Topological T-duality for Real circle bundles

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Talk about

Topological T-duality for Real circle bundle, in which a variant of K-theory $K_\pm$ appears.

Introduction: What is topological T-duality?

- A relation of twisted cohomology (twisted K-theory).
- More concretely, the duality relates a torus bundle with another one, so that twisted cohomology groups of the total spaces of these torus bundles are isomorphic.
- Originally, the idea came from T-duality in string theory:

  P. Bouwknegt, J. Evslin and V. Mathai,
  "T-duality: topology change from H-flux".
  Comm. Math. Phys. 249 (2004), no. 2, 383-415.
- Nowadays, there are a number of generalizations.
- A topological T-duality for circle bundles, following Bunke and Schick, is as follows:

Topological T-duality for circle bundles (former half)

Let X be a finite CW complex, and (E, h) a pair consisting of a principal circle bundle $\pi : E \to X$ and $h \in H^3(E; \mathbb{Z})$. Then, up to isomorphism, there uniquely exists a pair $(\hat{E}, \hat{h})$ consisting of a principal circle bundle $\hat{\pi} : \hat{E} \to X$ and $\hat{h} \in H^3(\hat{E}; \mathbb{Z})$ such that

$$\pi_* h = c_1(\hat{E}), \qquad \hat{\pi}_* \hat{h} = c_1(E), \qquad p^* h = \hat{p}^* \hat{h}.$$

((E, h) and $(\hat{E}, \hat{h})$ will be called T-dual to each other.)

$$\begin{tikzcd} & E \times_X \hat{E} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\ E \arrow[dr, "\pi"'] & & \hat{E} \arrow[dl, "\hat{\pi}"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} & H^3(E \times_X \hat{E}) & \\ H^3(E) \arrow[ur, "p^*"] \arrow[dr, "\pi_*"'] & & H^3(\hat{E}) \arrow[ul, "\hat{p}^*"'] \arrow[dl, "\hat{\pi}_*"] \\ & H^2(X) & \end{tikzcd}$$

Topological T-duality for circle bundles (latter half)

- Recall that $h \in H^3(X;\mathbb{Z})$ twists topological K-theory:

$$K^*(X) \quad \rightsquigarrow \quad K^{h+*}(X)$$

For T-dual pairs (E,h) and $(\hat{E},\hat{h})$, there is an isomorphism

$$T: \ K^{h+n}(E) \longrightarrow K^{\hat{h}+n-1}(\hat{E}).$$

(T will be called the T-transformation.)

$$\begin{tikzcd}
(E,h) \arrow[dr, "\pi"'] & & (\hat{E},\hat{h}) \arrow[dl, "\hat{\pi}"] \\
& X &
\end{tikzcd}$$

- A generalization of this duality is my main theorem.

Topological T-duality for Real circle bundles

Main Theorem [G, arXiv:1310.8446]

- Let X be a finite $\mathbb{Z}/2$-CW complex, and (E, h) a pair consisting of a Real circle bundle $\pi : E \to X$ and $h \in H^3_{\mathbb{Z}/2}(E; \mathbb{Z})$. Then, up to isomorphism, there uniquely exists a pair $(\hat{E}, \hat{h})$ consisting of a Real circle bundle $\hat{\pi} : \hat{E} \to X$ and $\hat{h} \in H^3_{\mathbb{Z}/2}(\hat{E}; \mathbb{Z})$ such that

$$\pi_* h = c_1^R(\hat{E}), \qquad \hat{\pi}_* \hat{h} = c_1^R(E), \qquad p^* h = \hat{p}^* \hat{h}.$$

- For (E, h) and $(\hat{E}, \hat{h})$ as above, there are isomorphisms

$$K^{h+n}_{\mathbb{Z}/2}(E) \xrightarrow{T} K^{\hat{h}+n-1}_{\pm}(\hat{E}), \quad K^{h+n}_{\pm}(E) \xrightarrow{T} K^{\hat{h}+n-1}_{\mathbb{Z}/2}(\hat{E}).$$

What are $\mathbb{Z}/2$-CW complex, Real circle bundle, c_1^R and $K_\pm$?

What are $\mathbb{Z}/2$-CW complex, Real circle bundle, c_1^R and $K_\pm$?

- A $\mathbb{Z}/2$-CW complex is a space with $\mathbb{Z}/2$-action ($\mathbb{Z}/2$-space), which has a structure like a CW complex. (For example, a smooth manifold with smooth $\mathbb{Z}/2$-action gives rise to a $\mathbb{Z}/2$-CW complex.)
- A Real circle bundle is a certain principal circle bundle on a $\mathbb{Z}/2$-space, which arises as the unit circle bundle of a complex line bundle with 'real structure' in the sense of Atiyah.
- c_1^R is a certain Chern class of Real circle bundles.

More details will follow.

What is $K_\pm$?

