Introduction to measurable rigidity of mapping class groups

Yoshikata Kida

Mathematical Institute, Tohoku University
Sendai 980-8758, Japan
email: kida@math.tohoku.ac.jp

2000 Mathematics Subject Classification: 20F38, 37A20.

Keywords: Mapping class groups, measure equivalence, orbit equivalence, lattice embeddings, discrete measured groupoids.

Contents

1 Introduction

A subgroup of a locally compact second countable group G is said to be a *lattice* if it is discrete in G and has cofinite measure with respect to the Haar measure on G. Classically, it is a basic problem to consider whether a lattice in a Lie group determines its ambient Lie group. More precisely, when Γ and Λ are lattices in Lie groups G and H, respectively, it is an interesting problem to study when the existence of an isomorphism between Γ and Λ implies the existence of an isomorphism between G and H. The Mostow-Prasad-Margulis rigidity theorem gives a complete answer to this question for semisimple Lie groups in a more sophisticated form.

In this chapter, by a discrete group we mean a discrete and countable group. Let us consider the following question: Given two discrete groups, under what conditions can they be realized as lattices in the same locally compact second countable group? Suppose that discrete groups Γ, Λ are lattices in a locally compact second countable group G. We shall observe some consequences of this situation. Consider the action of $\Gamma \times \Lambda$ on G given by

$$(\gamma, \lambda)g = \gamma g \lambda^{-1}, \quad \gamma \in \Gamma, \ \lambda \in \Lambda, \ g \in G.$$

It is easy to check the following:

- The actions $\Gamma(\simeq \Gamma \times \{e\}) \curvearrowright G$ and $\Lambda(\simeq \{e\} \times \Lambda) \curvearrowright G$ are both measure-preserving with respect to the (left) Haar measure m on G. Note that the existence of a lattice in G implies the invariance of m under right multiplication by each element of G.
- The action $\Gamma \curvearrowright G$ is free and admits a *fundamental domain* of finite measure, i.e., a Borel subset $F \subset G$ such that $m(F) < \infty$, $\bigcup_{\gamma \in \Gamma} \gamma F = G$, and $m(\gamma_1 F \cap \gamma_2 F) = 0$ for any distinct $\gamma_1, \gamma_2 \in \Gamma$. We can say the same thing for the action $\Lambda \curvearrowright G$.

In a general situation of the above one, Gromov introduced the notion of measure equivalence as follows.

Definition 1.1 ([23, 0.5.E]). We say that two discrete groups Γ and Λ are *measure equivalent (ME)* if there exists a measure-preserving action of $\Gamma \times \Lambda$ on a standard Borel space (Σ, m) with a σ-finite positive measure such that both of the actions $\Gamma(\simeq \Gamma \times \{e\}) \curvearrowright \Sigma$ and $\Lambda(\simeq \{e\} \times \Lambda) \curvearrowright \Sigma$ are essentially free and admit a fundamental domain of finite measure. The space (Σ, m) (equipped with the $(\Gamma \times \Lambda)$-action) is then called a *ME coupling* of Γ and Λ.

A *standard Borel space* is a Borel space arising from a separable complete metric space (see [34] for details of standard Borel spaces). An action of a discrete group on a measure space is said to be *essentially free* if the stabilizers of almost all points are trivial.

It is easy to see that ME defines an equivalence relation among discrete groups (see Section 2 in [16] or Remark 3.8 in this chapter). In the study of ME, it is fundamental to classify various discrete groups up to ME and to determine completely the class consisting of all discrete groups ME to a given group. We give three typical examples of ME couplings.

Example 1.2. Let G be a locally compact second countable group equipped with the Haar measure and let Γ, Λ be lattices in G. The action of $\Gamma \times \Lambda$ on G given by

$$(\gamma, \lambda)g = \gamma g \lambda^{-1}, \quad \gamma \in \Gamma, \ \lambda \in \Lambda, \ g \in G$$

defines an ME coupling of Γ and Λ.

Example 1.3. This is a special case of the above example. Let Γ be a discrete group and let Λ be a finite index subgroup of Γ. The action of $\Gamma \times \Lambda$ on Γ given by

$$(\gamma, \lambda)\gamma' = \gamma\gamma'\lambda^{-1}, \quad \gamma, \gamma' \in \Gamma, \ \lambda \in \Lambda$$

defines an ME coupling of Γ and Λ, where the measure on Γ is the counting one.

Example 1.4. Let Γ be a discrete group and let N be a finite normal subgroup of Γ. Choose an essentially free, measure-preserving action of Γ on a standard Borel space X with a finite positive measure (e.g., the Bernoulli action $\Gamma \curvearrowright \prod_\Gamma [0,1]$ when Γ is infinite). Then the action of $\Gamma \times (\Gamma/N)$ on $X \times (\Gamma/N)$ given by

$$(\gamma, \lambda)(x, \lambda') = (\gamma x, p(\gamma)\lambda'\lambda^{-1}), \quad \gamma \in \Gamma, \ \lambda, \lambda' \in \Gamma/N, \ x \in X$$

defines an ME coupling of Γ and Γ/N, where $p\colon \Gamma \to \Gamma/N$ is the quotient homomorphism. Note that we can find a fundamental domain F for the action $N \curvearrowright X$ since N is finite. It is easy to see that $F \times \{eN\} \subset \Gamma \times (\Gamma/N)$ is a fundamental domain for the action $\Gamma(\simeq \Gamma \times \{e\}) \curvearrowright X \times (\Gamma/N)$.

Commensurability up to finite kernels is the equivalence relation for discrete groups defined by declaring two groups in an exact sequence $1 \to A \to B \to C \to 1$ of discrete groups to be equivalent if the third group is finite. It follows from the last two examples that two commensurable groups up to finite kernels are ME. In particular, all finite groups are ME. Conversely, it is easy to see that a discrete group ME to a finite group is also finite.

Measure equivalence can be viewed as a measure-theoretic analogue of quasi-isometry (QI) between finitely generated groups. It is known that two finitely generated groups are QI if and only if there exists a continuous $(\Gamma \times \Lambda)$-action on some locally compact space Ω such that both of the actions of

$\Gamma(\simeq \Gamma \times \{e\})$ and $\Lambda(\simeq \{e\} \times \Lambda)$ on Ω are properly discontinuous and cocompact (see 0.2.C in [23]). On the other hand, there are examples of two ME groups which are not QI, and examples of two QI groups which are not ME. For example, $\mathbb{Z}$ and $\mathbb{Z}^2$ are ME but not QI (see Theorem 1.8). It is known that Kazhdan's property (T) is invariant under ME (see Corollary 1.4 in [17]) and that there exist two finitely generated groups Γ_1, Γ_2 satisfying the following (see Section 3.6 in [6]): Γ_1 and Γ_2 are QI; and Γ_1 satisfies property (T), whereas Γ_2 does not satisfy property (T). Hence, Γ_1 and Γ_2 are not ME.

ME rigidity for mapping class groups. In this chapter, we study mapping class groups of compact orientable surfaces from the viewpoint of ME, and consider a locally compact second countable group containing a lattice isomorphic to mapping class groups.

Let M be a connected compact orientable surface of genus g and with p boundary components. The mapping class group $\Gamma(M)$ of M is defined to be the group of isotopy classes of all orientation-preserving diffeomorphisms of M. Let $\Gamma(M)^\diamond$ be the extended mapping class group of M, i.e., the group of isotopy classes of all diffeomorphisms of M. The group $\Gamma(M)^\diamond$ contains $\Gamma(M)$ as a subgroup of index 2. Let $\kappa(M) = 3g+p-4$ be the complexity of M and assume that $\kappa(M) > 0$. Let $C = C(M)$ be the curve complex of M, on which $\Gamma(M)^\diamond$ naturally acts (see Definition 2.1). We denote by $\mathrm{Aut}(C)$ the automorphism group of the simplicial complex C. It is known that the kernel of the natural homomorphism $\pi\colon \Gamma(M)^\diamond \to \mathrm{Aut}(C)$ and the index $[\mathrm{Aut}(C) : \pi(\Gamma(M)^\diamond)]$ are both finite (see Theorem 2.3). Our first aim in this chapter is to survey the proof of the following rigidity theorem for $\Gamma(M)$, which completely determines the class of discrete groups ME to $\Gamma(M)$.

Theorem 1.5 ([36, Theorem 1.1]). *Let M be a surface with $\kappa(M) > 0$. If a discrete group Λ is ME to the mapping class group $\Gamma(M)$, then there exists a homomorphism $\rho\colon \Lambda \to \mathrm{Aut}(C)$ such that the kernel of ρ and the index $[\mathrm{Aut}(C) : \rho(\Lambda)]$ are both finite.*

Our second aim is to survey the proof of the following theorem, which determines all locally compact second countable groups containing a lattice isomorphic to mapping class groups. The idea of this work relies on Furman's paper [18] about the same problem for higher rank lattices. To the best of our knowledge, there exists no natural topological group containing the mapping class group as a lattice other than the mapping class group itself. The following theorem assures this observation. It has already been known that the mapping class group for a surface with positive complexity is not isomorphic to a lattice in any semisimple Lie group, by a result due to Kaimanovich and Masur [33]. (They also showed that any sufficiently large subgroup of the mapping class group is not isomorphic to a lattice in a semisimple Lie group with real rank at least 2.)

Theorem 1.6 ([36, Theorem 1.4]). *Let M be a surface with $\kappa(M) > 0$ and let Γ be a finite index subgroup of $\Gamma(M)^\diamond$. Let G be a locally compact second countable group and let $\sigma\colon \Gamma \to G$ be a lattice embedding, that is, σ is an injective homomorphism such that $\sigma(\Gamma)$ is a lattice in G. Then the following assertions hold:*

(i) *There exists a continuous homomorphism $\Phi_0\colon G \to \mathrm{Aut}(C)$ such that $\Phi_0(\sigma(\gamma)) = \pi(\gamma)$ for any $\gamma \in \Gamma$, where $\pi\colon \Gamma(M)^\diamond \to \mathrm{Aut}(C)$ is the natural homomorphism.*

(ii) *Let K be the kernel of Φ_0 and let Γ act on K by conjugation via σ. Let $\rho\colon \Gamma \ltimes K \to G$ be the homomorphism defined by $\rho(k) = k$ for $k \in K$ and $\rho(\gamma) = \sigma(\gamma)$ for $\gamma \in \Gamma$. Then the kernel of ρ and the index $[G : \rho(\Gamma \ltimes K)]$ are both finite.*

In particular, G admits infinitely many connected components, and $\sigma(\Gamma)$ is cocompact in G.

This theorem says that there exists no interesting lattice embedding of the mapping class group into a locally compact second countable group.

Amenability of the action $\Gamma(M)^\diamond \curvearrowright \partial C$. This property plays an important role in the proof of Theorems 1.5 and 1.6. Let M be a surface with $\kappa(M) \geq 0$. It is known that the curve complex $C = C(M)$ is a hyperbolic metric space in the sense of Gromov, by a result due to Masur and Minsky [43]. See also Hamenstädt's proof in Volume I of this handbook [27]. Hence, we can construct the Gromov boundary ∂C of C, which is known to be non-empty. Then $\Gamma(M)^\diamond$ acts on ∂C continuously with respect to the topology on ∂C as the Gromov boundary of C. It can be shown that ∂C is a standard Borel space with respect to the σ-field of subsets of ∂C generated by this topology (see Proposition 3.10 in [35]). We refer to [39], [24], [27] for more details of the boundary ∂C. The action $\Gamma(M)^\diamond \curvearrowright \partial C$ admits the following remarkable property:

Theorem 1.7 ([35, Theorem 3.29]). *Let M be a surface with $\kappa(M) \geq 0$ and let C be the curve complex for M. Let μ be a probability measure on the Gromov boundary ∂C such that the action of $\Gamma(M)^\diamond$ on $(\partial C, \mu)$ is non-singular. Then the action $\Gamma(M)^\diamond \curvearrowright (\partial C, \mu)$ is amenable (in a measurable sense).*

Here, when we are given a Borel action of a discrete group Γ on a Borel space S equipped with a positive measure ν, we say that the action $\Gamma \curvearrowright (S, \nu)$ is *non-singular* if $\nu(\gamma A) = 0$ for any $\gamma \in \Gamma$ and any Borel subset A of S with $\nu(A) = 0$. Amenability of group actions on measure spaces was first introduced by Zimmer [62] as a generalization of amenability of groups. Once it is shown that some action of a group is amenable, there are many applications to the study of that group from various aspects (see Section 8). In Section 4, we discuss the notion of amenable actions of groups and Theorem 1.7. We will apply Theorem 1.7 in the proof of Theorem 5.10 to show that IA subgroupoids are amenable.

Short description of history. The first magnificent result on ME is due to Ornstein and Weiss. Following Dye's results [11], [12] on some amenable groups from the viewpoint of orbit equivalence, Ornstein and Weiss obtained the following result (see Section 4 for the definition and elementary facts about amenable groups).

Theorem 1.8 ([52]). *An infinite discrete group is ME to $\mathbb{Z}$ if and only if it is amenable. In particular, all infinite solvable groups are ME to each other.*

It is natural to consider lattices in various Lie groups from the viewpoint of ME because of Example 1.2. Based on Zimmer's cocycle superrigidity theorem [63], Furman established the following rigidity result for higher rank lattices.

Theorem 1.9 ([16]). *Let G be a connected simple Lie group of non-compact type with finite center and real rank at least 2. Let Γ be a lattice in G. If a discrete group Λ is ME to Γ, then there exists a homomorphism $\rho\colon \Lambda \to \mathrm{Aut}(\mathrm{Ad}G)$ such that $\ker\rho$ is finite and $\rho(\Lambda)$ is a lattice in $\mathrm{Aut}(\mathrm{Ad}G)$.*

Note that the kernel of the natural composed map $G \to \mathrm{Ad}G \to \mathrm{Aut}(\mathrm{Ad}G)$ and the index of the image of G in $\mathrm{Aut}(\mathrm{Ad}G)$ are both finite. Thanks to this result, the class of discrete groups ME to a lattice in G is completely determined. At present, these two theorems and Theorem 1.5 are the only results which completely describe the class of discrete groups ME to a given infinite group. It is known that there exist continuously many discrete groups ME to a non-abelian free group (see Theorem 2.27 in [48]). Although no group-theoretic characterization of the class of such groups is known, some non-trivial examples of groups in that class are known (see [21]).

Gaboriau [20] proved that the sequence $\{\beta_n(\Gamma)\}_{n\in\mathbb{N}}$ of ℓ^2-Betti numbers for a discrete group Γ is an invariant for ME in the following sense: If two discrete groups Γ and Λ are ME, then there exists a positive real number c such that $\beta_n(\Gamma) = c\beta_n(\Lambda)$ for all n. This fact leads to big progress in the classification problem of discrete groups up to ME because this numerical invariant is defined for all discrete groups and is computable for various discrete groups arising geometrically.

The theory of ME is deeply linked with the theory of orbit equivalence. In fact, Ornstein and Weiss's original theorem is formulated in terms of orbit equivalence. Moreover, orbit equivalence is closely related to the theory of von Neumann algebras. There are many noteworthy results around these fields. We recommend the reader to consult [21], [58], [60] and the references therein for recent development of these fields.

Organization of this chapter. In Section 2, we recall fundamentals of mapping class groups, groupoids, and ME. It is important to know the construction of an isomorphism between two discrete measured groupoids from an ME coupling. Thanks to this construction, we can handle the classification problem of ME as an algebraic problem of groupoids arising from measure-preserving actions of discrete groups on measure spaces. To analyze the groupoid arising from an action of the mapping class group, we study its subgroupoids. Many facts about subgroups of mapping class groups reviewed in this section will help us to proceed to the analysis of subgroupoids because a groupoid is a generalization of a group.

In Section 3, we give an outline of the proof of Theorems 1.5 and 1.6. The main step in the proof is to consider a self ME coupling of the mapping class group $\Gamma(M)$, i.e., an ME coupling of $\Gamma(M)$ and $\Gamma(M)$. This corresponds to considering an isomorphism between groupoids arising from two actions of $\Gamma(M)$. We first explain what we can say about such an isomorphism, which will be formulated in Theorem 3.6. Here, we give only its statement, and will explain its proof in subsequent sections. Assuming Theorem 3.6, we show that any self ME coupling of $\Gamma(M)$ can be reduced to a much simpler self ME coupling of $\Gamma(M)$. We explain how to deduce the rigidity results in Theorems 1.5 and 1.6 from such a reduction. As another direct application, we prove a rigidity result in terms of orbit equivalence.

In Section 4, we recall amenability of discrete measured groupoids. This notion is often utilized in the study of groupoids and plays an important role in this work.

From Section 5 to Section 7, we study subgroupoids of a groupoid $\mathcal{G}$ arising from a measure-preserving action of $\Gamma(M)$ on a standard Borel space with a finite positive measure. In Section 5, we classify subgroupoids of $\mathcal{G}$, following the classification of subgroups of $\Gamma(M)$ due to McCarthy and Papadopoulos [45]. We introduce two types of subgroupoids of $\mathcal{G}$, which are called IA and reducible ones, respectively. In Section 6, we recall the definition of normal subgroupoids of a discrete measured groupoid, and study the normalizers in $\mathcal{G}$ of an IA or reducible subgroupoid. In Section 7, using results shown in the previous sections, we characterize various reducible subgroupoids in terms of amenability and normal subgroupoids. This characterization makes it possible to study an isomorphism between groupoids arising from two actions of $\Gamma(M)$ and to prove Theorem 3.6.

Finally, in Section 8, we briefly explain other related results shown in the series of papers [35], [36], [37], [38].

Acknowledgements. The author would like to express his deep gratitude to Athanase Papadopoulos and Charles Boubel for reading the first version of this chapter very carefully. Thanks to their valuable comments, this chapter was greatly improved. This chapter was written during the stay at Max Planck

Institute for Mathematics in Bonn. The author wishes to thank the institute for its warm hospitality.

2 Preliminaries

2.1 Mapping class groups

In this subsection, we recall fundamental facts about mapping class groups and several geometric objects related to them. We refer the reader to [13], [30], [32] or Sections 3.1, 3.2, 4.3 and 4.5 in [35] and the references therein for the material of this subsection. Chapter 8 of Volume I of this handbook ([49]) also deals with this material.

Let $M = M_{g,p}$ be a connected, compact and orientable surface of *type* (g, p), that is, of genus g and with p boundary components. Throughout the chapter, a surface is assumed to be connected, compact and orientable unless otherwise stated. Let $\Gamma(M)$ be the *mapping class group* of M, i.e., the group of isotopy classes of all orientation-preserving diffeomorphisms of M. The *extended mapping class group* $\Gamma(M)^{\diamond}$ of M is the group of isotopy classes of all diffeomorphisms of M, which contains $\Gamma(M)$ as a subgroup of index 2. Let $\kappa(M) = 3g+p-4$ be the *complexity* of M. We recall two geometric objects, the curve complex and the Thurston boundary, on which $\Gamma(M)^{\diamond}$ naturally acts.

