof of the L^p Baum-Connes conjecture consists of the technical construction which allows to produce out of any element $a \in K_0(C_r^{\times,p}(\Gamma))$ a quasi-projection P which represents the same element a in the group $K_0(C_r^{\times,p}(\Gamma,\mathcal{K}_\alpha))$ and has arbitrarily small propagation. All elements $a \in K_0(C_r^{\times,p}(\Gamma,\mathcal{K}_\alpha))$ with arbitrarily small propagation are in the image of the Baum-Connes map.

The crucial point in this construction is the use of the Bott-Dirac operator on the space $\mathcal{B}_\alpha$. Note that $q-1=(p-1)^{-1}$. Let F be a finite subset of Z, $\{e_i\}_{i\in F}$ the standard basis for V_F, and x_i the corresponding coordinate functions.

The Bott-Dirac operator is given by the formula:

$$D_{\alpha,t} =$$

$$\sum_{i\in Z} t\lambda_i((\partial/\partial x_i+\alpha x_i^{p-1})ext(e_i)$$

$$+(-(q-1)\partial/\partial x_i+\alpha x_i^{p-1})int(e_i)).$$

We omit the parameter t in the notation below, but it will play an important role in the proof.

The kernel of the operator D_α is the element of the space $\mathcal{B}_\alpha$ defined as the inductive limit of functions $f_{F_2-F_1}$.

An important remark is that the operator D_α^2 is *accretive*, i.e. for any $f \in \mathcal{B}_\alpha$ and f^* in the dual space B_α^*, with $||f^*|| = ||f||^{p-1}$ and $< f, f^* >= ||f||^p$, one has: $< D_\alpha^2(f), f^* >\geq 0$.

This allows to define the operator $(1 + D^2_{\alpha,F})^{-1}$ which is bounded and compact. This property remains true on the inductive limit space $\mathcal{B}_\alpha$ if the sequence of positive numbers $\{\lambda_i\} \to \infty$. In addition, the sequence λ_i has to be specifically chosen to insure the commutation properties of the operator D_α with the group action.

Finally, when $t \to \infty$, the operator $(1+D^2_\alpha)^{-1}$ uniformly converges to the projection onto the kernel of the operator D_α.

We give now a sketch of the proof of the main result. Let P_α be the operator D_α going from Λ^{ev} to Λ^{od}, and $Q_\alpha = D_\alpha/(1+D_\alpha^2)$ going in the opposite direction.

Lemma. The bounded operators $P_\alpha Q_\alpha - 1$, $Q_\alpha P_\alpha - 1$ and $P_\alpha(Q_\alpha P_\alpha - 1)$ can be approximated in norm by operators with arbitrarily small propagation when α and t are large. The same is true for the commutators of these operators with elements $g \in \Gamma$.

The idea of the proof: the projection p_α onto the kernel of D_α can be approximated in norm by operators with arbitrarily small propagation when $t > \alpha$ is large enough.

Let $P_{\alpha,n}$ and $Q_{\alpha,n}$ be respectively the direct sums of n copies of P_α and Q_α. Given an element in $K_0(C_r^{\times,p}(\Gamma))$, we represent it as a quasi-idempotent q in $M_n(\mathbf{C}\Gamma)$ for some n satisfying

$$||q|| \leq c_0, \quad ||q^2 - q|| < 1/100.$$

Let $P_{\alpha,n} \otimes I$ and $Q_{\alpha,n} \otimes I$ be the operators acting on the Banach space $\mathcal{B}_\alpha \otimes l^p(\Gamma)$. We define $P_{\alpha,n,q} = q(P_{\alpha,n} \otimes I)q$ and $Q_{\alpha,n,q} = q(Q_{\alpha,n} \otimes I)q$. We abbreviate $P_{\alpha,n,q}$ and $Q_{\alpha,n,q}$ respectively by $P_{\alpha,q}$ and $Q_{\alpha,q}$.

Let:

$$F_{\alpha,q} = P_{\alpha,q}Q_{\alpha,q},$$
$$G_{\alpha,q} = Q_{\alpha,q}P_{\alpha,q},$$
$$A_{\alpha,q} = F_{\alpha,q} + (q - F_{\alpha,q})F_{\alpha,q},$$
$$B_{\alpha,q} = P_{\alpha,q}(q - G_{\alpha,q})$$
$$+(q - F_{\alpha,q})P_{\alpha,q}(q - G_{\alpha,q}),$$
$$C_{\alpha,q} = (q - G_{\alpha,q})Q_{\alpha,q},$$
$$D_{\alpha,q} = (q - G_{\alpha,q})^2.$$

We put:

$$q_1 = \begin{pmatrix} A_{\alpha,q} & B_{\alpha,q} \\ C_{\alpha,q} & D_{\alpha,q} \end{pmatrix},$$

$$q_0 = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that

$$q_1 \in q_0(M_{2n}(C_r^{\times,p}(\Gamma, \mathcal{K}_\alpha)^+))q_0$$

(where $^+$ means adjoining a unit to the algebra), and

$$q_1 - q_0 \in M_{2n}(C_r^{\times,p}(\Gamma, \mathcal{K}_\alpha)).$$

We define $\psi_\alpha([q])$ as the K-theory difference class $[q_1] - [q_0]$ in $K_0(C_r^{\times,p}(\Gamma, \mathcal{K}_\alpha))$.

Proposition. The operator $q_1 - q_0$ can be approximated in norm by a bounded operator in $M_{2n}(C_r^{\times,p}(\Gamma, \mathcal{K}_\alpha))$ with norm bounded by a universal constant and with arbitrarily small propagation when α is large enough and $t >> \alpha$.

Remark. In the language of localization algebras this means that $q_1 - q_0$ is in the image of the corresponding Baum-Connes map.

Denote by ϕ_α the isomorphism $K_0(C^p_r(\Gamma, \mathcal{K}_\alpha)) \simeq K_0(C^p_r(\Gamma))$.

Proposition. In the assumptions of the main theorem, $\phi_\alpha \circ \psi_\alpha$ is the identity homomorphism.

The proof uses the homotopy of the affine action of Γ to the linear action (by contraction of the 1-cocycle to 0).

Corollary. $\phi_\alpha([q_1]-[q_0]) = [q]$.

This implies that q is in the image of the Baum-Connes map.