- $K_\pm$ is a variant of K-theory, defined by

$$K_{\pm}^{h+n}(X) = K_{\mathbb{Z}/2}^{h+n+1}(X \times \tilde{I}, X \times \partial \tilde{I}),$$

where $\tilde{I} = [-1, 1]$, the $\mathbb{Z}/2$-action $\tau : X \times \tilde{I} \to X \times \tilde{I}$ is given by $\tau(x, t) = (\tau_X(x), -t)$, and $h \in H^3_{\mathbb{Z}/2}(X; \mathbb{Z})$.

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- $K_{\pm}$ was originally introduced by Witten in a context of string theory.
- In a different notation, $K_{\pm}$ also appears in Rosenberg's Künneth theorem for $\mathbb{Z}/2$-equivariant K-theory,

More details will follow.

Main Theorem (again)

- Let X be a finite $\mathbb{Z}/2$-CW complex, and (E, h) a pair consisting of a Real circle bundle $\pi : E \to X$ and $h \in H^3_{\mathbb{Z}/2}(E; \mathbb{Z})$. Then, up to isomorphism, there uniquely exists a pair $(\hat{E}, \hat{h})$ consisting of a Real circle bundle $\hat{\pi} : \hat{E} \to X$ and $\hat{h} \in H^3_{\mathbb{Z}/2}(\hat{E}; \mathbb{Z})$ such that

$$\pi_* h = c_1^R(\hat{E}), \qquad \hat{\pi}_* \hat{h} = c_1^R(E), \qquad p^* h = \hat{p}^* \hat{h}.$$

- For (E, h) and $(\hat{E}, \hat{h})$ as above, there are isomorphisms

$$K^{h+n}_{\mathbb{Z}/2}(E) \xrightarrow{T} K^{\hat{h}+n-1}_{\pm}(\hat{E}), \quad K^{h+n}_{\pm}(E) \xrightarrow{T} K^{\hat{h}+n-1}_{\mathbb{Z}/2}(\hat{E}).$$

Related works

- If the $\mathbb{Z}/2$-action on X is free, then the main theorem recovers a result of D. Baraglia.
- His result is also generalized by Mathai and Rosenberg, in a different way.

Plan of talk

1. Introduction
2. Background from string
3. $H_{\pm}$
4. $K_{\pm}$
5. Point in the proof
6. Example
7. Closing

Background from string

- Superstring theory is a candidate of theory of everything, in which strings are the fundamental objects.
- There are five types of theories
  type I, type II (A, B), heterotic ($E_8 \times E_8$, $SO(32)$).
- These theories are related by various dualities.
- For example, IIA theory and IIB theory are originally formulated on the (Minkowski) spacetime $\mathbb{R}^{10}$.
- Then, by T-duality, IIA and IIB are 'equivalent', upon toroidal compactifications along 1-dimension:

$$\text{IIA on } \mathbb{R}^9 \times S^1 \overset{\text{T-dual}}{\longleftrightarrow} \text{IIB on } \mathbb{R}^9 \times S^1$$

D-brane and its charge

- D-branes are (at the 'classical' level) objects to which the ends of strings can be attached.
- D-branes are charged with respect to background Ramond-Ramond fields.
- K-theory arises as a home of the charges:

Type I	$KO(X)$
Type IIA	$K^1(X)$
Type IIB	$K^0(X)$

- We can anticipate that, by a string duality, possible D-branes and hence K-theories would be related.

From T-duality to topological T-duality

- This is the case for the T-duality:

$$\text{IIA on } \mathbb{R}^9 \times S^1 \overset{\text{T-dual}}{\longleftrightarrow} \text{IIB on } \mathbb{R}^9 \times S^1$$

- For any space X, we have

$$K^n(X \times S^1) \cong K^n(X) \oplus K^{n-1}(X),$$

so that, at least abstractly,

$$K^n(X \times S^1) \cong K^{n-1}(X \times S^1).$$

- An attempt to generalize the isomorphism above to non-trivial circle bundles on X lead Bouwknegt, Evslin and Mathai to the idea of topological T-duality.

Orbifold string theory

- Orbifolding is
  - a recipe producing a theory from a string theory.
  - to take into account a symmetry of string theory.
- Examples of symmetries:
  1. An action of a group G on $\mathbb{R}^{10}$.
  2. The $\mathbb{Z}/2$-symmetry Ω in type IIB theory, reversing the orientations of strings.
  3. The $\mathbb{Z}/2$-symmetry $(-1)^{F_L}$ in type II theory, acting according to the left moving spacetime fermion number.
- Generally, we combine symmetries to orbifold.
  (If Ω is included, orbifolding is called orientifolding.)

Orbifolding and K-theory

- Upon orbifolding, home of D-brane charges are modified.

$$\begin{aligned} G\text{-action}: &\quad K\text{-theory} \rightsquigarrow G\text{-equivariant } K\text{-theory}, \\ \Omega: &\quad K(X) \rightsquigarrow KR(X), \\ (-1)^{F_L}: &\quad K^n(X) \rightsquigarrow K^n_\pm(X), \end{aligned}$$

where $KR(X)$ is the K-theory of Real vector bundles, namely, complex vector bundles with 'real structure' in the sense of Atiyah.