The curve complex C. This simplicial complex was introduced by Harvey [28] and plays an indispensable role in this chapter. We recall some fundamental properties.

Definition 2.1. For a surface M, let $V(C) = V(C(M))$ be the set of all non-trivial isotopy classes of non-peripheral simple closed curves on M. Here, a simple closed curve on M is said to be non-peripheral if it is not isotopic to any boundary component of M. Let $S(M)$ denote the set of all non-empty finite subsets of $V(C)$ which can be realized disjointly on M at the same time.

When $\kappa(M) > 0$, we define the *curve complex* $C = C(M)$ as a simplicial complex such that the set of vertices is $V(C)$, and the set of simplices is $S(M)$.

When $\kappa(M) = 0$, that is, when M is either of type $(1,1)$ or $(0,4)$, we define the *curve complex* $C = C(M)$ as the one-dimensional simplicial complex such that the set of vertices is $V(C)$, the set of edges is defined as follows: A pair $\{\alpha, \beta\}$ of two distinct elements of $V(C)$ forms an edge if α and β have the lowest possible intersection number, that is, 1 for $M_{1,1}$ and 2 for $M_{0,4}$.

When $M = M_{0,3}$, let $C = C(M)$ be the empty set. For other surfaces, we do not need to define curve complexes because such surfaces do not appear as components of the surface obtained by cutting a surface with non-negative complexity along disjoint and mutually non-isotopic curves.

We immediately see that the curve complex is locally infinite (if it is non-empty). Although it is often difficult to treat the curve complex because of this property, it admits the following remarkable property.

Theorem 2.2 ([43], [47]). *If M is a surface with $\kappa(M) \geq 0$, then the curve complex $C = C(M)$ is connected. Moreover, when C is equipped with the natural simplicial metric, it has infinite diameter and is hyperbolic in the sense of Gromov.*

Let $\mathrm{Aut}(C)$ be the automorphism group of the simplicial complex C. Note that since $\Gamma(M)^\diamond$ acts on C simplicially, there is a natural homomorphism $\pi\colon \Gamma(M)^\diamond \to \mathrm{Aut}(C)$. It is natural to ask whether this natural homomorphism is an isomorphism or not. The following theorem answers this question completely in the case of $\kappa(M) > 0$. We refer to [40], [42], [47] for the case of $\kappa(M) = 0$.

Theorem 2.3 ([31], [40], [42]). *Let M be a surface with $\kappa(M) > 0$.*

(i) *If M is neither $M_{1,2}$ nor $M_{2,0}$, then π is an isomorphism.*

(ii) *If $M = M_{1,2}$, then the image of π is a subgroup of* $\mathrm{Aut}(C)$ *with index* 5 *and* $\ker(\pi)$ *is the subgroup generated by a hyperelliptic involution, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

(iii) *If $M = M_{2,0}$, then π is surjective and* $\ker(\pi)$ *is the subgroup generated by a hyperelliptic involution, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

(iv) *The two simplicial complexes $C(M_{0,5})$ and $C(M_{1,2})$ (resp. $C(M_{0,6})$ and $C(M_{2,0})$) are isomorphic.*

The Thurston boundary $\mathcal{PMF}$. Here, we recall some important facts on the Thurston boundary. We recommend the reader to consult [13] for details and proofs of the following facts.

Let M be a surface with $\kappa(M) \geq 0$ and let $\mathcal{R}(M)$ be the set of all non-negative real valued functions on $V(C)$, endowed with the product topology. We denote by $\mathcal{PR}(M)$ the quotient space of $\mathcal{R}(M) \setminus \{0\}$ by the natural diagonal action of the multiplicative group $\mathbb{R}^*_{>0}$ of all positive real numbers. Let $i\colon V(C) \times V(C) \to \mathbb{N}$ be the minimal geometric intersection number among representatives of two elements of $V(C)$. In particular, $i(\alpha,\alpha) = 0$ for all $\alpha \in V(C)$. For each $\alpha \in V(C)$, we can define an element of $\mathcal{R}(M) \setminus \{0\}$ by the function $V(C) \ni \beta \mapsto i(\alpha,\beta)$. The induced map $V(C) \to \mathcal{R}(M)$ is then injective. The closure of $\mathbb{R}^*_{>0} \cdot V(C)$ in $\mathcal{R}(M)$ is denoted by $\mathcal{MF} = \mathcal{MF}(M)$, and it is called the space of *measured foliations* on M. This space $\mathcal{MF}$ is homeomorphic to $\mathbb{R}^{6g-6+2p}$. In fact, it is known that each element of $\mathcal{MF}$

can be identified with a foliation with some singularities on M equipped with a transverse measure. The way to identify an element of $\mathcal{MF}$ and a measured foliation is not immediate. The reader should be referred to Exposé 5 in [13] for this identification.

Moreover, the composed map $V(C) \to \mathcal{R}(M) \setminus \{0\} \to \mathcal{PR}(M)$ is also injective. The closure of the image is denoted by $\mathcal{PMF} = \mathcal{PMF}(M)$, and it is called the *Thurston boundary* or the space of *projective measured foliations* on M. This space $\mathcal{PMF}$ is homeomorphic to the sphere of dimension $6g - 7 + 2p$. It is known that $S(M)$ can also naturally be embedded into $\mathcal{PMF}$ by using the minimal geometric intersection number $i\colon S(M) \times V(C) \to \mathbb{N}$ among representatives of elements of $S(M)$ and $V(C)$. This function i can be continuously extended to a function $\mathcal{MF} \times \mathcal{MF} \to \mathbb{R}_{\geq 0}$ which is $\mathbb{R}^*_{>0}$-homogeneous in the following sense:

$$i(r_1F_1, r_2F_2) = r_1r_2i(F_1, F_2)$$

for any $r_1, r_2 \in \mathbb{R}^*_{>0}$ and $F_1, F_2 \in \mathcal{MF}$. Hence, for two elements $F_1, F_2 \in \mathcal{PMF}$, whether $i(F_1, F_2) = 0$ or $\neq 0$ makes sense. As $\mathcal{R}(M)$ is endowed with the product topology, the group $\Gamma(M)^\diamond$ acts continuously on both $\mathcal{MF}$ and $\mathcal{PMF}$, and the equation

$$i(gF_1, gF_2) = i(F_1, F_2)$$

holds for any $g \in \Gamma(M)^\diamond$ and $F_1, F_2 \in \mathcal{MF}$ (or $\mathcal{PMF}$). Let

$$\mathcal{MIN} = \{F \in \mathcal{PMF} : i(F, \alpha) \neq 0 \text{ for any } \alpha \in V(C)\}$$

be the set of all *minimal measured foliations* on M, which is a $\Gamma(M)^\diamond$-invariant Borel subset of $\mathcal{PMF}$.

Each point of the Teichmüller space $\mathcal{T} = \mathcal{T}(M)$ also defines an element of $\mathcal{R}(M) \setminus \{0\}$. Indeed, once chosen a hyperbolic metric on M, there is exactly one geodesic in each free homotopy class of closed, non-peripheral curves on M. The lengths of these geodesics give a map $\mathcal{T} \to \mathcal{R}(M) \setminus \{0\}$. The induced map $\mathcal{T} \to \mathcal{PR}(M)$ is then injective, and $\mathcal{PMF}$ forms the boundary of the image of this map. The disjoint union $\overline{\mathcal{T}} = \mathcal{T} \cup \mathcal{PMF}$ is called the *Thurston compactification* of the Teichmüller space, which is homeomorphic to a closed Euclidean ball of dimension $6g-6+2p$ whose boundary corresponds to $\mathcal{PMF}$.

For $g \in \Gamma(M)$, let us denote by

$$\mathrm{Fix}(g) = \left\{x \in \overline{\mathcal{T}} : gx = x\right\}$$

the fixed point set of g. Each element $g \in \Gamma(M)$ is classified as follows in terms of its fixed points on $\overline{\mathcal{T}}$ (see Exposé 9, §V, Théorème and Exposé 11, §4, Théorème in [13]):

Theorem 2.4. *Let M be a surface with $\kappa(M) \geq 0$. Each element $g \in \Gamma(M)$ can be classified into the following three types:*

(i) *g has finite order and has a fixed point on $\mathcal{T}$.*

(ii) *g is* pseudo-Anosov*, that is,* Fix(g) *consists of exactly two points of $\mathcal{MIN}$.*

(iii) *g has infinite order and is* reducible*, that is, there exists $\sigma \in S(M)$ such that $g\sigma = \sigma$.*

Note that these three types are mutually exclusive. We say that $F \in \mathcal{PMF}$ is a *pseudo-Anosov foliation* if F is a fixed point for some pseudo-Anosov element. It is known that the set of all pseudo-Anosov foliations is dense in $\mathcal{PMF}$.

Dynamics of each element of $\Gamma(M)$ on $\mathcal{PMF}$. This information will help us to consider the problem of probability measures on $\mathcal{PMF}$ which are invariant for the action of a subgroup of $\Gamma(M)$ (see Subsection 5.1). Let M be a surface with $\kappa(M) \geq 0$. A pseudo-Anosov element $g \in \Gamma(M)$ has the following remarkable dynamics on $\overline{\mathcal{T}}$.

Theorem 2.5 ([32, Theorem 7.3.A]). *Let M be a surface with $\kappa(M) \geq 0$ and let $g \in \Gamma(M)$ be a pseudo-Anosov element. Then the two fixed points $F_{\pm}(g) \in \mathcal{MIN}$ of g satisfy the following: If U is an open neighborhood of $F_{+}(g)$ in $\overline{\mathcal{T}}$ and if K is a compact subset of $\overline{\mathcal{T}} \setminus \{F_{-}(g)\}$, then there exists $N \in \mathbb{N}$ such that $g^n(K) \subset U$ for all $n \geq N$.*

We call $F_{+}(g)$ (resp. $F_{-}(g)$) the *unstable* (resp. *stable*) *foliation* for g.

We next consider the dynamics of a reducible element. We say that $g \in \Gamma(M)$ is *pure* if the isotopy class g contains a diffeomorphism φ of M satisfying the following Condition (P):

We say that a diffeomorphism φ of M satisfies *Condition* (P) if there exists a closed one-dimensional submanifold c (may be empty) of M such that

- each component of c is neither homotopic on M to a point nor to ∂M;
- φ is the identity on c, and it does not rearrange the components of $M \setminus c$. Moreover, φ induces on each component of the surface M_c obtained by cutting M along c a diffeomorphism isotopic to either a pseudo-Anosov or the identity diffeomorphism.

We may assume that c does not have *superfluous components*, that is, we cannot discard any component of c without violating Condition (P). Note that if some component of c is on the boundary of two components on which the action of φ is isotopic to the identity, then the action of φ on the union of these two components is not necessarily isotopic to the identity. There exists a finite index subgroup of $\Gamma(M)$ consisting of pure elements (see Theorem 2.8 (i)).

Let $g \in \Gamma(M)$ be a pure element and let c be a one-dimensional submanifold of M satisfying Condition (P) for some diffeomorphism in the isotopy class g. Let $Q_1, \ldots, Q_n$ be the components of M_c on which g induces a pseudo-Anosov element, and let $F_+^1, \ldots, F_+^n$, $F_-^1, \ldots, F_-^n \in \mathcal{MF}$ be some representatives of the corresponding unstable and stable foliations. Let $\alpha_1, \ldots, \alpha_m$ be the isotopy classes of the components of c which are also boundary components of some Q_i. Let $\beta_1, \ldots, \beta_l$ be the isotopy classes of the remaining components of c. For $F \in \mathcal{MF}$, let $[F] \in \mathcal{PMF}$ denote the projection of F onto $\mathcal{PMF}$. Define two subsets Δ^u, Ψ^s of $\mathcal{PMF}$ by

$$\Delta^u = \left\{ \left[\sum_{i=1}^n m_i F_+^i + \sum_{j=1}^m a_j \alpha_j + \sum_{k=1}^l b_k \beta_k \right] \in \mathcal{PMF} : \right.$$

$$\left. m_i, a_j, b_k \geq 0,\ \sum_{i=1}^n m_i + \sum_{j=1}^m a_j + \sum_{k=1}^l b_k > 0 \right\},$$

$$\Psi^s = \left\{ [F] \in \mathcal{PMF} : i(F, F_-^i) = i(F, \beta_k) = 0 \text{ for all } i, k \right\}.$$

See Section 2.4 in [30] for the sum of disjoint foliations. These subsets Δ^u, Ψ^s are closed in $\mathcal{PMF}$. Moreover, if g is a reducible element of infinite order, then both Δ^u and Ψ^s are contained in $\mathcal{PMF} \setminus \mathcal{MIN}$ (see Corollary 2.16 in [30]). The following gives the behavior of the dynamics of a pure reducible element on $\mathcal{PMF}$.

Theorem 2.6 ([30, Theorem 3.5]). *Let M be a surface with $\kappa(M) \geq 0$ and let $g \in \Gamma(M)$ be a pure element. Let U be an open subset and let K be a compact subset of $\mathcal{PMF}$ such that $\Delta^u \subset U$ and $K \subset \mathcal{PMF} \setminus \Psi^s$. Then there exists $N \in \mathbb{N}$ such that $g^n(K) \subset U$ for all $n \geq N$.*

Classification of subgroups of $\Gamma(M)$. Let M be a surface with $\kappa(M) \geq 0$. Using the classification of elements of $\Gamma(M)$ in Theorem 2.4, McCarthy and Papadopoulos [45] classified subgroups of $\Gamma(M)$ as follows.

Theorem 2.7. *Let M be a surface with $\kappa(M) \geq 0$. Each subgroup Γ of $\Gamma(M)$ can be classified into the following four types:*

(i) *Γ is finite.*

(ii) *There exists a pseudo-Anosov element $g \in \Gamma$ such that $h\{F_\pm(g)\} = \{F_\pm(g)\}$ for any $h \in \Gamma$. In this case, Γ is virtually cyclic and we say that Γ is* IA *(= infinite, irreducible and amenable).*

(iii) *Γ is infinite and there exists $\sigma \in S(M)$ such that $g\sigma = \sigma$ for any $g \in \Gamma$.*

(iv) *There exist two pseudo-Anosov elements $g_1, g_2 \in \Gamma$ such that $\{F_\pm(g_1)\} \cap \{F_\pm(g_2)\} = \emptyset$. In this case, Γ contains a non-abelian free subgroup and is said to be* sufficiently large.

Note that these four types are mutually exclusive (use Theorem 2.5). A subgroup of $\Gamma(M)$ is said to be *reducible* if it fixes some element of $S(M)$.

Some special subgroups of finite index in $\Gamma(M)$. We finally introduce some finite index subgroups of $\Gamma(M)$ which satisfy nice properties. Since a discrete group and its finite index subgroup are ME as seen in Example 1.3, we may consider such special subgroups instead of $\Gamma(M)$ in the problem of ME. Thanks to the nice properties, many arguments technically get much easier.

For $\sigma \in S(M)$, we often denote by M_σ for simplicity the surface obtained by cutting M along a realization of curves in σ when a realization of σ is not specified. It is well-known that if $g \in \Gamma(M)$ satisfies the equation $g\sigma = \sigma$, then there exist a realization c of σ and a diffeomorphism φ of M whose isotopy class is g such that $\varphi(c) = c$ (see Theorem 5.2 in [41] for the proof). Then φ induces a diffeomorphism on the surface M_c obtained by cutting M along c. When φ preserves each component of M_c, we say that g preserves each component of M_σ. This definition depends only on the isotopy classes σ and g, and does not depend on the choice of c and φ. Likewise, we often identify an isotopy class and some representative of it for simplicity of the notation if no serious problem occurs. For an integer m, let $\Gamma(M; m)$ be the subgroup of $\Gamma(M)$ consisting of all elements which act trivially on the homology group $H_1(M; \mathbb{Z}/m\mathbb{Z})$. This subgroup has the following notable properties (see Theorem 1.2 and Corollaries 1.5, 1.8, 3.6 in [30]).

Theorem 2.8. *Let M be a surface with $\kappa(M) \geq 0$ and let $m \geq 3$ be an integer. Then the following assertions hold:*

(i) *$\Gamma(M; m)$ is a torsion-free subgroup of finite index in $\Gamma(M)$ and consists of pure elements.*

(ii) *If $g \in \Gamma(M; m)$ and $F \in \mathcal{PMF}$ satisfy $g^n F = F$ for some $n \in \mathbb{Z} \setminus \{0\}$, then $gF = F$.*

(iii) *If $g \in \Gamma(M; m)$ and $\sigma \in S(M)$ satisfy $g^n \sigma = \sigma$ for some $n \in \mathbb{Z} \setminus \{0\}$, then $g\alpha = \alpha$ for any $\alpha \in \sigma$, and g preserves each component of M_σ and preserves each component of the boundary of M.*

2.2 Discrete measured groupoids

This subsection is a short review of the notion of a discrete measured groupoid. We refer to [4], [5] and Chapter XIII, §3 in [59] for more details.

Measure theory. We first recall some basic terminology in measure theory. A *Borel space* X is a set equipped with a distinguished σ-field of subsets of X. A subset in the σ-field is called a *Borel subset*. A map $f\colon X \to Y$ between Borel spaces X and Y is said to be *Borel* if $f^{-1}(A)$ is a Borel subset of X for any Borel subset A of Y. In this chapter, we always assume a Borel space to be standard. A Borel space is *standard* if as a Borel space, it is isomorphic to a Borel space associated with a separable complete metric space. The following facts are known:

Theorem 2.9. (i) *If a Borel space X is a countable union of Borel subsets of X which are standard as a Borel space, then X is standard.*

(ii) *Any Borel subset of a standard Borel space is standard as a Borel space.*

(iii) *Any two standard Borel spaces with the same cardinality are isomorphic as a Borel space.*

(iv) *Let X, Y be standard Borel spaces and let $f\colon X \to Y$ be a Borel map such that $f^{-1}(y)$ is countable for each $y \in Y$. Then there exists a countable Borel partition $X = \bigsqcup_n X_n$ satisfying the following: Let f_n denote the restriction of f to X_n. The image $f_n(X_n)$ is a Borel subset of Y, and the map $f_n\colon X_n \to f_n(X_n)$ is a Borel isomorphism.*

We refer to 13.4, 15.6 and 18.14 in [34] for Assertions (ii), (iii) and (iv), respectively. The reader should consult [34] for more details of standard Borel spaces. By a *standard measure space* we mean a standard Borel space X equipped with a σ-finite positive measure μ. If μ is finite, i.e., if $\mu(X) < \infty$, then we say that (X, μ) is a *standard finite measure space*.

Let μ be a positive measure on a Borel space X. We say that a Borel subset A of X is (μ-)*null* (resp. *conull*) if $\mu(A) = 0$ (resp. $\mu(X \setminus A) = 0$). A property of points of X which holds for all x outside some μ-null Borel subset of X is said to hold for (μ-)*almost every* (or *a.e.*) $x \in X$. A point $x \in X$ with $\mu(\{x\}) > 0$ is called an *atom* for the measure space (X, μ). Two measures μ and ν on a Borel space X are said to be *equivalent* if the following holds: For a Borel subset A of X, $\mu(A) = 0$ if and only if $\nu(A) = 0$.

Let (X, μ), (Y, ν) be Borel spaces with a positive measure. By a *measure space isomorphism* $f\colon (X, \mu) \to (Y, \nu)$ we mean a Borel isomorphism $f\colon X' \to Y'$ between conull Borel subsets $X' \subset X$ and $Y' \subset Y$ such that $f_*\mu$ and ν are equivalent.

Groupoids. A groupoid is a generalization of a group. Given a set X, a groupoid $\mathcal{G}$ on X is, roughly speaking, the set of arrows whose end and initial points are in X satisfying several conditions. In the following definition, the maps $r, s\colon \mathcal{G} \to X$ assign to an arrow in $\mathcal{G}$ its end and initial points in X, respectively.

Definition 2.10. If two non-empty sets $\mathcal{G}$, X are equipped with two maps $r, s\colon \mathcal{G} \to X$ and the following operations, then $\mathcal{G}$ is called a *groupoid* on X:

(i) We put

$$\mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} : s(\gamma_1) = r(\gamma_2)\}.$$

There is a map $\mathcal{G}^{(2)} \ni (\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2 \in \mathcal{G}$ satisfying the two equations $r(\gamma_1\gamma_2) = r(\gamma_1)$ and $s(\gamma_1\gamma_2) = s(\gamma_2)$, and satisfying the associative law. The last condition means that the equation $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$ holds for all $(\gamma_1, \gamma_2), (\gamma_2, \gamma_3) \in \mathcal{G}^{(2)}$.

(ii) There is a map $X \ni x \mapsto e_x \in \mathcal{G}$ satisfying the following equations: $r(e_x) = s(e_x) = x$; $\gamma e_x = \gamma$ for any $\gamma \in \mathcal{G}$ with $s(\gamma) = x$; and $e_x\gamma' = \gamma'$ for any $\gamma' \in \mathcal{G}$ with $r(\gamma') = x$. It is easy to see that for each $x \in X$, e_x is an unique element of $\mathcal{G}$ satisfying these equations.

(iii) There is a map $\mathcal{G} \ni \gamma \mapsto \gamma^{-1} \in \mathcal{G}$ satisfying the following equations: $r(\gamma^{-1}) = s(\gamma)$; $s(\gamma^{-1}) = r(\gamma)$; $\gamma\gamma^{-1} = e_{r(\gamma)}$; and $\gamma^{-1}\gamma = e_{s(\gamma)}$. It is easy to see that for each $\gamma \in \mathcal{G}$, γ^{-1} is an unique element of $\mathcal{G}$ satisfying these equations.

In the above notation, X is called the *unit space*, and X is identified with the set of all units of $\mathcal{G}$ via the map $x \mapsto e_x$. The maps $r, s\colon \mathcal{G} \to X$ are called the *range*, *source* maps, respectively. For $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$, the element $\gamma_1\gamma_2 \in \mathcal{G}$ is called the *product* of two elements γ_1, γ_2. We refer to e_x as the *unit* on $x \in X$ and refer to γ^{-1} as the *inverse* of $\gamma \in \mathcal{G}$.

Consider a subset $\mathcal{H} \subset \mathcal{G}$ satisfying the following three conditions:

- If $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)} \cap (\mathcal{H} \times \mathcal{H})$, then $\gamma_1\gamma_2 \in \mathcal{H}$.
- If $\gamma \in \mathcal{H}$, then $\gamma^{-1} \in \mathcal{H}$.
- $e_x \in \mathcal{H}$ for all $x \in X$.

This subset $\mathcal{H}$ admits the structure of a groupoid on X induced from the one for $\mathcal{G}$. This groupoid $\mathcal{H}$ on X is called a *subgroupoid* of $\mathcal{G}$.

We say that a groupoid is *Borel* if all the associated spaces and maps are Borel. When we consider a Borel groupoid $\mathcal{G}$ on a standard Borel space, we always assume $\mathcal{G}$ to be also standard as a Borel space.

Notation. Let $\mathcal{G}$ be a groupoid on the unit space X with the range and source maps $r, s\colon \mathcal{G} \to X$, respectively. We denote by

$$I\colon \mathcal{G} \ni \gamma \mapsto \gamma^{-1} \in \mathcal{G}.$$

the inverse map. We write $\mathcal{G}^x = r^{-1}(x)$ and $\mathcal{G}_x = s^{-1}(x)$ for $x \in X$. Note that $\mathcal{G}^x = I(\mathcal{G}_x)$ for each $x \in X$. We say that $\mathcal{G}$ is *discrete* when $\mathcal{G}^x$ is countable for each $x \in X$. For $x, y \in X$, we write

$$\mathcal{G}^x_y = \{\gamma \in \mathcal{G} : r(\gamma) = x, s(\gamma) = y\}.$$

It is easy to see that for each $x \in X$, $\mathcal{G}^x_x$ admits the structure of a group induced from the structure of a groupoid on $\mathcal{G}$. This group $\mathcal{G}^x_x$ is called the isotropy group on $x \in X$.

Example 2.11. *Groups.* Let G be a group. Then G can be seen as a groupoid on the set consisting of a single point. Conversely, any groupoid on the set consisting of a single point is a group.

Example 2.12. *Equivalence relations.* Let X be a non-empty set. Let $\mathcal{R}$ be an equivalence relation on X, i.e., a subset of $X \times X$ satisfying the following three conditions:

- $(x, x) \in \mathcal{R}$ for all $x \in X$;
- If $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$;
- If $(x, y), (y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$.

Define two maps $r, s\colon \mathcal{R} \to X$ and the operations of products and inverses by

$$r(x, y) = x, \quad s(x, y) = y, \quad (x, y)(y, z) = (x, z), \quad (x, y)^{-1} = (y, x).$$

Then $\mathcal{R}$ is a groupoid on X. If each equivalence class for $\mathcal{R}$ is at most countable, then $\mathcal{R}$ is a discrete groupoid.

Measures on discrete Borel groupoids. If we are given a discrete Borel groupoid and a positive measure on the unit space, then we can define a natural measure on the groupoid as follows.

Definition 2.13. Given a discrete Borel groupoid $\mathcal{G}$ on a Borel space X, we say that a σ-finite positive measure μ on X is *quasi-invariant* for $\mathcal{G}$ if the two measures $\tilde{\mu}$ and $I_*\tilde{\mu}$ on $\mathcal{G}$ are equivalent. Here, the measure $\tilde{\mu}$ is defined by

$$\tilde{\mu}(A) = \int_X \sum_{\gamma \in \mathcal{G}_x} \chi_A(\gamma) d\mu(x),$$

for a Borel subset A of $\mathcal{G}$, where χ_A is the characteristic function on A. We say that μ is *invariant* for $\mathcal{G}$ if $I_*\tilde{\mu} = \tilde{\mu}$. A discrete Borel groupoid $\mathcal{G}$ equipped with a quasi-invariant measure μ on the unit space X is called a *discrete measured groupoid* on (X, μ). Given a discrete measured groupoid $\mathcal{G}$ on (X, μ), we always equip $\mathcal{G}$ with the measure $\tilde{\mu}$ defined above. This measure $\tilde{\mu}$ is a σ-finite positive measure on $\mathcal{G}$.

Notation. Let $\mathcal{G}$ be a discrete measured groupoid on a standard measure space (X, μ). If A is a Borel subset of X, then we denote by $\mathcal{G}A$ the *saturation* of A, which is the Borel subset of X defined by

$$\mathcal{G}A = \{r(\gamma) \in X : \gamma \in \mathcal{G}, s(\gamma) \in A\} = \{s(\gamma) \in X : \gamma \in \mathcal{G}, r(\gamma) \in A\}.$$

It can be shown that $\mathcal{G}A$ is a Borel subset of X and that $\mu(\mathcal{G}A) = 0$ when $\mu(A) = 0$ (use Theorem 2.9 (iv)). If $\mathcal{G}A = A$, then A is said to be *$\mathcal{G}$-invariant*. Note that if X' is a conull Borel subset of X, then $X' \setminus \mathcal{G}(X \setminus X')$ is a conull $\mathcal{G}$-invariant Borel subset of X contained in X'.

Definition 2.14. Let $\mathcal{G}$ be a discrete measured groupoid on a standard measure space (X, μ). If A is a Borel subset of X with positive measure, then the Borel subset

$$\{\gamma \in \mathcal{G} : r(\gamma), s(\gamma) \in A\}$$

has the natural structure of a groupoid on A induced from $\mathcal{G}$. This groupoid is called the *restriction* of $\mathcal{G}$ to A and is denoted by $(\mathcal{G})_A$.

Definition 2.15. Let $\mathcal{G}$, $\mathcal{H}$ be discrete measured groupoids on standard measure spaces (X, μ), (Y, ν), respectively. By a groupoid homomorphism $f\colon \mathcal{G} \to \mathcal{H}$ we mean a Borel map $f\colon (\mathcal{G})_A \to \mathcal{H}$ for some conull $\mathcal{G}$-invariant Borel subset A of X satisfying the following two conditions:

- $f_*\mu$ and ν are equivalent;
- f preserves the operation of products, i.e., the equation $f(\gamma_1\gamma_2) = f(\gamma_1)f(\gamma_2)$ holds for all $(\gamma_1, \gamma_2) \in ((\mathcal{G})_A)^{(2)}$.

When X is identified with the set of all units of $\mathcal{G}$, the map $f\colon (\mathcal{G})_A \to \mathcal{H}$ induces a Borel map $f\colon A \to Y$.

We do not distinguish two groupoid homomorphisms $f_1, f_2\colon \mathcal{G} \to \mathcal{H}$ such that $f_1 = f_2$ on $(\mathcal{G})_A$ for some conull $\mathcal{G}$-invariant Borel subset A of X.

Remark 2.16. For $i \in \{1, 2, 3\}$, let $\mathcal{G}_i$ be a discrete measured groupoid on a standard measure space (X_i, μ_i). Let $f\colon \mathcal{G}_1 \to \mathcal{G}_2$ and $g\colon \mathcal{G}_2 \to \mathcal{G}_3$ be groupoid homomorphisms. For $i \in \{1, 2\}$, take a conull $\mathcal{G}_i$-invariant Borel subset A_i of X_i such that f_i is defined on $(\mathcal{G}_i)_{A_i}$. It is easy to see that the Borel subset $A_1' = (A_1 \cap f^{-1}(A_2)) \setminus \mathcal{G}_1(A_1 \setminus f^{-1}(A_2))$ of X_1 is conull and $\mathcal{G}_1$-invariant. The composition of the two Borel maps $f\colon (\mathcal{G}_1)_{A_1'} \to \mathcal{G}_2$ and $g\colon (\mathcal{G}_2)_{A_2} \to \mathcal{G}_3$ is then defined. It is clear that this composition defines a groupoid homomorphism from $\mathcal{G}_1$ into $\mathcal{G}_3$. We denote it by $g \circ f\colon \mathcal{G}_1 \to \mathcal{G}_3$.

Definition 2.17. Let $\mathcal{G}$, $\mathcal{H}$ be discrete measured groupoids on standard measure spaces (X, μ), (Y, ν), respectively. A groupoid homomorphism $f\colon \mathcal{G} \to \mathcal{H}$ is called an *isomorphism* if there exists a groupoid homomorphism $g\colon \mathcal{H} \to \mathcal{G}$ such that the compositions $g \circ f\colon \mathcal{G} \to \mathcal{G}$ and $f \circ g\colon \mathcal{H} \to \mathcal{H}$ coincide with the identity homomorphisms on $\mathcal{G}$ and on $\mathcal{H}$, respectively. In this case, $\mathcal{G}$ and $\mathcal{H}$ are said to be *isomorphic*.

Though we often need to take $\mathcal{G}$-invariant Borel subsets of X in many situations in this chapter, we do not always mention it for simplicity of the notation.

As seen in Example 2.12, an equivalence relation on a set defines a groupoid on the set. We next introduce an equivalence relation on a Borel space which induces a discrete measured groupoid on the Borel space.

Definition 2.18. Let (X, μ) be a standard measure space. Let $\mathcal{R}$ be a Borel subset of $X \times X$ such that

- $\mathcal{R}$ defines an equivalence relation on X as in Example 2.12;
- for each $x \in X$, the equivalence class $\mathcal{R}_x = \{y \in X : (y, x) \in \mathcal{R}\}$ of x is at most countable.

Then $\mathcal{R}$ is a discrete Borel groupoid on X with respect to the structure introduced in Example 2.12. If μ is quasi-invariant for this groupoid, then $\mathcal{R}$ is called a *discrete measured equivalence relation* (or simply an *equivalence relation*) on (X, μ).

Definition 2.19. Let $\mathcal{G}$ be a discrete measured groupoid on a standard measure space (X, μ). It is easy to see that

$$\mathcal{R} = \{(r(\gamma), s(\gamma)) \in X \times X : \gamma \in \mathcal{G}\}$$

has the structure of a discrete measured groupoid on (X, μ) such that

$$r(x, y) = x, \quad s(x, y) = y, \quad (x, y)(y, z) = (x, z), \quad (x, y)^{-1} = (y, x).$$

This groupoid is called the *quotient equivalence relation* of $\mathcal{G}$. Note that if the isotropy group $\mathcal{G}_x^x$ is trivial for a.e. $x \in X$, then $\mathcal{G}$ and its quotient equivalence relation $\mathcal{R}$ are isomorphic via the following isomorphism:

$$\mathcal{G} \ni \gamma \mapsto (r(\gamma), s(\gamma)) \in \mathcal{R}.$$

In this case, $\mathcal{G}$ is said to be *principal*.

We give one typical example of discrete measured groupoids appearing in this chapter. We recommend the reader to see [5] for other examples of discrete measured groupoids.

Example 2.20. *Group actions.* Let G be a discrete group and assume that G admits a *non-singular* action on a standard measure space (X, μ), which means that $\mu(A) = 0$ if and only if $\mu(gA) = 0$ for any $g \in G$ and for any Borel subset $A \subset X$. The direct product $G \times X$ then has the structure of a groupoid such that

$$r(g, x) = gx, \quad s(g, x) = x, \quad (g, hx)(h, x) = (gh, x), \quad (g, x)^{-1} = (g^{-1}, gx).$$

This groupoid is often written as $G \ltimes (X, \mu)$ or $G \ltimes X$. Since the action $G \curvearrowright (X, \mu)$ is non-singular, μ is quasi-invariant for $G \ltimes X$.

It is easy to see that μ is invariant for the action $G \curvearrowright (X, \mu)$ if and only if it is an invariant measure for the groupoid $G \ltimes X$. For a Borel subset $A \subset X$, the saturation $(G \ltimes X)A$ is equal to the saturation $GA = \bigcup_{g \in G} gA$.

The quotient equivalence relation

$$\mathcal{R} = \{(gx, x) \in X \times X : g \in G, x \in X\}$$

of $G \ltimes X$ admits the structure of a discrete measured groupoid on (X, μ) as seen in Definition 2.19. This $\mathcal{R}$ can also be seen as a discrete measured equivalence relation on (X, μ) arising from the equivalence relation declaring that two points of X are equivalent if and only if they are in the same G-orbit.

Note that the action $G \curvearrowright (X, \mu)$ is *essentially free*, that is, the stabilizer of almost every point $x \in X$ is trivial if and only if $G \ltimes X$ is principal.

In this chapter, we mainly treat groupoids isomorphic to subgroupoids of a groupoid arising from a measure-preserving action of a discrete group on a standard finite measure space. In Section 4, we however treat discrete measured groupoids arising from non-singular actions of discrete groups which are never measure-preserving (see Theorems 4.20 and 4.21).

Conjugacy and orbit equivalence. Given two actions $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$, when are the two associated groupoids isomorphic? We shall give two equivalence relations for non-singular actions of discrete groups on measure spaces, called conjugacy and orbit equivalence. It is shown that when two actions are both essentially free, they are orbit equivalent if and only if the associated groupoids are isomorphic.

Definition 2.21. Let Γ, Λ be discrete groups and let (X, μ), (Y, ν) be standard measure spaces. Consider non-singular actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$. The two actions are said to be *conjugate* if there exist an isomorphism $F \colon \Gamma \to \Lambda$ and a measure space isomorphism $f \colon (X, \mu) \to (Y, \nu)$ such that

$$f(\gamma x) = F(\gamma) f(x) \quad \text{for any } \gamma \in \Gamma \text{ and a.e. } x \in X.$$

More precisely, this means that we can take conull Borel subsets $X' \subset X$ and $Y' \subset Y$ and a Borel isomorphism $f \colon X' \to Y'$ satisfying the following: the two measures $f_*\mu$ and ν are equivalent; and for any $\gamma \in \Gamma$ and a.e. $x \in X'$, γx belongs to X' and the equation $f(\gamma x) = F(\gamma) f(x)$ holds.

Orbit equivalence is a weaker equivalence relation than conjugacy.

Definition 2.22. Let Γ, Λ be discrete groups and let (X, μ), (Y, ν) be standard measure spaces. Consider non-singular actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$. The two actions are said to be *orbit equivalent (OE)* if there exists a measure space isomorphism $f \colon (X, \mu) \to (Y, \nu)$ such that

$$f(\Gamma x) = \Lambda f(x) \quad \text{for a.e. } x \in X.$$

More precisely, this means that we can take conull Borel subsets $X' \subset X$ and $Y' \subset Y$ and a Borel isomorphism $f \colon X' \to Y'$ satisfying the following: the two measures $f_*\mu$ and ν are equivalent; and for a.e. $x \in X'$, Γx is contained in X' and the equation $f(\Gamma x) = \Lambda f(x)$ holds. It is easy to see that this f induces an isomorphism between the two quotient equivalence relations of

$\Gamma \ltimes X$ and $\Lambda \ltimes Y$, which is defined by $(x, y) \mapsto (f(x), f(y))$. Conversely, if an isomorphism between the two quotient equivalence relations of $\Gamma \ltimes X$ and $\Lambda \ltimes Y$ is given, then the associated map f between their unit spaces satisfies the above condition of OE.