- In this context, $K_\pm$ was originally introduced:

  E. Witten, "D-branes and K-theory".
  J. High Energy Phys. 1998, no. 12, Paper 19, 41 pp.

Orbifolding relates string theory

- **Like dualities, orbifolding happens to relate theories.**
  1. **the orbifolding of type IIB theory by Ω is equivalent to type I theory.**

     $$\text{Type IIB on } \mathbb{R}^{10} \ / \ \Omega \longleftrightarrow \text{Type I on } \mathbb{R}^{10}$$

  2. **the orbifolding of type IIB theory by $(-1)^{F_L}$ is equivalent to type IIA theory.**

     $$\text{Type IIB on } \mathbb{R}^{10} \ / \ (-1)^{F_L} \longleftrightarrow \text{Type IIA on } \mathbb{R}^{10}$$

- **The relations are compatible with those of K-theories: If $\mathbb{Z}/2$ acts on a space X trivially, then**

$$KR(X) \cong KO(X), \qquad K^n_\pm(X) \cong K^{n-1}(X).$$

Duality and orbifolding

- The compatibility of duality and orbifolding is an issue of physicists, and is tested in various cases.
- In some cases, they are not compatible: S-duality in IIB theory transforms Ω to $(-1)^{F_L}$, but:

$$\begin{array}{ccc} \text{Type IIB on } \mathbb{R}^{10} \ / \ \Omega & \not\leftrightarrow & \text{Type IIB on } \mathbb{R}^{10} \ / \ (-1)^{F_L} \\ \| & & \| \\ \text{Type I on } \mathbb{R}^{10} & & \text{Type IIA on } \mathbb{R}^{10} \end{array}$$

- A compatible case motivated the main theorem.
- Let T^ℓ be the ℓ-dimensional torus $\mathbb{R}^\ell/\mathbb{Z}^\ell$ with the $\mathbb{Z}/2$-action $I_\ell(\vec{x}) = -\vec{x}$.
- The orbifolding of IIA compactified on T^{2k} by I_{2k} is T-dual to that of IIB compactified on T^{2k} by $(-1)^{F_L} I_{2k}$.

Motivating duality and orbifolding

IIA on $\mathbb{R}^{10-2k} \times T^{2k}/I_{2k} \leftrightarrow$ IIB on $\mathbb{R}^{10-2k} \times T^{2k}/(-1)^{F_L} I_{2k}$

- This duality is consistent with K-theory: For any space X with $\mathbb{Z}/2$-action, there are isomorphisms

$$K^n_{\mathbb{Z}/2}(X \times T^\ell) \cong (K^n_{\mathbb{Z}/2}(X) \oplus K^{n-1}_{\pm}(X))^{\oplus 2^{\ell-1}},$$
$$K^n_{\pm}(X \times T^\ell) \cong (K^n_{\pm}(X) \oplus K^{n-1}_{\mathbb{Z}/2}(X))^{\oplus 2^{\ell-1}},$$

  so that, at least abstractly,

$$K^n_{\mathbb{Z}/2}(X \times T^\ell) \cong K^{n-1}_{\pm}(X \times T^\ell).$$

- In view of the 'usual' topological T-duality, we can anticipate the main theorem.

Main Theorem (again)

- Let X be a finite $\mathbb{Z}/2$-CW complex, and (E, h) a pair consisting of a Real circle bundle $\pi : E \to X$ and $h \in H^3_{\mathbb{Z}/2}(E; \mathbb{Z})$. Then, up to isomorphism, there uniquely exists a pair $(\hat{E}, \hat{h})$ consisting of a Real circle bundle $\hat{\pi} : \hat{E} \to X$ and $\hat{h} \in H^3_{\mathbb{Z}/2}(\hat{E}; \mathbb{Z})$ such that

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- For (E, h) and $(\hat{E}, \hat{h})$ as above, there are isomorphisms

$$K^{h+n}_{\mathbb{Z}/2}(E) \xrightarrow{T} K^{\hat{h}+n-1}_{\pm}(\hat{E}), \quad K^{h+n}_{\pm}(E) \xrightarrow{T} K^{\hat{h}+n-1}_{\mathbb{Z}/2}(\hat{E}).$$

Plan of talk

$H_\pm$

- $H_\pm$ is defined in a way similar to $K_\pm$.

Definition

For a space X with $\mathbb{Z}/2$-action, $H^n_\pm(X) = H^n_\pm(X;\mathbb{Z})$ is defined as the following Borel equivariant cohomology:

$$\begin{aligned} H^n_\pm(X) &= H^{n+1}_{\mathbb{Z}/2}(X \times \tilde{I}, X \times \partial\tilde{I}; \mathbb{Z}) \\ &= H^{n+1}(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (X \times \tilde{I}), E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (X \times \partial\tilde{I}); \mathbb{Z}), \end{aligned}$$

where $\tilde{I} = [-1,1]$, the $\mathbb{Z}/2$-action $\tau : X \times \tilde{I} \to X \times \tilde{I}$ is given by $\tau(x,t) = (\tau_X(x), -t)$, and $E\mathbb{Z}_2 \to B\mathbb{Z}_2$ is the universal $\mathbb{Z}/2$-bundle.