If the action $\Lambda \curvearrowright (Y, \nu)$ is essentially free, then we can define a Borel map

$$\alpha\colon \Gamma \times X \to \Lambda \quad \text{so that} \quad f(\gamma x) = \alpha(\gamma, x) f(x)$$

for $\gamma \in \Gamma$ and a.e. $x \in X$. This map α satisfies the following cocycle identity

$$\alpha(\gamma_1, \gamma_2 x)\alpha(\gamma_2, x) = \alpha(\gamma_1\gamma_2, x)$$

for any $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $x \in X$. Thus, α is a groupoid homomorphism from $\Gamma \ltimes X$ into Λ. We call α the *OE cocycle* associated with f.

The reader can check that when the actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are both essentially free, they are OE via f if and only if the two groupoids $\Gamma \ltimes X$ and $\Lambda \ltimes Y$ are isomorphic under the groupoid homomorphism $(\gamma, x) \mapsto (\alpha(\gamma, x), f(x))$ associated with f.

We next introduce a slightly weaker equivalence relation than OE, called weak orbit equivalence (WOE). It is known that two discrete groups are measure equivalent (ME) if and only if the two groups admit ergodic, measure-preserving and essentially free actions which are WOE (see Corollary 2.34).

Definition 2.23. Let Γ, Λ be discrete groups and let (X, μ), (Y, ν) be standard measure spaces. Consider non-singular actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$. The two actions are said to be *weakly orbit equivalent (WOE)* if there exist Borel subsets $A \subset X$, $B \subset Y$ and a Borel isomorphism $f\colon A \to B$ satisfying the following three conditions:

(i) $\Gamma A = X$, $\Lambda B = Y$ up to null sets;

(ii) The two measures $f_*(\mu|_A)$ and $\nu|_B$ are equivalent;

(iii) $f(\Gamma x \cap A) = \Lambda f(x) \cap B$ for a.e. $x \in A$.

As in the case of OE, this f induces an isomorphism between the two quotient equivalence relations of $(\Gamma \ltimes X)_A$ and $(\Lambda \ltimes Y)_B$. Conversely, if an isomorphism between the two quotient equivalence relations of $(\Gamma \ltimes X)_A$ and $(\Lambda \ltimes Y)_B$ for Borel subsets A, B of X satisfying Condition (i) is given, then the associated map f between their unit spaces A and B satisfies Conditions (ii), (iii).

Groupoids of infinite type. In most sections of this chapter, we study a groupoid $\mathcal{G}$ associated with a measure-preserving action of a discrete group Γ on a standard finite measure space and study its subgroupoids. In particular, we mainly study its subgroupoids of infinite type. When the unit space of $\mathcal{G}$ consists of a single atom and $\mathcal{G}$ is then isomorphic to Γ, subgroupoids of $\mathcal{G}$ of infinite type correspond to infinite subgroups of Γ. Before defining the notion of groupoids of infinite type, we introduce recurrence of a discrete measured equivalence relation. Recall that a discrete measured equivalence relation can be seen as a discrete measured groupoid (see Definition 2.18).

Definition 2.24. Let $\mathcal{R}$ be a discrete measured equivalence relation on a standard finite measure space (X, μ) with an invariant measure μ for $\mathcal{R}$. We say that $\mathcal{R}$ is *recurrent* if a.e. equivalence class for $\mathcal{R}$ is infinite, that is, for a.e. $x \in X$, the set $\mathcal{R}_x = \{y \in X : (y, x) \in \mathcal{R}\}$ is infinite.

Let $\mathcal{R}$ be a discrete measured equivalence relation on a standard finite measure space (X, μ) with an invariant measure μ for $\mathcal{R}$. It is known that if $\mathcal{R}$ is recurrent and A is a Borel subset of X with positive measure, then the restriction $(\mathcal{R})_A$ is also recurrent (see the proof of Lemma 2.5 in [2]). Moreover, it is shown that there exists an essentially unique Borel partition $X = A_1 \sqcup A_2$ such that

- $(\mathcal{R})_{A_1}$ is recurrent;
- $(\mathcal{R})_{A_2}$ admits a *fundamental domain*, i.e., there exists a Borel subset B of A_2 such that for a.e. $x \in A_2$, $(\mathcal{R})_{A_2}x \cap B$ consists of exactly one point, where $(\mathcal{R})_{A_2}x$ denotes the equivalence class for $(\mathcal{R})_{A_2}$ containing x.

See Lemma 2.12 in [2] for the proof of this fact. The reader can check that $\mathcal{R}A_1 = A_1$ and $\mathcal{R}A_2 = A_2$ up to null sets. It is easy to treat an equivalence relation which admits a (Borel) fundamental domain because the space of orbits of the equivalence relations can be identified with its fundamental domain. The above fact means that any equivalence relation can be divided into an easy part and a non-trivial part. Hence, it is often enough to consider only recurrent equivalence relations in the study of equivalence relations. We refer to Section 2 in [2] for fundamental properties of discrete measured equivalence relations, where the recurrence of discrete measured equivalence relations such that μ is not necessarily invariant is also discussed. The notion of groupoids of infinite type is defined as follows.

Definition 2.25. Let $\mathcal{G}$ be a discrete measured groupoid on a standard finite measure space (X, μ) with an invariant measure μ for $\mathcal{G}$. Then we say that $\mathcal{G}$ is of *infinite type* if there exists a Borel partition $X = X_1 \sqcup X_2$ such that

- the isotropy group $\mathcal{G}_x^x$ is infinite for a.e. $x \in X_1$;
- the quotient equivalence relation of $(\mathcal{G})_{X_2}$ is recurrent.

By definition, if $\mathcal{G}$ is of infinite type, then $(\mathcal{G})_A$ is also of infinite type for any Borel subset $A \subset X$ with positive measure. The next proposition shows that a measure-preserving action of an infinite discrete group on a standard finite measure space always gives rise to a groupoid of infinite type.

Proposition 2.26. *Let Γ be an infinite discrete group and suppose that Γ admits a measure-preserving action on a standard finite measure space (X, μ). Then the associated groupoid $\Gamma \ltimes X$ is of infinite type, and thus so is the restriction $(\Gamma \ltimes X)_A$ for any Borel subset $A \subset X$ with positive measure.*

Proof. Let $\mathcal{R}$ be the quotient equivalence relation of $\Gamma \ltimes X$. Choose a Borel partition $X = A_1 \sqcup A_2$ such that $(\mathcal{R})_{A_1}$ is recurrent and $(\mathcal{R})_{A_2}$ admits a fundamental domain $B \subset A_2$ (see the comment right after Definition 2.24). Both A_1 and A_2 are then invariant under the action $\Gamma \curvearrowright (X, \mu)$. For a.e. $x \in A_2$, the orbit Γx consists of only finitely many points because the action $\Gamma \curvearrowright (X, \mu)$ is measure-preserving and $\mu(A_2) < \infty$. Therefore, the stabilizer of a.e. $x \in A_2$ is infinite. □

2.3 ME and isomorphism of groupoids

In this subsection, we construct from an ME coupling of discrete groups Γ and Λ an isomorphism of groupoids associated with some measure-preserving actions of Γ and Λ on standard finite measure spaces. This construction was essentially given in Section 3 in [17]. Thanks to this construction, we can reduce the problem of ME to an algebraic problem of groupoids arising from group actions.

Let (Σ, m) be an ME coupling of discrete groups Γ and Λ, i.e., (Σ, m) is a standard Borel space with a σ-finite positive measure, and there is a measure-preserving action $\Gamma \times \Lambda \curvearrowright (\Sigma, m)$ such that both of the actions $\Gamma(\simeq \Gamma \times \{e\}) \curvearrowright (\Sigma, m)$ and $\Lambda(\simeq \{e\} \times \Lambda) \curvearrowright (\Sigma, m)$ are essentially free and admit a fundamental domain of finite measure (see Definition 1.1). Choose fundamental domains $Y \subset \Sigma$ for the action $\Gamma \curvearrowright \Sigma$, and $X \subset \Sigma$ for the action $\Lambda \curvearrowright \Sigma$. Remark that we have a natural measure-preserving action of Γ on X equipped with the restricted finite measure μ of m to X because X can be identified with the quotient space Σ/Λ as a Borel space. Similarly, we have a natural measure-preserving action of Λ on Y with a finite measure ν. In order to distinguish from the original actions of Γ and Λ on Σ, we denote the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ by $\gamma \cdot x$, $\lambda \cdot y$, respectively, using a dot.

Lemma 2.27. *In the above notation, one can choose X and Y so that $A = X \cap Y$ satisfies the following two conditions:*

- *$\Gamma \cdot A = X$ up to null sets when A is regarded as a subset of X;*
- *$\Lambda \cdot A = Y$ up to null sets when A is regarded as a subset of Y.*

Proof. Let S be the set of all Borel subsets $B \subset \Sigma$ such that $m(\gamma_1 B \triangle \gamma_2 B) = 0$ for all distinct $\gamma_1, \gamma_2 \in \Gamma$, and $m(\lambda_1 B \triangle \lambda_2 B) = 0$ for all distinct $\lambda_1, \lambda_2 \in \Lambda$. Here, $C \triangle D$ denotes the symmetric difference of two sets C and D. If we find

$A \in S$ such that the equation $(\Gamma \times \Lambda)A = \Sigma$ holds up to null sets, then choose fundamental domains $X \subset \Sigma$ for the action $\Lambda \curvearrowright \Sigma$, and $Y \subset \Sigma$ for the action $\Gamma \curvearrowright \Sigma$ such that $A \subset X$ and $A \subset Y$. The above two conditions are then satisfied for these X and Y. Hence, we will find $A \in S$ satisfying the equation $(\Gamma \times \Lambda)A = \Sigma$ up to null sets.

Put $M = \sup_{B \in S} m(B)$. Then $M < \infty$. Since we can always take fundamental domains of the actions of Γ and Λ on Σ whose intersection has positive measure, the number M is positive. Let $\{B_n\}_{n \in \mathbb{N}}$ be a sequence of elements of S such that $m(B_n) \to M$ as $n \to \infty$. Put $A_1 = B_1$ and define $A_n = (B_n \setminus (\Gamma \times \Lambda)A_{n-1}) \cup A_{n-1}$ for $n \geq 2$ inductively. Then $A_n \in S$ and $A = \bigcup_n A_n$ is also in S. It is easy to see that $m((\Gamma \times \Lambda)B_n \setminus (\Gamma \times \Lambda)A_n) = 0$. In particular, $m((\Gamma \times \Lambda)B_n \setminus (\Gamma \times \Lambda)A) = 0$ for all n.

We claim that $(\Gamma \times \Lambda)A = \Sigma$ up to null sets. If $\Sigma \setminus (\Gamma \times \Lambda)A$ had positive measure, then it would be an ME coupling of Γ and Λ. There exists a Borel subset $B \subset \Sigma \setminus (\Gamma \times \Lambda)A$ which is in S as a Borel subset of Σ and has positive measure. Take $n \in \mathbb{N}$ so that $M - m(B_n) < m(B)$. Then $B \cup B_n \in S$ and $m(B \cup B_n) = m(B) + m(B_n) > M$, which is a contradiction. $\square$

In what follows, suppose that X and Y satisfy the conditions of Lemma 2.27. Let $\mathcal{G} = \Gamma \ltimes (X, \mu)$ (resp. $\mathcal{H} = \Lambda \ltimes (Y, \nu)$) be the groupoid associated with the action $\Gamma \curvearrowright X$ (resp. $\Lambda \curvearrowright Y$). We can define a Borel map

$$\alpha\colon \Gamma \times X \to \Lambda \ \text{ so that } \ \gamma \cdot x = \alpha(\gamma, x)\gamma x \in X$$

for any $\gamma \in \Gamma$ and a.e. $x \in X$ because X is a fundamental domain of the action $\Lambda \curvearrowright \Sigma$. Similarly, we can define a Borel map

$$\beta\colon \Lambda \times Y \to \Gamma \ \text{ so that } \ \lambda \cdot y = \beta(\lambda, y)\lambda y \in Y$$

for $\lambda \in \Lambda$ and a.e. $y \in Y$.

Lemma 2.28. *The map $\alpha\colon \Gamma \times X \to \Lambda$ is a cocycle, that is, the cocycle identity*

$$\alpha(\gamma_1, \gamma_2 \cdot x)\alpha(\gamma_2, x) = \alpha(\gamma_1\gamma_2, x)$$

is satisfied for each $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $x \in X$. The map $\beta\colon \Lambda \times Y \to \Gamma$ also satisfies a similar identity.

This cocycle identity implies that α is a groupoid homomorphism from $\Gamma \ltimes X$ into Λ. We call α (resp. β) the *ME cocycle* associated with X (resp. Y).

Proof. This follows from the following equality:

$$\alpha(\gamma_1, \gamma_2 \cdot x)\alpha(\gamma_2, x)\gamma_1\gamma_2 x = \alpha(\gamma_1, \gamma_2 \cdot x)\gamma_1(\gamma_2 \cdot x) = \gamma_1 \cdot (\gamma_2 \cdot x) = (\gamma_1\gamma_2) \cdot x$$

for $\gamma_1, \gamma_2 \in \Gamma$ and $x \in X$, where the right hand side is in X. $\square$

Let $p\colon X \to Y$ and $q\colon Y \to X$ be the Borel maps defined by

$$p(x) = \Gamma x \cap Y, \quad q(y) = \Lambda y \cap X$$

for $x \in X$ and $y \in Y$. Note that both p and q are the identity on $A = X \cap Y$. Then we can show that

$$p(\gamma \cdot x) = \alpha(\gamma, x) \cdot p(x), \quad q(\lambda \cdot y) = \beta(\lambda, y) \cdot q(y)$$

for any $\gamma \in \Gamma$, $\lambda \in \Lambda$ and a.e. $x \in X$, $y \in Y$ as follows: Since $\gamma \cdot x = \alpha(\gamma, x)\gamma x$, there exists a unique $\gamma_1 \in \Gamma$ such that

$$p(\gamma \cdot x) = \gamma_1 \alpha(\gamma, x) x \in Y. \tag{2.1}$$

Let $\gamma_2, \gamma_3 \in \Gamma$ be unique elements such that $p(x) = \gamma_2 x \in Y$ and

$$\alpha(\gamma, x) \cdot p(x) = \gamma_3 \alpha(\gamma, x) \gamma_2 x \in Y. \tag{2.2}$$

Comparing (2.1) and (2.2), we see that $\gamma_1 = \gamma_3\gamma_2$ since Y is a fundamental domain of the action $\Gamma \curvearrowright \Sigma$. This proves the claim.

Define groupoid homomorphisms

$$f\colon (\mathcal{G})_A \ni (\gamma, x) \mapsto (\alpha(\gamma, x), p(x)) \in (\mathcal{H})_A,$$
$$g\colon (\mathcal{H})_A \ni (\lambda, y) \mapsto (\beta(\lambda, y), q(y)) \in (\mathcal{G})_A.$$

Note that $\beta(\alpha(\gamma, x), x) = \gamma$ for any $\gamma \in \Gamma$ and a.e. $x \in A$ with $\gamma \cdot x \in A$ because $\gamma\alpha(\gamma, x)x = \gamma \cdot x \in A \subset Y$. Similarly, $\alpha(\beta(\lambda, y), y) = \lambda$ for any $\lambda \in \Lambda$ and a.e. $y \in A$ with $\lambda \cdot y \in A$. Therefore, we obtain the following

Proposition 2.29. *In the above notation, the groupoid homomorphisms*

$$f\colon (\mathcal{G})_A \to (\mathcal{H})_A, \quad g\colon (\mathcal{H})_A \to (\mathcal{G})_A$$

satisfy $g \circ f = \mathrm{id}$ *and* $f \circ g = \mathrm{id}$.

This is a construction of an isomorphism between two (restrictions of) groupoids generated by actions of Γ and Λ from an ME coupling of Γ and Λ. In particular, if we can show that Γ and Λ admit no actions which generate isomorphic groupoids (even after restricting to any Borel subsets with positive measure), then this implies that Γ and Λ are *not* ME. Therefore, when we consider the problem of ME, it is effective to study algebraic properties of groupoids arising from specific groups. Such a groupoid $\Gamma \ltimes X$ often behaves like the group Γ. More precisely, suppose that Γ admits some "nice" action on a space S. The action is given by a homomorphism $\Gamma \to \mathrm{Aut}(S)$. The projection $\Gamma \ltimes X \to \Gamma$ is a groupoid homomorphism. We can then view the groupoid homomorphism $\Gamma \ltimes X \to \Gamma \to \mathrm{Aut}(S)$ as an action of $\Gamma \ltimes X$ on S, and we observe that this action of $\Gamma \ltimes X$ often gives rise to phenomena similar to the ones for the action of Γ on S. This idea greatly helps us to study the groupoid $\Gamma \ltimes X$ (see also the beginning of Section 5).

Return to the situation before Proposition 2.29 and consider the action of $\Gamma \times \Lambda$ on $X \times \Lambda$ defined by

$$(\gamma, \lambda)(x, \lambda') = (\gamma \cdot x, \alpha(\gamma, x)\lambda'\lambda^{-1}), \quad \gamma \in \Gamma,\ \lambda, \lambda' \in \Lambda,\ x \in X.$$

It is easy to check the following lemma, which means that we can reconstruct an ME coupling from the cocycle α.

Lemma 2.30. *In the above notation, the Borel map $\Sigma \to X \times \Lambda$ defined by $\lambda x \mapsto (x, \lambda^{-1})$ for $x \in X$ and $\lambda \in \Lambda$ is Borel isomorphic and $(\Gamma \times \Lambda)$-equivariant.*

Note that Proposition 2.29 implies that the two actions of Γ on X and Λ on Y are WOE (see Definition 2.23). Conversely, the following theorem is known. This states that given WOE actions of Γ and Λ, we can construct the corresponding ME coupling of Γ and Λ.

Definition 2.31. For simplicity, by a *standard action* of a discrete group we mean an essentially free, measure-preserving Borel action of that group on a standard finite measure space.

Theorem 2.32 ([17, Theorem 3.3]). *Suppose that two discrete groups Γ and Λ admit ergodic standard actions on (X, μ) and (Y, ν), respectively, which are WOE. Then we can construct an ME coupling (Σ, m) of Γ and Λ such that the Γ-actions on X and on $\Lambda \backslash \Sigma$ (resp. the Λ-actions on Y and on $\Gamma \backslash \Sigma$) are conjugate.*

In particular, if the two actions $\Gamma \curvearrowright (X, \mu)$ and $Y \curvearrowright (Y, \nu)$ are OE via a Borel isomorphism f between conull Borel subsets of X and Y, then we can construct the above ME coupling (Σ, m) so that the ME cocycle associated with some fundamental domain of the Λ-action on Σ, which is identified with X under the above conjugacy of the Γ-actions on X and on $\Lambda \backslash \Sigma$, is equal to the OE cocycle associated with f.

Remark 2.33. In the case of WOE, we can also define an associated WOE cocycle and prove a statement similar to the latter assertion in Theorem 2.32 (see Theorem 3.3 in [17]).

Corollary 2.34. *Two discrete groups are ME if and only if they admit ergodic standard actions on standard finite measure spaces which are WOE.*

Proof. The "if" part follows from Theorem 2.32. We prove the "only if" part. Let (Σ, m) be an ME coupling of discrete groups Γ and Λ. Let $\Gamma \curvearrowright (X, \mu)$ be a standard action. For example, the Bernoulli action $\Gamma \curvearrowright \prod_\Gamma [0, 1]$ given by

$$\gamma(x_g)_{g \in \Gamma} = (x_{\gamma^{-1}g})_{g \in \Gamma}, \quad \gamma \in \Gamma,\ (x_g)_{g \in \Gamma} \in \prod_\Gamma [0, 1]$$

is standard (see Section 2 in [38]). The action of $\Gamma \times \Lambda$ on $\Sigma \times X$ given by

$$(\gamma, \lambda)(x, x') = ((\gamma, \lambda)x, \gamma x'), \quad \gamma \in \Gamma, \ \lambda \in \Lambda, \ x \in \Sigma, x' \in X$$

defines an ME coupling of Γ and Λ such that the action $\Gamma \times \Lambda \curvearrowright (\Sigma \times X, m \times \mu)$ is essentially free. By utilizing the ergodic decomposition for the action $\Gamma \times \Lambda \curvearrowright (\Sigma \times X, m \times \mu)$, we can construct an ME coupling (Σ_0, m_0) of Γ and Λ such that the action $\Gamma \times \Lambda \curvearrowright (\Sigma_0, m_0)$ is essentially free and ergodic (see Lemma 2.2 in [16]). Thus, the two actions $\Gamma \curvearrowright \Lambda \backslash \Sigma_0$ and $\Lambda \curvearrowright \Gamma \backslash \Sigma_0$ are both ergodic and standard. Proposition 2.29 implies that the two actions are WOE. □

3 ME rigidity for mapping class groups

In this section, we state two key theorems for the proof of the ME rigidity result of the mapping class group $\Gamma(M)$. The first one, Theorem 3.1, is reduction of a self ME coupling of $\Gamma(M)$ (i.e., an ME coupling of $\Gamma(M)$ and itself) to a simpler self ME coupling of $\Gamma(M)$. This reduction is a very important step for the proof of measurable rigidity. As stated in Proposition 2.29, a self ME coupling of $\Gamma(M)$ gives rise to an isomorphism between groupoids arising from two measure-preserving actions of $\Gamma(M)$. The second key theorem 3.6 states a certain important property of such an isomorphism. The proof of this theorem will be explained in subsequent sections. In Subsection 3.1, assuming Theorem 3.6, we show Theorem 3.1. To establish ME rigidity from these theorems, we need one more step, which is explained in Subsection 3.2. In Subsection 3.3, we give another immediate application of the reduction of self ME couplings. We prove an OE rigidity result for ergodic standard actions of the mapping class group.

3.1 Reduction of self ME couplings of mapping class groups

We first give an outline to prove Theorem 1.5, an ME rigidity result for the mapping class group, and give three steps (1), (2), (3) for the proof. This outline is similar to Furman's one for the proof of Theorem 1.9, an ME rigidity result for higher rank lattices. Step (2) is devoted to the first key theorem 3.1 noted above. Here is the most important and difficult step. We give three steps (a), (b), (c) for the proof of this theorem. Some of the steps are formulated in terms of groupoids arising from actions of mapping class groups, and it seems complicated for beginners of groupoids. Before giving an explicit formulation of these steps, we explain how the steps are formulated when the unit spaces of the groupoids consist of a single atom, i.e., when the groupoids are groups.

Under this assumption, these steps are formulated in terms of groups, which is much easier to understand.

We shall give three steps (1), (2), (3) for the proof of Theorem 1.5. Let M be a surface with $\kappa(M) > 0$. Let (Σ, m) be an ME coupling of the mapping class group $\Gamma = \Gamma(M)$ and a discrete group Λ. Let $C = C(M)$ be the curve complex of M and let $\mathrm{Aut}(C)$ be its automorphism group. Let $\pi\colon \Gamma(M)^\diamond \to \mathrm{Aut}(C)$ be the natural homomorphism.

(1) Let Ω be the quotient space of $\Sigma \times \Lambda \times \Sigma$ by the $(\Lambda \times \Lambda)$-action on $\Sigma \times \Lambda \times \Sigma$ given by

$$(\lambda_1, \lambda_2)(x, \lambda, y) = (\lambda_1 x, \lambda_1 \lambda \lambda_2^{-1}, \lambda_2 y), \quad \lambda_1, \lambda_2, \lambda \in \Lambda, \ x, y \in \Sigma.$$

We define a $(\Gamma \times \Gamma)$-action on $\Sigma \times \Lambda \times \Sigma$ by

$$(\gamma_1, \gamma_2)(x, \lambda, y) = (\gamma_1 x, \lambda, \gamma_2 y), \quad \gamma_1, \gamma_2 \in \Gamma, \ \lambda \in \Lambda, \ x, y \in \Sigma.$$

This $(\Gamma \times \Gamma)$-action then induces a $(\Gamma \times \Gamma)$-action on Ω. It is easy to check that Ω is a self ME coupling of Γ, i.e., an ME coupling of Γ and Γ.

(2) We construct an almost $(\Gamma \times \Gamma)$-equivariant Borel map $\Phi\colon \Omega \to \mathrm{Aut}(C)$, i.e.,

$$\Phi((\gamma_1, \gamma_2)z) = \pi(\gamma_1)\Phi(z)\pi(\gamma_2)^{-1}$$

for any $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $z \in \Omega$.

(3) Using the map Φ, we construct a representation ρ of the group Λ on $\mathrm{Aut}(C)$. Moreover, this homomorphism $\rho\colon \Lambda \to \mathrm{Aut}(C)$ has finite kernel, and $\rho(\Lambda)$ is a finite index subgroup of $\mathrm{Aut}(C)$. This proves Theorem 1.5.

In this subsection, details of Step (2) are discussed. We explain Step (3) in Subsection 3.2. Step (2) is a consequence of the following theorem.

Theorem 3.1 ([36, Corollary 5.9]). *Let M be a surface with $\kappa(M) > 0$. Let Γ_1 and Γ_2 be finite index subgroups of $\Gamma(M)^\diamond$ and suppose that there is an ME coupling (Ω, ω) of Γ_1 and Γ_2. Then there exists an almost $(\Gamma_1 \times \Gamma_2)$-equivariant Borel map $\Phi\colon \Omega \to \mathrm{Aut}(C)$, i.e.,*

$$\Phi((\gamma_1, \gamma_2)z) = \pi(\gamma_1)\Phi(z)\pi(\gamma_2)^{-1}$$

for any $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$ and a.e. $z \in \Omega$.

This theorem means that all ME coupling of Γ_1 and Γ_2 can be reduced to the simpler ME coupling $\mathrm{Aut}(C)$ of Γ_1 and Γ_2 on which $\Gamma_1 \times \Gamma_2$ acts as follows:

$$(\gamma_1, \gamma_2)g = \pi(\gamma_1)g\pi(\gamma_2)^{-1}, \quad \gamma_1 \in \Gamma_1, \ \gamma_2 \in \Gamma_2, \ g \in \mathrm{Aut}(C).$$

By a technical lemma (see Lemma 5.8 in [36]), if we can construct an almost $(\Gamma_1' \times \Gamma_2')$-equivariant Borel map $\Phi\colon \Omega \to \mathrm{Aut}(C)$ for some finite index subgroups Γ_i' of Γ_i for $i \in \{1,2\}$, then Φ is in fact an almost $(\Gamma_1 \times \Gamma_2)$-equivariant. It follows that in the proof of Theorem 3.1, we may assume that both Γ_1 and Γ_2 are finite index subgroups of $\Gamma(M;m)$ with an integer $m \geq 3$ (see Theorem 2.8 for the subgroup $\Gamma(M;m)$ of $\Gamma(M)$). In what follows in this subsection, we always assume this condition (because the key theorem, Theorem 3.6, is proved under this assumption). To state an outline of the proof of Theorem 3.1, we fix the notation as follows.

Notation. We refer to the following assumption as $(\bullet)$:

- Let M be a surface with $\kappa(M) > 0$ and let $m \geq 3$ be an integer. Let Γ_1 and Γ_2 be finite index subgroups of $\Gamma(M;m)$. Let (Ω, ω) be an ME coupling of Γ_1 and Γ_2
- Take fundamental domains $X_1 \subset \Omega$ for the Γ_2-action on Ω, and $X_2 \subset \Omega$ for the Γ_1-action on Ω. Recall that the natural actions $\Gamma_1 \curvearrowright X_1$ and $\Gamma_2 \curvearrowright X_2$ are denoted by $(\gamma, x) \mapsto \gamma \cdot x$ by using a dot. By Lemma 2.27, we can choose X_1, X_2 so that $Y = X_1 \cap X_2$ satisfies that for $i \in \{1,2\}$, $\Gamma_i \cdot Y = X_i$ up to null sets when Y is regarded as a subset of X_i.
- For $i \in \{1,2\}$, set $\mathcal{G}^i = \Gamma_i \ltimes X_i$ and let $\rho_i \colon \mathcal{G}^i \to \Gamma_i$ be the projection, which is a groupoid homomorphism. By Proposition 2.29, there exists a groupoid isomorphism
$$f \colon (\mathcal{G}^1)_Y \to (\mathcal{G}^2)_Y.$$
Note that f is the identity on the unit space Y.
- For $i \in \{1,2\}$ and $\alpha \in V(C)$, let D_α^i be the intersection of Γ_i with the subgroup of $\Gamma(M)$ generated by the Dehn twist $t_\alpha \in \Gamma(M)$ about α. Let $\mathcal{G}_\alpha^i$ be the subgroupoid of $\mathcal{G}^i$ generated by the action of D_α^i, i.e.,
$$\mathcal{G}_\alpha^i = \{(\gamma, x) \in \mathcal{G}^i : \gamma \in D_\alpha^i,\ x \in X_i\}.$$

An outline of the proof of Theorem 3.1 is as follows.

(a) f preserves subgroupoids generated by Dehn twists up to a countable Borel partition (see Theorem 3.6 for a precise statement).

(b) Using Step (a), we construct a Borel map $\Psi \colon Y \to \mathrm{Aut}(C)$ associated with f.

(c) The Borel map $Y \ni x \mapsto \Psi(x)^{-1} \in \mathrm{Aut}(C)$ can be extended to an almost $(\Gamma_1 \times \Gamma_2)$-equivariant Borel map $\Phi \colon X_1 \times \Gamma_2 \to \mathrm{Aut}(C)$. Here, Y is identified with the Borel subset $Y \times \{e\}$ of $X_1 \times \Gamma_2$. Note that $(\Gamma_1 \times \Gamma_2)(Y \times \{e\}) = X_1 \times \Gamma_2$ up to null sets and that $X_1 \times \Gamma_2$ can be identified with Ω as an ME coupling of Γ_1 and Γ_2 (see Lemma 2.30).

In what follows, we explain an explicit statement of Step (a) and details of Steps (b) and (c).

Ivanov's argument. Before discussing details of Steps (a), (b) and (c), we study these steps in the degenerate case, that is, in the case where both X_1 and X_2 consist of a single atom. In this case, our groupoids $\mathcal{G}^1$ and $\mathcal{G}^2$ degenerate into the groups Γ_1 and Γ_2, respectively. Therefore, the argument for these steps gets much easier and clearer.

In the above argument, we obtained an isomorphism between the two restricted groupoids $(\Gamma_1 \ltimes X_1)_Y$ and $(\Gamma_2 \ltimes X_2)_Y$ arising from an ME coupling of finite index subgroups Γ_1 and Γ_2 of the mapping class group. What happens if we assume that each of X_1 and X_2 consists of a single atom? In this case, we obtain a group isomorphism $f\colon \Gamma_1 \to \Gamma_2$. Conversely, if $f\colon \Gamma_1 \to \Gamma_2$ is an isomorphism, then the action of $\Gamma_1 \times \Gamma_2$ on Γ_2 given by

$$(\gamma_1, \gamma_2)\gamma = f(\gamma_1)\gamma\gamma_2^{-1}, \quad \gamma_1 \in \Gamma_1,\ \gamma_2, \gamma \in \Gamma_2$$

defines an ME coupling of Γ_1 and Γ_2 such that $\{e\}$ is a fundamental domain for both of the actions of Γ_1 and Γ_2 on Γ_2, and the isomorphism between Γ_1 and Γ_2 given in Proposition 2.29 is equal to f. Ivanov showed the following theorem about an isomorphism between finite index subgroups of the mapping class group.

Theorem 3.2 ([32, Theorem 8.5.A]). *Let M be a surface with $\kappa(M) > 0$ and $M \neq M_{1,2}, M_{2,0}$. Let Γ_1 and Γ_2 be finite index subgroups of $\Gamma(M)^\diamond$. If $f\colon \Gamma_1 \to \Gamma_2$ is an isomorphism, then there exists a unique $g \in \Gamma(M)^\diamond$ such that $f(\gamma) = g\gamma g^{-1}$ for any $\gamma \in \Gamma_1$.*

Outline of the proof. The first step of the proof is to show that f maps sufficiently high powers of Dehn twists into powers of Dehn twists. Namely, for each $\alpha \in V(C)$, there exist non-zero integers N, M and $\beta \in V(C)$ such that $t_\alpha^N \in \Gamma_1$, $t_\beta^M \in \Gamma_2$ and $f(t_\alpha^N) = t_\beta^M$. This fact is a consequence of the following theorem, which characterizes a non-trivial power of a Dehn twist algebraically.

Theorem 3.3 ([32, Theorem 7.5.B]). *Let M be a surface with $\kappa(M) > 0$ and let $m \geq 3$ be an integer. Let G be a finite index subgroup of $\Gamma(M; m)$. An element $g \in G$ is a non-trivial power of some Dehn twist (i.e., there are $n \in \mathbb{Z} \setminus \{0\}$ and $\alpha \in V(C)$ such that $g = t_\alpha^n$) if and only if the center of the centralizer of g in G is isomorphic to $\mathbb{Z}$ and is not equal to the centralizer of g in G.*

Note that in the notation of Theorem 3.3, if $g = t_\alpha^n$ for $n \in \mathbb{Z} \setminus \{0\}$ and $\alpha \in V(C)$, then

- the centralizer $C_G(g)$ of g in G is equal to the stabilizer of α in G, i.e., $\{h \in G : h\alpha = \alpha\}$;
- the center of $C_G(g)$ is equal to $G \cap \langle t_\alpha \rangle$, where $\langle t_\alpha \rangle$ is the subgroup of $\Gamma(M)$ generated by t_α.

Return to the proof of Theorem 3.2. It follows from Theorem 3.3 that the isomorphism $f\colon \Gamma_1 \to \Gamma_2$ induces a map $\varphi\colon V(C) \to V(C)$ determined by $f(t_\alpha^N) = t_{\varphi(\alpha)}^M$ for $\alpha \in V(C)$ and some non-zero integers N, M. Such an element $\varphi(\alpha)$ is uniquely determined by the following fact.

Lemma 3.4. *Let M be a surface with $\kappa(M) \geq 0$. For $\alpha, \beta \in V(C)$ and $(k,l) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, if $t_\alpha^k = t_\beta^l$, then $\alpha = \beta$ and $k = l$.*

For the proof of Lemma 3.4, we use the following lemma, which is shown by using the dynamics of Dehn twists on the Thurston boundary.

Lemma 3.5 ([36, Lemma 5.3]). *Let M be a surface with $\kappa(M) \geq 0$.*

(i) *If two curves $\alpha, \beta \in V(C)$ satisfy $i(\alpha, \beta) = 0$, then the subgroup of $\Gamma(M)$ generated by the Dehn twists t_α, $t_\beta \in \Gamma(M)$ about them is a free abelian group of rank 2. In particular, it is amenable.*

(ii) *On the other hand, if $i(\alpha, \beta) \neq 0$, then the subgroup of $\Gamma(M)$ generated by t_α^n and t_β^m is a non-abelian free group of rank 2 for all sufficiently large $n, m \in \mathbb{N}$.*

Proof of Lemma 3.4. Suppose that $t_\alpha^k = t_\beta^l$ for $\alpha, \beta \in V(C)$ and $(k,l) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. It is enough to prove that $\alpha = \beta$ because any Dehn twist is an element of infinite order. If $i(\alpha, \beta) \neq 0$, then it would contradict Lemma 3.5 (ii). Thus, $i(\alpha, \beta) = 0$. When $\kappa(M) = 0$, two distinct elements of $V(C)$ always have non-zero geometric intersection number. This shows that $\alpha = \beta$. Suppose that $\kappa(M) > 0$. If $\alpha \neq \beta$, then there would exist $\gamma \in V(C)$ such that $i(\alpha, \gamma) = 0$ and $i(\beta, \gamma) \neq 0$. This also contradicts Lemma 3.5. □

Return to the proof of Theorem 3.2. Since f is an isomorphism, it is easy to see that the map $\varphi\colon V(C) \to V(C)$ is a bijection. By using Lemma 3.5, one can show that φ induces an automorphism of the curve complex C, which comes from some $g \in \Gamma(M)^\diamond$ by Theorem 2.3. Namely, for each $\alpha \in V(C)$, there exist non-zero integers N, M such that $f(t_\alpha^N) = t_{g(\alpha)}^M$. Note that

$$g t_\alpha g^{-1} = t_{g\alpha}^\varepsilon \text{ for } \alpha \in V(C) \text{ and } g \in \Gamma(M)^\diamond, \tag{3.1}$$

where ε is 1 if $g \in \Gamma(M)$, and -1 otherwise (see Lemma 4.1.C in [32]). Let $\gamma \in \Gamma_1$. For each $\alpha \in V(C)$, we have

$$f(\gamma t_\alpha^N \gamma^{-1}) = f(\gamma) f(t_\alpha^N) f(\gamma)^{-1} = f(\gamma) t_{g(\alpha)}^M f(\gamma)^{-1} = t_{f(\gamma)g(\alpha)}^{\varepsilon M}$$

for some non-zero integers N, M and $\varepsilon \in \{\pm 1\}$. On the other hand,

$$f(\gamma t_\alpha^{N'} \gamma^{-1}) = f(t_{\gamma\alpha}^{\varepsilon' N'}) = t_{g(\gamma\alpha)}^{M'}$$

for some non-zero integers N', M' and $\varepsilon' \in \{\pm 1\}$. These equations imply that $f(\gamma) g(\alpha) = g(\gamma\alpha)$ by Lemma 3.4, and thus $f(\gamma)\beta = g(\gamma g^{-1}(\beta))$ for any

$\beta \in V(C)$. By Theorem 2.3, $f(\gamma) = g\gamma g^{-1}$ for any $\gamma \in \Gamma_1$. Uniqueness of g satisfying this equation follows from the fact that the center of Γ_1 is trivial (use the equation (3.1) and Theorem 2.3). This proves Theorem 3.2. □

To sum up, Ivanov's proof of Theorem 3.2 is outlined as follows. Steps (A), (B) correspond to our Steps (a), (b), respectively. We use the notation of Theorem 3.2.