Basic properties of $H_{\pm}$ to be explained

- $H_{\pm}$ has a multiplicative structure;
- $H_{\pm}$ constitutes an equivariant cohomology theory;
- $H_{\pm}$ is a cohomology with local coefficients;
- Thom isomorphism for Real line bundles;
- Classification of Real circle bundles.

$H_\pm$ has a multiplicative structure

- $H^*_{\mathbb{Z}/2}(X) = \bigoplus_n H^n_{\mathbb{Z}/2}(X)$ forms a (graded) ring.
- In particular, $H^*_{\mathbb{Z}/2}(X)$ is a module over

$$H^*_{\mathbb{Z}/2}(\mathrm{pt}) = \mathbb{Z}[t]/(2t),$$

where $t \in H^2_{\mathbb{Z}/2}(\mathrm{pt}) = \mathbb{Z}/2$ is the generator.

n	0	1	2	3	4	5	6	7
$H^n_{\mathbb{Z}/2}(\mathrm{pt})$	$\mathbb{Z}$	0	$\mathbb{Z}_2 t$	0	$\mathbb{Z}_2 t^2$	0	$\mathbb{Z}_2 t^3$	0

$H_\pm$ has a multiplicative structure

- $H^*_\pm(X)$ is a module over $H^*_{\mathbb{Z}/2}(X)$ and $H^*_{\mathbb{Z}/2}(\mathrm{pt})$.
- Further, there is a multiplication

$$\cup : H^n_\pm(X) \times H^m_\pm(X) \to H^{n+m}_{\mathbb{Z}/2}(X).$$

- Consequently, we get a $\mathbb{Z} \oplus \mathbb{Z}_2$-graded ring:

$$\mathbb{H}^*(X) := H^*_{\mathbb{Z}/2}(X) \oplus H^*_\pm(X),$$

which is a module over

$$\mathbb{H}^*(\mathrm{pt}) \cong \mathbb{Z}[t^{1/2}]/(2t^{1/2}),$$

where $t^{1/2} \in H^1_\pm(\mathrm{pt}) = \mathbb{Z}/2$ is the generator.

n	0	1	2	3	4	5
$H^n_{\mathbb{Z}/2}(\mathrm{pt})$	$\mathbb{Z}$	0	$\mathbb{Z}_2 t$	0	$\mathbb{Z}_2 t^2$	0
$H^n_\pm(\mathrm{pt})$	0	$\mathbb{Z}_2 t^{1/2}$	0	$\mathbb{Z}_2 t^{3/2}$	0	$\mathbb{Z}_2 t^{5/2}$

$H_\pm$ constitutes an equivariant cohomology theory

- Since $H^*_{\mathbb{Z}/2}(X)$ constitutes a generalized cohomology theory, so does $H^*_\pm(X)$. (The homotopy, excision, exactness and additivity axioms are satisfied.)
- Aside the axioms above, there are the exact sequences:

$$\cdots \to H^n_{\mathbb{Z}_2}(X) \xrightarrow{f} H^n(X) \to H^n_\pm(X) \xrightarrow{\delta} H^{n+1}_{\mathbb{Z}_2}(X) \to \cdots,$$
$$\cdots \to H^n_\pm(X) \xrightarrow{f} H^n(X) \to H^n_{\mathbb{Z}_2}(X) \xrightarrow{\delta} H^{n+1}_\pm(X) \to \cdots,$$

where f is to 'forget' the $\mathbb{Z}/2$-action, and $\delta = t^{1/2}\cup$.

n	0	1	2	3	4	5
$H^n_{\mathbb{Z}/2}(\mathrm{pt})$	$\mathbb{Z}$	0	$\mathbb{Z}_2 t$	0	$\mathbb{Z}_2 t^2$	0
$H^n(\mathrm{pt})$	$\mathbb{Z}$	0	0	0	0	0
$H^n_\pm(\mathrm{pt})$	0	$\mathbb{Z}_2 t^{1/2}$	0	$\mathbb{Z}_2 t^{3/2}$	0	$\mathbb{Z}_2 t^{5/2}$

$H_\pm$ as cohomology with local coefficients

- Consider the $\mathbb{Z}/2$-equivariant real line bundle $\underline{\mathbb{R}}_1 = X \times \mathbb{R}$ on X, whose $\mathbb{Z}/2$-action is $(x, t) \mapsto (\tau_X(x), -t)$.
- The Thom isomorphism theorem for $\underline{\mathbb{R}}_1$ gives us:

There is a natural isomorphism

$$\begin{aligned} H^n_\pm(X) &:= H^{n+1}_{\mathbb{Z}/2}(X \times \tilde{I}, X \times \partial\tilde{I}; \mathbb{Z}) \\ &\cong H^n_{\mathbb{Z}/2}(X; \mathbb{Z}(1)) := H^n(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} X; \mathbb{Z}(1)), \end{aligned}$$

where $\mathbb{Z}(1)$ is the $\mathbb{Z}/2$-module whose underlying group is $\mathbb{Z}$ and $\pi_1(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} X)$ acts non-trivially through the homotopy exact sequence for the fibration $X \to E\mathbb{Z}_2 \times_{\mathbb{Z}_2} X \to B\mathbb{Z}_2$:

$$\pi_1(X) \to \pi_1(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} X) \to \pi_1(B\mathbb{Z}_2) = \mathbb{Z}_2.$$

Thom isomorphism for Real line bundles

- Recall 'Real vector bundles' in the sense of Atiyah:

Definition

A complex vector bundle with real structure, or a Real vector bundle, on a $\mathbb{Z}/2$-space X is a complex vector bundle $\pi : V \to X$ equipped with a lift $\tau : V \to V$ of the $\mathbb{Z}/2$-action on X such that:

$$\tau^2(v) = v, \qquad \tau(z_1 v_1 + z_2 v_2) = \bar{z}_1 \tau(v_1) + \bar{z}_2 \tau(v_2)$$

for all $x \in X$, $v, v_i \in \pi^{-1}(x)$ and $z_i \in \mathbb{C}$.

- For example, the trivial Real vector bundle is defined by $X \times \mathbb{C}^n$ with its $\mathbb{Z}/2$-action $\tau(x, z) = (\tau_X(x), \bar{z})$.

Thom isomorphism for Real line bundles

- For any Real line bundle $\pi : R \to X$, we have

$$w_1^{\mathbb{Z}/2}(R) = w_1^{\mathbb{Z}/2}(\underline{\mathbb{R}}_1) \in H^2_{\mathbb{Z}/2}(X; \mathbb{Z}/2).$$

Thom isomorphism

There are natural isomorphisms of $H^*_{\mathbb{Z}/2}(X)$-modules

$$H^n_{\mathbb{Z}/2}(X) \cong H^{n+2}_{\pm}(D(R), S(R)),$$
$$H^n_{\pm}(X) \cong H^{n+2}_{\mathbb{Z}/2}(D(R), S(R)),$$

where $D(R)$ and $S(R)$ are the disk and circle bundles of R, with respect to a $\mathbb{Z}/2$-invariant Hermitian metric on R.

- This induces the Gysin sequence for Real circle bundles.

Thom isomorphism for Real line bundles

Definition

A **Real circle bundle** $\pi : E \to X$ on a $\mathbb{Z}/2$-space X is a principal circle bundle equipped with a lift $\tau : E \to E$ of the $\mathbb{Z}/2$-action on X such that

$$\tau^2(\xi) = \xi, \qquad \tau(\xi u) = \tau(\xi)\bar{u}$$

for all $\xi \in E$ and $u \in S^1 \subset \mathbb{C}$.

- The unit sphere bundle of a Real line bundle is a Real circle bundle.
- Essentially, the notion of Real line bundle is equivalent to that of Real circle bundle.

Thom isomorphism for Real line bundles

Gysin sequence

For any Real circle bundle $\pi : E \to X$, there are natural exact sequences of $H^*_{\mathbb{Z}/2}(X)$-modules:

$$\cdots \to H^{n-2}_{\pm}(X) \xrightarrow{\chi_R} H^n_{\mathbb{Z}_2}(X) \xrightarrow{\pi^*} H^n_{\mathbb{Z}_2}(E) \xrightarrow{\pi_*} H^{n-1}_{\pm}(X) \to \cdots,$$

$$\cdots \to H^{n-2}_{\mathbb{Z}_2}(X) \xrightarrow{\chi_R} H^n_{\pm}(X) \xrightarrow{\pi^*} H^n_{\pm}(E) \xrightarrow{\pi_*} H^{n-1}_{\mathbb{Z}_2}(X) \to \cdots,$$

where $\chi_R = \chi_R(E) \in H^2_{\pm}(X)$ is the Euler class of R.

Classification of Real circle bundles

- It is well-known that complex line bundles (or principal circle bundles) on a space X are classified by $H^2(X;\mathbb{Z})$.
- It is also known that $\mathbb{Z}/2$-equivariant line bundles (or $\mathbb{Z}/2$-equivariant circle bundles) on a $\mathbb{Z}/2$-space are classified by $H^2_{\mathbb{Z}/2}(X;\mathbb{Z})$.

Proposition [Kahn, 1987]

Real line bundles are classified by $H^2_{\mathbb{Z}/2}(X;\mathbb{Z}(1)) \cong H^2_\pm(X)$.