(A) Characterize Dehn twists algebraically, and show that the isomorphism $f\colon \Gamma_1 \to \Gamma_2$ preserves Dehn twists.

(B) Define a bijection $\varphi\colon V(C) \to V(C)$ by the equation $f(t_\alpha^N) = t_{\varphi(\alpha)}^M$ for each $\alpha \in V(C)$ and some non-zero integers N, M. Show that φ defines an element of $\mathrm{Aut}(C)$. Let $g \in \Gamma(M)^\diamond \simeq \mathrm{Aut}(C)$ be the corresponding element.

(C) By a direct calculation, show the equation $f(\gamma) = g\gamma g^{-1}$ for all $\gamma \in \Gamma_1$.

The case of groupoids. Return to our situation. Our Step (a) corresponds to Ivanov's Step (A) and is stated explicitly as follows.

Theorem 3.6. *Under Assumption* ($\bullet$), *for each* $\alpha \in V(C)$, *there exist a countable Borel partition* $Y = \bigsqcup Y_n$ *and* $\beta_n \in V(C)$ *such that*

$$f((\mathcal{G}_\alpha^1)_{Y_n}) = (\mathcal{G}_{\beta_n}^2)_{f(Y_n)} \quad \textit{for each } n.$$

As explained in Step (a), this equation means that f preserves subgroupoids generated by Dehn twists after taking some countable Borel partition of Y.

Remark 3.7. Note that if a countable Borel partition $Y = \bigsqcup Y'_m$ and $\beta'_m \in V(C)$ also satisfy the equation in Theorem 3.6 and if $Z = f(Y_n \cap Y'_m)$ has positive measure for some n and m, then $(\mathcal{G}_{\beta_n}^2)_Z = (\mathcal{G}_{\beta_{m'}}^2)_Z$. It follows from Proposition 2.26 that there exist non-zero integers N, M and $x \in Z$ such that $(t_{\beta_n}^N, x) = (t_{\beta_{m'}}^M, x) \in (\mathcal{G}_{\beta_n}^2)_Z = (\mathcal{G}_{\beta_{m'}}^2)_Z$, which implies that $\beta_n = \beta_{m'}$ by Lemma 3.4.

The subsequent sections of this chapter will be devoted to the proof of Theorem 3.6. In Section 7, our plan of the proof will be presented. In this section, assuming Theorem 3.6, we proceed to Step (b).

About Step (b). Assuming Theorem 3.6, we construct the map Ψ in Step (b). We use the notation in Assumption ($\bullet$). Let $\Psi\colon Y \times V(C) \to V(C)$ be the Borel map defined by

$$\Psi(x, \alpha) = \beta_n \quad \text{if} \quad x \in Y_n$$

for $\alpha \in V(C)$, where $Y = \bigsqcup Y_n$ and β_n are chosen for α as in Theorem 3.6. By Remark 3.7, this definition does not depend on the choice of the countable

Borel partition of Y. It can be shown that the map $\Psi(x,\cdot)\colon V(C)\to V(C)$ is a bijection (because f is an isomorphism), and moreover $\Psi(x,\cdot)$ defines an automorphism of the curve complex C for a.e. $x\in Y$. Namely, $\Psi(x,\cdot)$ satisfies the following two conditions for a.e. $x\in Y$:

- If $\alpha,\beta\in V(C)$ satisfy $i(\alpha,\beta)=0$, then $i(\Psi(x,\alpha),\Psi(x,\beta))=0$;
- If $\alpha,\beta\in V(C)$ satisfy $i(\alpha,\beta)\neq 0$, then $i(\Psi(x,\alpha),\Psi(x,\beta))\neq 0$.

This fact can be shown by utilizing Lemma 3.5 and some elementary facts about amenable discrete measured groupoids. (In Section 4, amenability of a discrete measured groupoid will be introduced, which is an isomorphism invariant of discrete measured groupoids.)

Therefore, we can define a Borel map $\Psi\colon Y\to \mathrm{Aut}(C)$ by $\Psi(x)=\Psi(x,\cdot)$ for $x\in Y$. Using the equation (3.1) in Ivanov's proof, we can show that this map Ψ satisfies the following equality:

$$\Psi(r(\delta))=\pi\circ\rho_2(f(\delta))\Psi(s(\delta))\pi\circ\rho_1(\delta)^{-1},$$

or equivalently,

$$\Psi(\gamma\cdot x)=\pi\circ\rho_2(f(\gamma,x))\Psi(x)\pi(\gamma)^{-1} \tag{3.2}$$

for a.e. $\delta=(\gamma,x)\in(\mathcal{G}^1)_Y$ (see Lemma 5.5 in [36]), where $\pi\colon\Gamma(M)^\diamond\to \mathrm{Aut}(C)$ is the natural homomorphism. Note that Ψ, $\pi\circ\rho_2(f(\delta))$ and $\pi\circ\rho_1(\delta)$ correspond to g, $f(\gamma)$ and γ in Ivanov's argument, respectively. Thus, the equation (3.2) corresponds to his conclusion $g=f(\gamma)g\gamma^{-1}$.

About Step (c). Recall that the action of $\Gamma_1\times\Gamma_2$ on $X_1\times\Gamma_2$ was defined by

$$(\gamma_1,\gamma_2)(x,\gamma)=(\gamma_1\cdot x,\alpha(\gamma_1,x)\gamma\gamma_2^{-1}),\quad \gamma_1\in\Gamma_1,\ \gamma_2,\gamma\in\Gamma_2,\ x\in X_1$$

(see Lemma 2.30). Here, $\alpha\colon\Gamma_1\times X_1\to\Gamma_2$ is the ME cocycle associated with a fundamental domain X_1 of the Γ_2-action on Ω. Note that $\rho_2\circ f=\alpha$ on $(\mathcal{G}^1)_Y$ (see the definition of f introduced right before Proposition 2.29). Recall that the equation $(\Gamma_1\times\Gamma_2)(Y\times\{e\})=X_1\times\Gamma_2$ holds up to null sets. Therefore, we define a Borel map $\Phi\colon X_1\times\Gamma_2\to\mathrm{Aut}(C)$ by

$$\Phi((\gamma_1,\gamma_2)(x,e))=\pi(\gamma_1)\Psi(x)^{-1}\pi(\gamma_2)^{-1}$$

for $\gamma_1\in\Gamma_1$, $\gamma_2\in\Gamma_2$ and $x\in Y$. If it is well-defined, then it is easy to see that Φ is almost $(\Gamma_1\times\Gamma_2)$-equivariant. Take $\gamma_1,\gamma_1'\in\Gamma_1$, $\gamma_2,\gamma_2'\in\Gamma_2$ and $x,x'\in Y$ satisfying the equality

$$(\gamma_1,\gamma_2)(x,e)=(\gamma_1',\gamma_2')(x',e).$$

This equality implies that

$$(x,e)=(\gamma_1^{-1}\gamma_1',\gamma_2^{-1}\gamma_2')(x',e)=((\gamma_1^{-1}\gamma_1')\cdot x',\alpha(\gamma_1^{-1}\gamma_1',x')(\gamma_2^{-1}\gamma_2')^{-1}).$$

Hence, $(\gamma_1^{-1}\gamma_1') \cdot x' = x \in Y$. By using the equation (3.2) and the equation $\rho_2 \circ f = \alpha$ on $(\mathcal{G}^1)_Y$, we see that

$$\begin{aligned}\Psi(x) &= \Psi((\gamma_1^{-1}\gamma_1') \cdot x') = \pi \circ \rho_2(f(\gamma_1^{-1}\gamma_1', x'))\Psi(x')\pi(\gamma_1^{-1}\gamma_1')^{-1} \\ &= \pi \circ \alpha(\gamma_1^{-1}\gamma_1', x')\Psi(x')\pi(\gamma_1^{-1}\gamma_1')^{-1} = \pi(\gamma_2^{-1}\gamma_2')\Psi(x')\pi(\gamma_1^{-1}\gamma_1')^{-1}.\end{aligned}$$

This implies that $\pi(\gamma_1)\Psi(x)^{-1}\pi(\gamma_2)^{-1} = \pi(\gamma_1')\Psi(x')^{-1}\pi(\gamma_2')^{-1}$ and that the map Φ is well-defined. This shows Step (c).

Therefore, the remaining problem is to show Theorem 3.6, which will be explained in the subsequent sections. The first Step (A) for Ivanov's proof is to show that the isomorphism $f\colon \Gamma_1 \to \Gamma_2$ preserves powers of Dehn twists. To prove this, he characterized a power of a Dehn twist algebraically as in Theorem 3.3. In our case, to prove that the groupoid isomorphism $f\colon (\mathcal{G}^1)_Y \to (\mathcal{G}^2)_Y$ preserves subgroupoids generated by Dehn twists as in Theorem 3.6, we characterize such a subgroupoid algebraically in terms of discrete measured groupoids. However, we cannot expect a characterization similar to that of Theorem 3.3 because there is no notion corresponding to centralizers and centers in the theory of discrete measured groupoids. In the subsequent sections, we give a characterization of a subgroupoid generated by a Dehn twist from another point of view. This is formulated in terms of amenable, non-amenable subgroups and normal subgroups (see Propositions 7.7 and 7.8). Since amenability of a discrete measured groupoid and normality of a subgroupoid are invariant under isomorphism of groupoids, subgroupoids generated by Dehn twists are preserved by f thanks to this characterization.

In Sections 4, 5 and 6, we introduce many notions necessary for the formulation of this characterization. In Section 7, the characterization is given.

3.2 Deriving ME rigidity from reduction of self ME couplings

As an application of Theorem 3.1, we prove Theorems 1.5 and 1.6.

ME rigidity. The process to deduce ME rigidity from reduction of self ME couplings has already been developed by Furman [16], and Monod and Shalom [48]. We review their techniques here. Recall the following two operations to construct a new ME coupling from a given ME coupling.

An opposite coupling. Let (Σ, m) be an ME coupling of discrete groups Γ and Λ. Then an ME coupling $(\check{\Sigma}, \check{m})$ of Λ and Γ is defined as follows: As a measure space, $(\check{\Sigma}, \check{m}) = (\Sigma, m)$. The action of $\Lambda \times \Gamma$ on $(\check{\Sigma}, \check{m})$ is defined via the canonical isomorphism between $\Gamma \times \Lambda$ and $\Lambda \times \Gamma$.

A composed coupling. If (Σ, m) is an ME coupling of discrete groups Γ and Λ and if (Ω, n) is an ME coupling of discrete groups Λ and Δ, then an ME coupling $\Sigma \times_\Lambda \Omega$ of Γ and Δ is defined to be the quotient space of $\Sigma \times \Omega$ by the diagonal Λ-action, equipped with the induced action of $\Gamma \times \Delta$.

Remark 3.8. By using the above two associated couplings, we see that ME is an equivalence relation among discrete groups (see Section 2 in [16]). Note that a discrete group Γ is itself an ME coupling of Γ and Γ as in Example 1.3.

Let M be a surface with $\kappa(M) > 0$ and let $\Gamma = \Gamma(M)$ be the mapping class group. Let (Σ, m) be an ME coupling of Γ and an unknown group Λ. Construct the self ME coupling

$$\Omega = \Sigma \times_\Lambda \Lambda \times_\Lambda \check{\Sigma}$$

of Γ. We denote by $[x, \lambda, y] \in \Omega$ the projection of $(x, \lambda, y) \in \Sigma \times \Lambda \times \check{\Sigma}$. By applying Theorem 3.1, we obtain an almost $(\Gamma \times \Gamma)$-equivariant Borel map $\Phi\colon \Omega \to \mathrm{Aut}(C)$, i.e.,

$$\Phi((\gamma_1, \gamma_2)z) = \pi(\gamma_1)\Phi(z)\pi(\gamma_2)^{-1}$$

for any $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $z \in \Omega$, where $\pi\colon \Gamma(M)^\diamond \to \mathrm{Aut}(C)$ is the natural homomorphism. From this map, we want to construct a representation ρ of the unknown group Λ on $\mathrm{Aut}(C)$. We first consider the following special case.

Example 3.9 ([16, Example 5.1]). Let G be a locally compact second countable group and let Γ and Λ be lattices in G. Then G equipped with its Haar measure is an ME coupling of Γ and Λ as in Example 1.2. Define a Borel map

$$\Phi\colon G \times_\Lambda \Lambda \times_\Lambda \check{G} \to G$$

by $\Phi([x, \lambda, y]) = x\lambda y^{-1}$. This map is $(\Gamma \times \Gamma)$-equivariant. Observe that the map

$$\lambda \mapsto \Phi([x, \lambda, y])\Phi([x, e, y])^{-1} = (x\lambda y^{-1})(xy^{-1})^{-1} = x\lambda x^{-1}$$

does not depend on y, and defines a representation of Λ on G for a fixed x.

From this observation, in our case, we can also expect that the map

$$\lambda \mapsto \Phi([x, \lambda, y])\Phi([x, e, y])^{-1} \tag{3.3}$$

does not depend on y, and defines a representation of Λ on $\mathrm{Aut}(C)$ for a.e. $x \in \Sigma$. In fact, we can show these claims by using the following notable fact.

Theorem 3.10 ([36, Theorem 2.6]). *Let C be the curve complex of a surface M with $\kappa(M) > 0$. Let Γ be a finite index subgroup of* $\mathrm{Aut}(C)$. *Then the set*

$$\{\gamma g \gamma^{-1} \in \mathrm{Aut}(C) : \gamma \in \Gamma\}$$

is infinite for any $g \in \mathrm{Aut}(C) \setminus \{e\}$.

We do not here present how to use this theorem. It can be shown that the kernel of the representation

$$\rho_x\colon \Lambda \to \mathrm{Aut}(C), \quad \rho_x(\lambda) = \Phi([x, \lambda, y])\Phi([x, e, y])^{-1}$$

and the index $[\mathrm{Aut}(C) : \rho_x(\Lambda)]$ are both finite, which implies Theorem 1.5. To construct this representation, we do not use special properties of the mapping class group other than the one in Theorem 3.10. In fact, this construction can be applied to a more general setting (see Theorem 6.1 in [36]).

Lattice embeddings of mapping class groups. We briefly give an outline of the proof of Theorem 1.6. We explain only the construction of Φ_0 in the statement. The reader is referred to Section 8 in [36] for more details.

Let M be a surface with $\kappa(M) > 0$ and let Γ be a finite index subgroup of $\Gamma(M)^\diamond$. Let G be a locally compact second countable group and let $\sigma\colon \Gamma \to G$ be a lattice embedding, i.e., an injective homomorphism such that the image $\sigma(\Gamma)$ is a lattice in G. As in Example 1.2, G is a self ME coupling of Γ (via σ). By Theorem 3.1, there exists an almost $(\Gamma \times \Gamma)$-equivariant Borel map $\Phi\colon G \to \mathrm{Aut}(C)$. By using Theorem 3.10 and the fact that the self ME coupling G of Γ is not only a measure space but a group, we can show that $\Phi(g_1 g_2) = \Phi(g_1)\Phi(g_2)$ for a.e. $(g_1, g_2) \in G \times G$. Recall the following theorem.

Theorem 3.11 ([63, Theorems B.2, B.3]). *If H_1, H_2 are locally compact second countable groups and $f\colon H_1 \to H_2$ is a Borel map such that $f(hh') = f(h)f(h')$ for a.e. $(h, h') \in H_1 \times H_1$, then there exists a continuous homomorphism $f_0\colon H_1 \to H_2$ such that f and f_0 are equal a.e. on H_1.*

It follows that there exists a continuous homomorphism $\Phi_0\colon G \to \mathrm{Aut}(C)$ such that Φ and Φ_0 are equal a.e. on G. It is easy to check that $K = \ker\Phi_0$ admits a finite invariant measure. Therefore, K is compact. After several easy observations, we see that this Φ_0 is a desired homomorphism.

3.3 OE rigidity

In this subsection, we briefly give another application of Theorem 3.1. We prove a rigidity result for ergodic standard actions of mapping class groups in terms of OE.

Corollary 3.12. *Let M be a surface with $\kappa(M) > 0$ and $M \neq M_{1,2}, M_{2,0}$. Put $\Gamma = \Lambda = \Gamma(M)^\diamond$. Let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ be ergodic standard (i.e., measure-preserving and essentially free) actions on standard finite measure spaces. If the two actions are OE, then they are conjugate.*

Proof. Since the two actions are OE, there exists a measure space isomorphism $f\colon (X,\mu)\to(Y,\nu)$ such that

$$f(\Gamma x)=\Lambda f(x) \quad \text{for a.e. } x\in X.$$

One can then construct the OE cocycle $\alpha\colon \Gamma\times X\to\Lambda$ associated with f by the equation

$$f(\gamma x)=\alpha(\gamma,x)f(x) \quad \text{for } \gamma\in\Gamma \text{ and a.e. } x\in X.$$

By Theorem 2.32, we can construct an ME coupling (Ω,m) of Γ and Λ such that the ME cocycle associated with some fundamental domain of the Λ-action on Ω, which can be identified with X, is equal to α. In what follows, we denote the action $\Gamma\curvearrowright X$ by $(\gamma,x)\mapsto\gamma\cdot x$, using a dot.