- Consequently, Real circle bundles $\pi : E \to X$ on a $\mathbb{Z}/2$-space X are classified by $c_1^R(E) \in H^2_\pm(X)$.
- It holds that $c_1^R(E) = \chi_R(E)$.

Plan of talk

1. Introduction
2. Background from string
3. $H_\pm$
4. $K_\pm$ (Next)
5. Point in the proof
6. Example
7. Closing

$K_{\pm}$

Definition(again)

For a space X with $\mathbb{Z}/2$-action, $K^n_{\pm}(X)$ is defined by

$$K^n_{\pm}(X) = K^{n+1}_{\mathbb{Z}/2}(X \times \tilde{I}, X \times \partial\tilde{I}),$$

where $\tilde{I} = [-1, 1]$, the $\mathbb{Z}/2$-action $\tau : X \times \tilde{I} \to X \times \tilde{I}$ is given by $\tau(x, t) = (\tau_X(x), -t)$.

- $K_{\pm}$ has properties similar to $H_{\pm}$.
  - $K_{\pm}$ has a multiplicative structure;
  - $K_{\pm}$ constitutes an equivariant cohomology theory;
  - $K_{\pm}$ is a cohomology with local coefficients;
  - Thom isomorphism for Real line bundles;
- The similarity and the difference will be explained.

$K_\pm$ has a multiplicative structure

- As in the case of $H_\pm$, we have a ring.

$$\mathbb{K}^*(X) := K^*_{\mathbb{Z}/2}(X) \oplus K^*_\pm(X),$$

graded by $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ due to the Bott periodicity.

- On pt, we have

$$K^0_{\mathbb{Z}/2}(\mathrm{pt}) = R \cong \mathbb{Z}^2, \qquad K^1_{\mathbb{Z}/2}(\mathrm{pt}) = 0,$$
$$K^0_\pm(\mathrm{pt}) = 0, \qquad K^1_\pm(\mathrm{pt}) = R/J \cong \mathbb{Z},$$

where $R = \mathbb{Z}[t]/(t^2 - 1)$ is the representation ring of $\mathbb{Z}/2$, and $J = (1 + t)$ is an ideal in R.

- A generator $\sigma \in K^1_\pm(\mathrm{pt})$ satisfies the relations:

$$t\sigma = -\sigma, \qquad \sigma^2 = 1 - t,$$

so that there is a ring isomorphism

$$\mathbb{K}^*(\mathrm{pt}) \cong \mathbb{Z}[\sigma]/(\sigma^3 - 2\sigma).$$

$K_{\pm}$ constitutes an equivariant cohomology theory

- $K_{\pm}$ constitutes a generalized cohomology theory.
- There are the exact sequences:

$$\cdots \to K^n_{\mathbb{Z}_2}(X) \xrightarrow{f} K^n(X) \to K^n_{\pm}(X) \xrightarrow{\delta} K^{n+1}_{\mathbb{Z}_2}(X) \to \cdots,$$

$$\cdots \to K^n_{\pm}(X) \xrightarrow{f} K^n(X) \to K^n_{\mathbb{Z}_2}(X) \xrightarrow{\delta} K^{n+1}_{\pm}(X) \to \cdots,$$

where f is to 'forget' the $\mathbb{Z}/2$-action, and δ is to multiply the generator $\sigma \in K^1_{\pm}(\mathrm{pt}) \cong R/J \cong \mathbb{Z}$.

$K_\pm$ as cohomology with local coefficients

- $K_\pm$ is also a cohomology with local coefficients, known as twisted K-theory.
- In general, we use an element in the cohomology

$$H^1_{\mathbb{Z}/2}(X;\mathbb{Z}/2) \times H^3_{\mathbb{Z}/2}(X;\mathbb{Z})$$

  to twist $\mathbb{Z}/2$-equivariant K-theory $K^*_{\mathbb{Z}/2}(X)$.
- The Thom isomorphism theorem for the $\mathbb{Z}/2$-equivariant (real) line bundle $\underline{\mathbb{R}}_1 \to X$ provides us:

$$K^n_\pm(X) \cong K^{w_1^{\mathbb{Z}/2}(\underline{\mathbb{R}}_1)+n}_{\mathbb{Z}/2}(X)$$

- More generally, for $h \in H^3_{\mathbb{Z}/2}(X;\mathbb{Z})$, we have

$$K^{h+n}_\pm(X) \cong K^{w_1^{\mathbb{Z}/2}(\underline{\mathbb{R}}_1)+h+n}_{\mathbb{Z}/2}(X).$$

Thom isomorphism for Real line bundles

$\pi : R \to X$ a Real line bundle, $h \in H^3_{\mathbb{Z}/2}(X;\mathbb{Z})$.