By Theorem 3.1, there exists an almost $(\Gamma\times\Lambda)$-equivariant Borel map $\Phi\colon\Omega\to G$, where $G=\Gamma(M)^\diamond$ and $\mathrm{Aut}(C)$ are identified via the natural isomorphism $\pi\colon\Gamma(M)^\diamond\to\mathrm{Aut}(C)$ (see Theorem 2.3). Let $\varphi\colon X\to G$ be the Borel map defined by $\varphi(x)=\Phi(x)$ for $x\in X$. Then

$$\begin{aligned}&\varphi(\gamma\cdot x)\alpha(\gamma,x)\varphi(x)^{-1}=\Phi(\gamma\cdot x)\alpha(\gamma,x)\Phi(x)^{-1}\\ =&\Phi(\gamma\alpha(\gamma,x)x)\alpha(\gamma,x)\Phi(x)^{-1}=\gamma\Phi(x)\Phi(x)^{-1}=\gamma\end{aligned}$$

for any $\gamma\in\Gamma$ and a.e. $x\in X$. Define a Borel map $f_\varphi\colon X\to Y$ by $f_\varphi(x)=\varphi(x)f(x)$ for $x\in X$. Then for any $\gamma\in\Gamma$ and a.e. $x\in X$,

$$\begin{aligned}f_\varphi(\gamma\cdot x)&=\varphi(\gamma\cdot x)f(\gamma\cdot x)=\varphi(\gamma\cdot x)\alpha(\gamma,x)f(x)\\&=\gamma\varphi(x)f(x)=\gamma f_\varphi(x).\end{aligned}$$

Since the actions $\Gamma\curvearrowright(X,\mu)$ and $\Lambda\curvearrowright(Y,\nu)$ are both essentially free and f is a measure space isomorphism, the above equation implies that $f_\varphi\colon X\to Y$ is a measure space isomorphism. □

Remark 3.13. We can show the following much stronger rigidity theorem than Corollary 3.12.

Theorem 3.14 ([37, Theorem 1.1]). *Let M be a surface with $\kappa(M)>0$. If an ergodic standard action of a finite index subgroup Γ of $\Gamma(M)^\diamond$ and an ergodic standard action of a discrete group Λ are WOE, then the two actions are virtually conjugate.*

See Definition 1.3 in [37] for the definition of virtual conjugacy. In particular, the conclusion of this theorem implies that Γ and Λ are commensurable up to finite kernel. We refer to [17], [19], [37], [48], [55], [56], [60] for other rigidity results in terms of OE. See also the fourth remark in Section 8. These rigidity theorems and Theorem 3.14 sharply contrast with the following theorem due to Ornstein and Weiss.

Theorem 3.15 ([52]). *Let G_1 and G_2 be infinite amenable groups and suppose that G_i admits an ergodic standard action on a standard finite measure space (X_i, μ_i) for $i \in \{1, 2\}$. Then the two actions are OE.*

It is known that amenability of the acting group is preserved under OE. More precisely, let $G_i \curvearrowright (X_i, \nu_i)$ be an ergodic standard action of a discrete group G_i for $i \in \{1, 2\}$. If the two actions are OE and G_1 is amenable, then G_2 is also amenable (see Theorem 4.18 (i), (ii)). Therefore, Theorem 1.8 is a consequence of Corollary 2.34 and Theorem 3.15. Connes, Feldman, and Weiss [10] proved a generalization of Theorem 3.15 in terms of discrete measured equivalence relations (see Theorem 4.17).

It is well-known that there are many non-conjugate ergodic standard actions of $\mathbb{Z}$ as follows. Let Γ be a discrete group and let (X_0, μ_0) be a standard probability space, i.e., a standard Borel space with a probability measure. We assume that (X_0, μ_0) may contain an atom, whereas (X_0, μ_0) does not consist of a single atom. The Bernoulli action of Γ associated with (X_0, μ_0) is the action of Γ on the product space $(X_0, \mu_0)^\Gamma = \prod_\Gamma (X_0, \mu_0)$ given by

$$\gamma(x_g)_{g\in\Gamma} = (x_{\gamma^{-1}g})_{g\in\Gamma}, \quad \gamma \in \Gamma,\ (x_g)_{g\in\Gamma} \in X_0^\Gamma.$$

It is a natural question to understand when Bernoulli actions of $\mathbb{Z}$ arising from two different standard probability spaces are conjugate. Kolmogorov and Sinaĭ introduced a conjugacy invariant for actions of $\mathbb{Z}$, called entropy, and showed that the entropy of Bernoulli actions of $\mathbb{Z}$ can be computed in terms of (X_0, μ_0) and assumes all non-negative values. In particular, there exist continuously many conjugacy classes of ergodic actions of $\mathbb{Z}$. As the culmination of the study on this conjugacy problem, Ornstein [50], [51] proved that entropy is a complete invariant for Bernoulli actions of $\mathbb{Z}$, that is, two Bernoulli actions of $\mathbb{Z}$ which have the same entropy are conjugate. Moreover, this theory of entropy was extended to the setting of Bernoulli actions of infinite amenable groups by Ornstein and Weiss [53].

4 Amenable discrete measured groupoids

In the study of discrete measured groupoids, amenability is one of the most important notions like amenability of groups. One can construct a discrete measured groupoid from a non-singular action of a discrete group on a standard measure space (see Example 2.20). If the groupoid associated with a non-singular action of a discrete group is amenable, then the action is said to be amenable. This notion was first introduced by Zimmer [62]. One advantage of studying amenability of a group action is that (the groupoid arising from) an amenable action of a group behaves like an amenable group even if the acting group is non-amenable. We can thus apply various techniques for amenable groups to study a non-amenable group via its amenable action. Another advantage of the study of amenable groupoids is that under a certain condition, we can easily decide whether a groupoid is amenable or not. Since amenability is invariant under isomorphism of groupoids, this property is often used to distinguish two groupoids.

In this section, we recall the definition of amenable discrete measured groupoids and some of their fundamental properties. References for the material of this section is [4], [5] and Chapter 4 in [63]. We recommend the reader to consult [54] for applications of amenable actions of groups.

As discussed in Section 3, our final goal is to prove Theorem 3.6. For this purpose, we analyze various subgroupoids of the groupoid arising from a measure-preserving action of the mapping class group. It will be often necessary to prove amenability of some subgroupoids. To prove it, we make use of the amenability (in a measurable sense) of the action of the mapping class group on the boundary ∂C of the curve complex C. This fact will be explained at the end of this section. We note here that in this chapter, this amenability of the boundary action will be used only in the proof of Theorem 5.10.

Amenable groups. We first recall the notion of amenability of discrete groups. Although we can proceed to most parts of this section under the assumption that a group is locally compact and second countable, we always assume that a group is discrete for simplicity. We refer to Section 4.1 in [63] for amenability of locally compact second countable groups. Although there are many equivalent definitions of amenability of groups, we recall only the definition which motivated Zimmer to define amenability of a group action.

Let G be a discrete group. Let A be a non-empty compact convex subset in the closed unit ball of E^*, where E is a separable Banach space and its dual E^* is equipped with the weak*-topology. Suppose that G acts on E by isometric isomorphisms and that A is invariant for the induced action of G on E^*. Such an action of G is called an *affine action* on A.

Definition 4.1. Let G be a discrete group. We say that G is *amenable* if for every affine action of G on a space A like above, there exists a fixed point, that is, $a \in A$ such that $ga = a$ for any $g \in G$.

Example 4.2. We refer to Section 4 in [63] for the proof of the following facts.

(i) Finite groups and abelian groups are both amenable.

(ii) Let $1 \to A \to B \to C \to 1$ be an exact sequence of discrete groups. Then A and C are both amenable if and only if B is amenable. Hence, all solvable groups are amenable.

(iii) If G is a discrete group and $\{H_i\}_{i\in I}$ is a directed set of amenable subgroups of G, then the union $\bigcup_{i\in I} H_i$ is also amenable. For example, the following groups are amenable:

- The direct product $\oplus_{n\in\mathbb{N}} H_n$. Here, H_n is an amenable group.
- The infinite symmetric group $\mathfrak{S}_\infty = \bigcup_{n\in\mathbb{N}} \mathfrak{S}_n$. Here, $\mathfrak{S}_n$ is the symmetric group on n letters, and $\mathfrak{S}_n$ is identified with the subgroup of $\mathfrak{S}_{n+1}$ fixing the $(n+1)$-st letter.

(iv) Non-abelian free groups are typical examples of non-amenable groups. Therefore, every group containing a non-abelian free subgroup is non-amenable.

Example 4.3. Let G be an amenable group and suppose that G acts on a separable compact space K continuously. We denote by $M(K)$ the space of all probability measures on K with the weak*-topology, on which G acts continuously. Note that $M(K)$ is a weak*-closed, convex subset of the closed unit ball of $C(K)^*$, where $C(K)$ is the Banach space of $\mathbb{C}$-valued continuous functions on K with the sup norm. By the definition of amenability, there exists $\mu \in M(K)$ such that $g\mu = \mu$ for any $g \in G$.

If G is an infinite amenable subgroup of the mapping class group $\Gamma(M)$ for a surface M with $\kappa(M) \geq 0$, then it follows from Theorem 2.7 that there exists a non-empty finite subset $S \subset \mathcal{PMF}$ such that $gS = S$ for any $g \in G$. More explicitly, if G is IA, then we put $S = \{F_\pm(g)\}$ for some pseudo-Anosov element $g \in G$. If G is reducible, then there exists $\sigma \in S(M)$ fixed by all elements of G, and we put $S = \{\sigma\}$. The uniformly distributed probability measure on S is then a fixed point for the action of G on $M(\mathcal{PMF})$. Therefore, in this case, we can explicitly find an invariant probability measure on $\mathcal{PMF}$ for each amenable subgroup of $\Gamma(M)$.

Amenable groupoids. Zimmer [62] defined amenability of a group action as an analogue of Definition 4.1. The following definition of an amenable discrete measured groupoid is introduced in Chapter 4 of [4], which is a generalization of Zimmer's definition. A precise definition of amenable discrete measured groupoids is somehow complicated. After giving it, we recall several fundamental facts. The readers unfamiliar with this notion are recommended to consult [5], where a survey of amenability of groupoids is given.

When we defined amenability of groups, we considered an action of it on a separable Banach space. In the definition of amenability of groupoids $\mathcal{G}$, it is necessary to consider measurable bundles over the unit space of $\mathcal{G}$ whose fiber is an object appearing in the definition of amenability of groups. We first introduce an object on which a groupoid acts, called a measurable Banach bundle. A reference for the material in the following Definitions 4.4 and 4.6 is Chapter II in [15]. In the first definition, we shall recall basic terminology in measure theory. Recall that we refer to a standard Borel space X equipped with a σ-finite positive measure μ as a standard measure space. If $\mu(X) < \infty$, then we say that (X, μ) is a standard finite measure space.

Definition 4.4 ([15, II.1]). Let (X, μ) be a standard measure space. We denote by $\mathcal{B}$ the set of all Borel subsets of X.

(i) A subset A of X is *μ-null* if there exists a countable family $\{A_n\}_n$ of elements of $\mathcal{B}$ such that $A \subset \bigcup_n A_n$ and $\mu(A_n) = 0$ for all n.

(ii) A subset A of X is *μ-measurable* if the symmetric difference $A \triangle B$ is μ-null for some $B \in \mathcal{B}$.

(iii) A property of points of X which holds for all x outside some μ-null subset of X is said to hold for *μ-almost every* (or *μ-a.e.*) x.

(iv) A map $f\colon X \to Y$ into a standard Borel space Y is *μ-measurable* if $f^{-1}(A)$ is μ-measurable for any Borel subset A of Y.

The following lemma is an easy exercise. For the proof, note that the σ-field of Borel subsets of a standard Borel space is generated by countably many Borel subsets of it as a σ-field.

Lemma 4.5. *Let (X, μ) be a standard measure space. If $\varphi\colon X \to Y$ is a μ-measurable map into a standard Borel space Y, then there exist a Borel map $\psi\colon X \to Y$ and a Borel subset X' of X such that $\mu(X \setminus X') = 0$ and $\varphi(x) = \psi(x)$ for all $x \in X'$.*

We next introduce the notion of measurable Banach bundles over a standard measure space (X, μ). Suppose that for each $x \in X$, we are given a Banach space E_x. We refer to a function f on X such that $f(x) \in E_x$ for each $x \in X$ as a *vector field* on X. We will define measurability of such a vector field. We equip the complex field $\mathbb{C}$ with the structure of a standard Borel space associated with the usual topology of $\mathbb{C}$.

Definition 4.6 ([15, II.4]). In the above notation, a *μ-measurable structure* for the family $\{E_x\}_{x \in X}$ is a non-empty family $\mathcal{M}$ of vector fields on X satisfying the following four conditions:

(i) If $f, g \in \mathcal{M}$, then the vector field $x \mapsto f(x) + g(x)$ is also in $\mathcal{M}$.

(ii) If $f \in \mathcal{M}$ and a map $\phi\colon X \to \mathbb{C}$ is μ-measurable, then the vector field $x \mapsto \phi(x) f(x)$ is also in $\mathcal{M}$.

(iii) If $f \in \mathcal{M}$, then the function $x \mapsto \|f(x)\|$ is μ-measurable, where $\|\cdot\|$ is the norm on E_x.

(iv) Suppose that f is a vector field on X such that there exists a sequence $\{g_n\}$ of elements of $\mathcal{M}$ such that $g_n(x) \to f(x)$ in E_x as $n \to \infty$ for μ-a.e. $x \in X$. Then $f \in \mathcal{M}$.

The family $\{E_x\}_{x\in X}$ endowed with this structure $\mathcal{M}$ is called a *measurable Banach bundle* over (X,μ), and is denoted by E. We refer to an element of $\mathcal{M}$ as a *measurable section* for the bundle E.

In the next definition, we introduce the notion of separability for a measurable Banach bundle.

Definition 4.7 ([4, Definition A.3.4]). Let $E=(\{E_x\}_{x\in X},\mathcal{M})$ be a measurable Banach bundle over a standard measure space (X,μ). We say that E is *separable* if there exists a sequence $\{g_n\}_n$ of elements of $\mathcal{M}$ such that the set $\{g_n(x)\}_n$ is total in E_x for μ-a.e. $x\in X$, that is, the set of all finite $\mathbb{C}$-linear combinations of elements in $\{g_n(x)\}_n$ is dense in E_x.

Remark 4.8. Let $E=(\{E_x\}_{x\in X},\mathcal{M})$ be a measurable Banach bundle over a standard measure space (X,μ).

(i) Let (Y,ν) be a standard measure space and suppose that we are given a Borel map $\pi\colon Y\to X$ such that $\pi_*\nu$ and μ are equivalent. The set $\mathcal{N}=\{f\circ\pi : f\in\mathcal{M}\}$ generates a ν-measurable structure $\pi^*\mathcal{M}$ for the family $\{E_{f(y)}\}_{y\in Y}$. We denote by π^*E the corresponding bundle over (Y,ν) and call it the *pull-back* of E by π. If E is separable, then so is π^*E (see Example (3) of Appendix A in [4]).

(ii) Consider the family $\{E_x^*\}_{x\in X}$ of the duals. We denote by $\mathcal{M}^*$ the set of all vector fields φ for this family such that the function $x\mapsto\langle\varphi(x),f(x)\rangle$ is μ-measurable for all $f\in\mathcal{M}$. The following fact is known (see Lemma A.3.7 in [4]): If E is separable, then $E^*=(\{E_x^*\}_{x\in X},\mathcal{M}^*)$ is a measurable Banach bundle.

(iii) When E is separable, we denote by $L^\infty(X,E^*)$ the space of all $\varphi\in\mathcal{M}^*$ such that the function $x\mapsto\|\varphi(x)\|$ belongs to $L^\infty(X)$, and we denote by $\|\varphi\|_\infty$ the μ-essential supremum for this function. It is known that $L^\infty(X,E^*)$ is a Banach space with respect to the norm $\|\cdot\|_\infty$ (see Proposition A.3.9 in [4]).

We next define an action of a discrete measured groupoid on a measurable Banach bundle.

Definition 4.9 ([4, Definition 4.1.1]). Let $\mathcal{G}$ be a discrete measured groupoid on a standard measure space (X,μ). A *measurable $\mathcal{G}$-bundle* over (X,μ) is a pair (E,L), where $E=(\{E_x\}_{x\in X},\mathcal{M})$ is a measurable Banach bundle over (X,μ), and L is a linear isometric representation of $\mathcal{G}$ on E. Namely,

- for each $\gamma\in\mathcal{G}$, L gives an isometric isomorphism $L(\gamma)\colon E_{s(\gamma)}\to E_{r(\gamma)}$;
- L preserves products, i.e., $L(\gamma_1\gamma_2)=L(\gamma_1)L(\gamma_2)$ for all $(\gamma_1,\gamma_2)\in\mathcal{G}^{(2)}$;

- L is measurable in the sense that for each $f \in \mathcal{M}$, the vector field $\gamma \mapsto L(\gamma)f(s(\gamma))$ for the family $\{E_{r(\gamma)}\}_{\gamma\in\mathcal{G}}$ is in $r^*\mathcal{M}$, where $r\colon \mathcal{G} \to X$ is the range map.

Remark 4.10. In Definition 4.9, assume that E is separable. The pair (E^*, L^*) defined by the following equation then gives a measurable $\mathcal{G}$-bundle over (X, μ), and we call it the *dual $\mathcal{G}$-bundle* of the $\mathcal{G}$-bundle E:

$$\langle L^*(\gamma)e^*, e\rangle = \langle e^*, L(\gamma^{-1})e\rangle \quad \text{for } \gamma \in \mathcal{G},\ e^* \in E^*_{s(\gamma)},\ e \in E_{r(\gamma)}.$$

The next definition introduces the notion corresponding to convex, weak*-closed subsets contained in the closed unit ball of the dual of a separable Banach space appearing in the definition of amenability of groups.

Definition 4.11 ([4, Definitions 4.2.1, 4.2.5]). Let (X, μ) be a standard measure space.

(i) Let $E = (\{E_x\}_{x\in X}, \mathcal{M})$ be a separable measurable Banach bundle over (X, μ). Suppose that for each $x \in X$, we are given a subset A_x of the closed unit ball of the dual E^*_x. We refer to the family $A = \{A_x\}_{x\in X}$ as a *measurable field* for the dual E^* if there exists a sequence $\{\psi_n\}_n$ of elements of $L^\infty(X, E^*)$ such that A_x is the closed convex hull of the set $\{\psi_n(x)\}_n$ for μ-a.e. $x \in X$.

(ii) Let $\mathcal{G}$ be a discrete measured groupoid on (X, μ) and let (E, L) be a separable measurable $\mathcal{G}$-bundle over (X, μ). A measurable field $A = \{A_x\}_{x\in X}$ for the dual E^* is called a *$\mathcal{G}$-field* if $L^*(\gamma)A_{s(\gamma)} = A_{r(\gamma)}$ for $\tilde{\mu}$-a.e. $\gamma \in \mathcal{G}$, where $\tilde{\mu}$ is the measure on $\mathcal{G}$ introduced in Definition 2.13.

Finally, we define amenability of discrete measured groupoids as follows.

Definition 4.12. A discrete measured groupoid $\mathcal{G}$ on a standard measure space (X, μ) is *amenable* if the following holds: For any separable measurable $\mathcal{G}$-bundle (E, L) over (X, μ) and for any $\mathcal{G}$-field $A = \{A_x\}_{x\in X}$ for the dual E^*, there exists $\varphi \in L^\infty(X, E^*)$ such that

- $\varphi(x) \in A_x$ for μ-a.e. $x \in X$;
- $L^*(\gamma)\varphi(s(\gamma)) = \varphi(r(\gamma))$ for $\tilde{\mu}$-a.e. $\gamma \in \mathcal{G}$.