Thom isomorphism

There are natural isomorphisms of $K^*_{\mathbb{Z}/2}(X)$-modules

$$K^{h+n}_{\mathbb{Z}/2}(X) \cong K_\pm^{\pi^*(W_3^{\mathbb{Z}/2}(R)+h)+n}(D(R), S(R)),$$

$$K^{h+n}_{\pm}(X) \cong K_{\mathbb{Z}/2}^{\pi^*(W_3^{\mathbb{Z}/2}(R)+h)+n}(D(R), S(R)).$$

- In general, $W_3^{\mathbb{Z}/2}(R) \in H^3_{\mathbb{Z}/2}(X;\mathbb{Z})$ has the expression:

$$W_3^{\mathbb{Z}/2}(R) = t^{1/2} \cup c_1^R(R).$$

- There is the corresponding Gysin sequence, in which the push-forward picks up $W_3^{\mathbb{Z}/2}$.

Plan of talk

Main Theorem (again)

- **Let X be a finite $\mathbb{Z}/2$-CW complex, and (E, h) a pair consisting of a Real circle bundle $\pi : E \to X$ and $h \in H^3_{\mathbb{Z}/2}(E; \mathbb{Z})$. Then, up to isomorphism, there uniquely exists a pair $(\hat{E}, \hat{h})$ consisting of a Real circle bundle $\hat{\pi} : \hat{E} \to X$ and $\hat{h} \in H^3_{\mathbb{Z}/2}(\hat{E}; \mathbb{Z})$ such that**

$$\pi_* h = c_1^R(\hat{E}), \qquad \hat{\pi}_* \hat{h} = c_1^R(E), \qquad p^* h = \hat{p}^* \hat{h}.$$

- **For (E, h) and $(\hat{E}, \hat{h})$ as above, there are isomorphisms**

$$K^{h+n}_{\mathbb{Z}/2}(E) \xrightarrow{T} K^{\hat{h}+n-1}_{\pm}(\hat{E}), \quad K^{h+n}_{\pm}(E) \xrightarrow{T} K^{\hat{h}+n-1}_{\mathbb{Z}/2}(\hat{E}).$$

- **The idea of the proof is parallel to those in works of Baraglia, Bunke and Schick.**

- The first part is shown by using Leray-Serre spectral sequences carefully.
- The second part constructs the module maps T first.

$$
\begin{aligned}
K^{h+n}_{\mathbb{Z}/2}(E) &\xrightarrow{p^*} K^{p^*h+n}_{\mathbb{Z}/2}(E \times_X \hat{E}) && \text{(pull-back)}\\
&\xrightarrow{\cong} K^{\hat{p}^*\hat{h}+n}_{\mathbb{Z}/2}(E \times_X \hat{E}) && (p^*h = \hat{p}^*\hat{h})\\
&\xrightarrow{\hat{p}_*} K^{\hat{h}+W_3^{\mathbb{Z}/2}(\hat{\pi}^*E)+n-1}_{\pm}(\hat{E}) && \text{(push-forward)}\\
&\xrightarrow{\cong} K^{\hat{h}+n-1}_{\pm}(\hat{E}). && ((-1)^*)
\end{aligned}
$$

$$
\begin{tikzcd}
 & E \times_X \hat{E} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\
E \arrow[dr, "\pi"'] & & \hat{E} \arrow[dl, "\hat{\pi}"] \arrow[loop right] \\
 & X &
\end{tikzcd}
\quad -1 \in S^1 \subset \mathbb{C}
$$

- To prove that T is an isomorphism, we use
  - the assumption that X is a $\mathbb{Z}/2$-CW complex,
  - the axioms of cohomology theory

  to reduce the problem to the case of pt.
- Then, it suffices to prove that

$$T: \ \mathbb{K}^*(S^1) \longrightarrow \mathbb{K}^*(S^1)$$

  is an isomorphism. ($\tau(u) = \bar{u}$ for $u \in S^1 \subset \mathbb{C}$.)
- This is directly verified.

$$\mathbb{K}^*(\mathrm{pt}) = \mathbb{Z}[\sigma]/(\sigma^3 - 3\sigma),$$
$$\mathbb{K}^*(S^1) = \mathbb{Z}[\sigma, \chi]/(\sigma^3 - 3\sigma, \chi^2 - \sigma\chi),$$

$$T(1) = (1 - \sigma^2)\chi, \qquad T(\chi) = 1 - \sigma\chi.$$

Example

- Let S^1 be the circle with the trivial $\mathbb{Z}/2$-action.
- Since $H^2_{\pm}(S^1) = \mathbb{Z}/2$, there are essentially two Real circle bundles:

$$\pi_0 : E_0 \to S^1, \qquad c_1^R(E_0) = 0,$$
$$\pi_1 : E_1 \to S^1. \qquad c_1^R(E_1) = c \neq 0$$

- There are essentially five pairs on S^1:

$$(E_0, 0) \quad (E_0, \pi_0^*(t^{1/2}c)) \quad (E_0, h_0) \qquad (E_1, 0) \quad (E_1, h_1)$$

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- There are essentially five pairs on S^1:

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$$K^0_{\mathbb{Z}/2}(E_0) \cong R \oplus R/J, \qquad K^1_{\mathbb{Z}/2}(E_0) \cong R \oplus R/J,$$
$$K^0_{\pm}(E_0) \cong R \oplus R/J, \qquad K^1_{\pm}(E_0) \cong R \oplus R/J.$$

$$K^{h+0}_{\mathbb{Z}/2}(E_0) \cong R/I, \qquad K^{h+1}_{\mathbb{Z}/2}(E_0) \cong R/J \oplus I/2I,$$
$$K^{h+0}_{\pm}(E_0) \cong R/J \oplus I/2I, \qquad K^{h+1}_{\pm}(E_0) \cong R/I,$$

where $h = \pi_0^*(t^{1/2}c)$.

$$R = \mathbb{Z}[t]/(t^2 - 1), \ I = (1 - t), \ J = (1 + t).$$

$$\begin{tikzcd}
(E_0, 0) \arrow[loop below] & (E_0, \pi_0^*(t^{1/2}c)) \arrow[loop below] & {\color{red}(E_0, h_0)} \arrow[r, leftrightarrow] & {\color{red}(E_1, 0)} & {\color{red}(E_1, h_1)} \arrow[loop below]
\end{tikzcd}$$

$$K^{h_0+0}_{\mathbb{Z}/2}(E_0) \cong R/I \oplus R/J, \qquad K^{h_0+1}_{\mathbb{Z}/2}(E_0) \cong R,$$
$$K^{h_0+0}_{\pm}(E_0) \cong R/I \oplus R/J, \qquad K^{h_0+1}_{\pm}(E_0) \cong R.$$

$$K^{0}_{\mathbb{Z}/2}(E_1) \cong R, \qquad K^{1}_{\mathbb{Z}/2}(E_1) \cong R/I \oplus R/J,$$
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$$K^{h_1+0}_{\mathbb{Z}/2}(E_1) \cong R/I, \qquad K^{h_1+1}_{\mathbb{Z}/2}(E_1) \cong R/I,$$
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Plan of talk

Closing: a possible topological T-duality

- **In the main theorem, a Real circle bundle comes as a T-dual of a Real circle bundle.**

Real

$$\begin{tikzcd} E \arrow[dr, "\pi"'] & & \hat{E} \arrow[dl, "\hat{\pi}"] \\ & X & \end{tikzcd}$$

Real

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- In the main theorem, a Real circle bundles comes as a T-dual of a Real circle bundle.
- It would be possible to generalize the proof to get a '$\mathbb{Z}/2$-equivariant topological T-duality' in which:
  1. a $\mathbb{Z}/2$-equivariant circle bundle comes as a T-dual of a $\mathbb{Z}/2$-equivariant circle bundle;
  2. $\mathbb{Z}/2$-equivariant K-groups of their total spaces are isomorphic.

$$\underset{\mathbb{Z}/2\text{-equivariant}}{E} \xrightarrow{\pi} X \xleftarrow{\hat{\pi}} \underset{\mathbb{Z}/2\text{-equivariant}}{\hat{E}}$$

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- In these dualities, 'Real world' and 'equivariant world' are parallel.

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  2. $\mathbb{Z}/2$-equivariant K-groups of their total spaces are isomorphic.
- In these dualities, 'Real world' and 'equivariant world' are parallel.
- Is there a duality mixing these two parallel worlds?

$\mathbb{Z}/2$-equivariant

$$\begin{tikzcd} E \arrow[dr, "\pi"'] & & \hat{E} \arrow[dl, "\hat{\pi}"] \\ & X & \end{tikzcd}$$

Real

- **If there were such a duality, the third cohomology classes on the total spaces should live in $H_\pm$:**

$$\begin{array}{ccc} (E, h) & \longleftrightarrow & (\hat{E}, \hat{h}) \\ \begin{cases} E : \mathbb{Z}/2\text{-equiv} \\ h \in H^3_\pm(E; \mathbb{Z}) \end{cases} & & \begin{cases} \hat{E} : \text{Real} \\ \hat{h} \in H^3_\pm(\hat{E}; \mathbb{Z}) \end{cases} \end{array}$$

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- KR-theory can be twisted by $H^3_\pm$. [Moutuou]
- There may exist an isomorphism:

$$T : \ KR^{h+n}(E) \longrightarrow KR^{\hat{h}+n-1}(\hat{E}).$$

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- **The anticipation is justified by two examples:**
  - **The trivial case:**
$$\begin{cases} E = X \times S^1 \text{ the trivial } \mathbb{Z}/2\text{-equivariant circle bundle,} \\ \hat{E} = X \times \tilde{S}^1 \text{ the trivial Real circle bundle,} \\ h = 0, \hat{h} = 0. \end{cases}$$

$$KR^n(X \times S^1) \cong KR^n(X) \oplus KR^{n-1}(X),$$
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  - **The case where the $\mathbb{Z}/2$-action is free [Baraglia].**

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  - The case where the $\mathbb{Z}/2$-action is free [Baraglia].

Thank you very much.