When the discrete measured groupoid arising from a non-singular action of a discrete group on a standard finite measure space is amenable, we say that the action is *amenable*.

Note that in Zimmer's definition of amenable actions of groups, only constant Banach bundles (i.e., bundles $\{E_x\}_{x\in X}$ such that E_x is the same for all $x \in X$) are considered instead of general Banach bundles as above. However, Zimmer's definition is equivalent to the above one. The proof of this fact is given in Theorem 4.2.7 in [4] and Section 3 in [3].

Compared with the definition of amenability of groups in Definition 4.1, the second condition for φ in Definition 4.12 can be phrased by saying that φ is a fixed point for the action L of the groupoid $\mathcal{G}$ on E. In general, given a group G and a space S, we refer to a homomorphism $G \to \mathrm{Aut}(S)$ as an action of G on S. Hence, given a groupoid $\mathcal{G}$, we should refer to a groupoid homomorphism $\mathcal{G} \to \mathrm{Aut}(S)$ as an action of $\mathcal{G}$ on S. We next define a fixed point for such an action of a groupoid. However, for a standard Borel space S, we know no natural Borel structure on $\mathrm{Aut}(S)$, the group of Borel automorphisms of S. Hence, we consider nothing but the following special action of a groupoid when the groupoid admits a Borel structure. In what follows, a groupoid homomorphism from a discrete measured groupoid $\mathcal{G}$ into a discrete group Γ is always assumed to be Borel as a map from $\mathcal{G}$ into Γ.

Definition 4.13. Let $\mathcal{G}$ be a discrete measured groupoid on a standard measure space (X, μ). Let S be a standard Borel space. Suppose that we are given a Borel action of a discrete group Γ on S and a groupoid homomorphism $\rho\colon \mathcal{G} \to \Gamma$. Then a Borel map $\varphi\colon X \to S$ satisfying the equation

$$\rho(\gamma)\varphi(s(\gamma)) = \varphi(r(\gamma)) \quad \text{for a.e. } \gamma \in \mathcal{G}$$

is called an *invariant Borel map* for $\mathcal{G}$. We say that φ is *ρ-invariant* for $\mathcal{G}$ when we specify ρ.

More generally, if A is a Borel subset of X and if a Borel map $\varphi\colon A \to S$ satisfies the above equation for a.e. $\gamma \in (\mathcal{G})_A$, then we say for simplicity that φ is *invariant* for $\mathcal{G}$ although we should say that φ is invariant for $(\mathcal{G})_A$.

Given an action of a groupoid, we often use amenability of the groupoid to obtain an invariant Borel map for the action as shown in the following proposition. Recall that for a separable compact space K, we denote by $M(K)$ the space of probability measures on K. This space is a convex, weak*-closed subset contained in the closed unit ball of the dual of $C(K)$, the Banach space of $\mathbb{C}$-valued continuous functions on K with the sup norm.

Proposition 4.14. *Let $\mathcal{G}$ be a discrete measured groupoid on a standard measure space (X, μ). Let Γ be a discrete group and suppose that Γ acts on a separable compact space K continuously. Let $\rho\colon \mathcal{G} \to \Gamma$ be a groupoid homomorphism. If $\mathcal{G}$ is amenable, then there exists a ρ-invariant Borel map $\varphi\colon X \to M(K)$, i.e., a Borel map satisfying the equation*

$$\rho(\gamma)\varphi(s(\gamma)) = \varphi(r(\gamma)) \quad \text{for a.e. } \gamma \in \mathcal{G}.$$

Here, the action $\Gamma \curvearrowright M(K)$ is given by the induced one from the action $\Gamma \curvearrowright K$.

Proof. We put $E_x = C(K)$ for all $x \in X$. We define a μ-measurable structure $\mathcal{M}$ for the family $\{E_x\}_{x\in X}$ as the one generated by the constant vector fields $x \mapsto e$ for all $e \in C(K)$. Then $E = (\{E_x\}_{x\in X}, \mathcal{M})$ is a separable measurable Banach bundle. For $\gamma \in \mathcal{G}$, we define an isometric isomorphism $L(\gamma)\colon E_{s(\gamma)} \to E_{r(\gamma)}$ by $L(\gamma)e = \rho(\gamma)e$ for $e \in E_{s(\gamma)}$. The pair (E, L) is then a $\mathcal{G}$-bundle. Since the family $A = \{A_x\}_{x\in X}$ given by $A_x = M(K)$ defines a $\mathcal{G}$-field for E^*, we get $\varphi \in L^\infty(X, E^*)$ such that $\varphi(x) \in M(K)$ for μ-a.e. $x \in X$, and $L^*(\gamma)\varphi(s(\gamma)) = \varphi(r(\gamma))$ for $\tilde{\mu}$-a.e. $\gamma \in \mathcal{G}$. This equation is equivalent to $\rho(\gamma)\varphi(s(\gamma)) = \varphi(r(\gamma))$. The proposition follows from Lemma 4.5. □

Example 4.15. Let Γ be a discrete group and suppose that Γ admits a non-singular action on a standard measure space (X, μ). We denote by $\mathcal{G}$ the associated groupoid $\Gamma \ltimes X$. Then

$$\rho\colon \mathcal{G} \to \Gamma, \quad (g, x) \mapsto g$$

defines a groupoid homomorphism. Suppose that we are given a separable compact space K on which Γ acts continuously. It follows from Proposition 4.14 that if $\mathcal{S}$ is an amenable subgroupoid of $\mathcal{G}$, then there exists a Borel map $\varphi\colon X \to M(K)$ such that $\rho(g, x)\varphi(x) = \varphi(gx)$, that is, $g\varphi(x) = \varphi(gx)$ for a.e. $(g, x) \in \mathcal{S}$.

We give fundamental properties of amenable discrete measured groupoids.

Theorem 4.16. *Let $\mathcal{G}$ be a discrete measured groupoid on a standard measure space (X, μ).*

(i) *$\mathcal{G}$ is amenable if and only if its quotient equivalence relation*

$$\{(r(\gamma), s(\gamma)) \in X \times X : \gamma \in \mathcal{G}\}$$

is amenable and for a.e. $x \in X$, the isotropy group $\mathcal{G}_x^x = \{\gamma \in \mathcal{G} : r(\gamma) = s(\gamma) = x\}$ is amenable.

(ii) *Any subgroupoid of an amenable discrete measured groupoid is amenable.*

(iii) *Let $A \subset X$ be a Borel subset with positive measure. If $\mathcal{G}$ is amenable, then so is the restricted groupoid $(\mathcal{G})_A$. If $\mathcal{G}A = X$ up to null sets, then the converse also holds.*

For Assertion (i), we refer to Corollary 5.3.33 in [4]. Assertion (ii) follows from hyperfiniteness of amenable equivalence relations shown in [10] and Assertion (i). The former part of Assertion (iii) can be shown by using Assertion (ii) because $(\mathcal{G})_A$ is identified with the subgroupoid $(\mathcal{G})_A \cup \{e_x : x \in X \setminus A\}$ of $\mathcal{G}$, where $e_x \in \mathcal{G}$ denotes the unit on x. The latter part can also be proved directly by using this identification. The following is one of the most highlighted theorems about principal discrete measured groupoids, and it is a generalization of Theorem 3.15. Recall that $\mathcal{G}$ is said to be principal if the isotropy group

$\mathcal{G}_x^x$ is trivial for each $x \in X$. A principal groupoid is isomorphic to its quotient equivalence relation.

Theorem 4.17 ([10]). *For $i \in \{1, 2\}$, let $\mathcal{G}_i$ be ergodic principal discrete measured groupoids on a standard finite measure space (X_i, μ_i). For $i \in \{1, 2\}$, we suppose that μ_i is invariant for $\mathcal{G}_i$ and that μ_i has no atom, that is, there exists no point $x \in X_i$ with $\mu_i(\{x\}) > 0$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic.*

Here, a discrete measured groupoid $\mathcal{G}$ on (X, μ) is said to be *ergodic* if the following holds: If a Borel subset $A \subset X$ satisfies the equation $\mathcal{G}A = A$ up to null sets, then either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. In the next theorem, we particularly consider a groupoid arising from a group action.

Theorem 4.18. *Let (X, μ) and (Y, ν) be standard finite measure spaces.*

- (i) *Let G be a discrete group and suppose that we have a non-singular action of G on (X, μ). Let $\mathcal{G}$ be the associated groupoid. If G is amenable, then $\mathcal{G}$ is amenable.*
- (ii) *Conversely, in Assertion* (i)*, if the action $G \curvearrowright (X, \mu)$ is measure-preserving and $\mathcal{G}$ is amenable, then G is amenable.*
- (iii) *Let G be a discrete group and suppose that we have non-singular actions $G \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$. If there exists a G-equivariant Borel map $f\colon X \to Y$ such that $f_*\mu = \nu$ and if the action $G \curvearrowright (Y, \nu)$ is amenable, then the action $G \curvearrowright (X, \mu)$ is also amenable.*

Assertion (i) follows from Propositions 4.2.2 and A.3.9 in [4]. For Assertions (ii) and (iii), we refer to Proposition 4.3.3 in [63] and [3], respectively. This subsection will end with several examples of amenable discrete measured groupoids.

Example 4.19. *Groupoids admitting fundamental domains.* Let G be a discrete group. Then the action of G on G by left multiplication is amenable, where a measure μ on G is given by $f \in \ell^1(G)$ such that $f(g) > 0$ for each $g \in G$. More generally, suppose that G admits a non-singular action on a standard finite measure space (X, μ) and suppose that the action admits a *fundamental domain*, that is, there exists a Borel subset $F \subset X$ such that $\mu(\bigcup_{g \in G} F) = \mu(X)$ and $Gx \cap F$ consists of a single point for a.e. $x \in X$. Then the action $G \curvearrowright (X, \mu)$ is amenable. Note that if G is infinite and G admits an essentially free, measure-preserving action on a standard finite measure space, then there exists no Borel fundamental domain for the action. A discrete measured equivalence relation which admits a fundamental domain is also amenable.

The following is an example of amenable actions of non-amenable groups.

Theorem 4.20 ([1], [4, Appendix B]). *Let Γ be an infinite hyperbolic group in the sense of Gromov. Let μ be a probability measure on the Gromov boundary $\partial\Gamma$ such that the action of Γ on $(\partial\Gamma, \mu)$ is non-singular. Then the action $\Gamma \curvearrowright (\partial\Gamma, \mu)$ is amenable.*

In the proof of this result, approximately invariant means for (the groupoid arising from) the boundary action of Γ are constructed. Recall that for a discrete group G, a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\ell^1(G)$ is called *approximately invariant means* for G if

- for each n, $f_n(g) \geq 0$ for all $g \in G$ and $\sum_{g\in G} f_n(g) = 1$;
- for each $g \in G$, $\sum_{h\in G} |f_n(g^{-1}h) - f_n(h)| \to 0$ as $n \to \infty$.

It is well-known that a discrete group G is amenable if and only if G admits approximately invariant means. We can also define approximately invariant means for a discrete measured groupoid as an analogue of the above definition (see Chapter 3 in [4]), and we can show that a discrete measured groupoid is amenable if and only if there exist such means for it (see Theorem 4.2.7 in [4]). When we are given a group action and we want to show that it is amenable, we often prove that it admits approximately invariant means, for it is often difficult to prove the fixed point property in Definition 4.12 directly for concrete examples of group actions.

In Example 3.8 of [5] and Example 2.2 of [54], approximately invariant means for the boundary action of non-abelian free groups are constructed explicitly. This construction can be generalized to the case of hyperbolic groups by using the uniform thinness of all geodesic triangles on their Cayley graphs. Since the curve complex C for a surface M with $\kappa(M) \geq 0$ is hyperbolic (see Theorem 2.2), this proof motivates the following theorem. We denote by ∂C the Gromov boundary of C. It is known that ∂C is a non-empty standard Borel space (see Proposition 3.10 in [35]). We refer to [39], [24], [27] for details of the boundary ∂C.

Theorem 4.21 ([35, Theorem 3.29]). *Let M be a surface with $\kappa(M) \geq 0$ and let C be the curve complex for M. Let μ be a probability measure on the Gromov boundary ∂C such that the action of $\Gamma(M)^\diamond$ on $(\partial C, \mu)$ is non-singular. Then the action $\Gamma(M)^\diamond \curvearrowright (\partial C, \mu)$ is amenable.*

Since C is hyperbolic, we expect a construction of approximately invariant means for the action of $\Gamma(M)^\diamond$ on ∂C similar to the one for hyperbolic groups noted above. However, we can not apply the construction directly because C is locally infinite. To avoid this difficulty, we use the finiteness property of tight geodesics on the curve complex established by Masur and Minsky [44], and Bowditch [9]. A tight geodesic is a geodesic in C with a special property. Roughly speaking, the finiteness property of tight geodesics says that the set of tight geodesics behaves like the set of geodesics on a locally finite hyperbolic graph. Thanks to this property, we can construct approximately invariant means for the action of $\Gamma(M)^\diamond$ on ∂C as in the case of hyperbolic groups. Geometric properties of the curve complex we use in the proof of Theorem 4.21 are only the hyperbolicity and this finiteness property. We omit the proof of Theorem 4.21 and note here that the finiteness property of tight geodesics will also be used in the construction of several natural Borel maps associated with the curve complex (see Remark 5.9).

In Section 5, we will use the following corollary, which is an immediate consequence of Theorem 4.21. We denote by $\partial_2 C$ the quotient space of $C \times C$ by the coordinate exchanging action of the symmetric group on two letters.

Corollary 4.22 ([35, Lemma 4.32]). *Let M be a surface with $\kappa(M) \geq 0$. Let μ be a probability measure on $\partial_2 C$ such that the action of $\Gamma(M)^\diamond$ on $(\partial_2 C, \mu)$ is non-singular. Then the action $\Gamma(M)^\diamond \curvearrowright (\partial_2 C, \mu)$ is amenable.*

5 Two types of subgroupoids: IA and reducible ones

Let M be a surface with $\kappa(M) > 0$ and let $m \geq 3$ be an integer. Let Γ be a finite index subgroup of $\Gamma(M; m)$ (see Theorem 2.8 for the subgroup $\Gamma(M; m)$ of $\Gamma(M)$). Let $\mathcal{G}$ be the discrete measured groupoid on a standard finite measure space (X, μ) which arises from a measure-preserving action $\Gamma \curvearrowright (X, \mu)$. The final goal of Sections 5, 6 and 7 is to prove Theorem 3.6. This theorem states that any isomorphism between such groupoids arsing from actions of mapping class groups preserves subgroupoids generated by actions of Dehn twists. To characterize such subgroupoids algebraically in terms of groupoids, we introduce two types of subgroupoids of $\mathcal{G}$. The first one is called IA subgroupoids, which correspond to IA (= infinite, irreducible and amenable) subgroups in the classification theorem of subgroups of mapping class groups (see Theorem 2.7). The second one is called reducible subgroupoids, which correspond to infinite reducible subgroups.

Let $\rho\colon \mathcal{G} \to \Gamma$ be the groupoid homomorphism given by $(g, x) \mapsto g$ for $g \in \Gamma$ and $x \in X$. We denote by $M(\mathcal{PMF})$ the space of probability measures on the Thurston boundary $\mathcal{PMF}$. Each element $\gamma \in \mathcal{G}$ then acts on $M(\mathcal{PMF})$ via ρ. We can regard this assignment as the action of $\mathcal{G}$ on $M(\mathcal{PMF})$. We define the above two classes of subgroupoids $\mathcal{S}$ of $\mathcal{G}$ in terms of Borel maps $\varphi\colon X \to M(\mathcal{PMF})$ which is ρ-invariant for $\mathcal{S}$, i.e., $\rho(\gamma)\varphi(s(\gamma)) = \varphi(r(\gamma))$ for a.e. $\gamma \in \mathcal{S}$. These ρ-invariant Borel maps play a role of fixed points for the action of $\mathcal{G}$ on $M(\mathcal{PMF})$ (see Definition 4.13 and the comment right before it).

In Subsection 5.1, we characterize IA and reducible subgroups in terms of their fixed points in $M(\mathcal{PMF})$. This will help us to understand the motivation of the definition of IA and reducible subgroupoids. In Subsection 5.2, we analyze IA subgroupoids and study properties of Borel maps into $M(\mathcal{PMF})$ which are ρ-invariant for them. It is shown that IA subgroupoids are in fact amenable as groupoids. In Subsection 5.3, we study reducible subgroupoids $\mathcal{S}$ and give the definition of canonical reduction systems for $\mathcal{S}$. This is an essentially unique Borel map into $S(M)$ which is ρ-invariant for $\mathcal{S}$ and which satisfies nice properties. This system is a generalization of canonical reduction systems for reducible subgroups introduced by Birman, Lubotzky, and McCarthy [8], and Ivanov [30].

5.1 IA and reducible subgroups

Let M be a surface with $\kappa(M) \geq 0$ and let Γ be an infinite subgroup of $\Gamma(M)$. Recall that Γ is said to be IA if there exists a pseudo-Anosov element $g \in \Gamma$ such that $\{F_{\pm}(g)\}$, the set of its pseudo-Anosov foliations, is fixed by all elements of Γ. In this case, Γ is virtually cyclic. If there exists $\sigma \in S(M)$ fixed by all elements of Γ, then Γ is said to be reducible. In the next two propositions, we characterize these two classes of subgroups in terms of their fixed points on the space $M(\mathcal{PMF})$ of probability measures on the Thurston boundary $\mathcal{PMF}$. We say that $\nu \in M(\mathcal{PMF})$ is *invariant* for a subgroup Γ of $\Gamma(M)$ if $g\nu = \nu$ for each $g \in \Gamma$.

Proposition 5.1. *Let M be a surface with $\kappa(M) \geq 0$ and let Γ be an infinite subgroup of $\Gamma(M)$. Then the following assertions hold:*

(i) *The subgroup Γ is IA if and only if there exists an invariant measure $\nu \in M(\mathcal{PMF})$ for Γ such that $\nu(\mathcal{MIN}) = 1$.*

(ii) *If Γ is IA, then any invariant measure $\nu \in M(\mathcal{PMF})$ for Γ satisfies that $\nu(\{F_{\pm}(g)\}) = 1$ for some pseudo-Anosov element $g \in \Gamma$.*

Proof. The "only if" part of Assertion (i) has already been seen in Example 4.3. Assertion (ii) follows from the dynamics of pseudo-Anosov elements on $\mathcal{PMF}$ (see Theorem 2.5).

We may assume that Γ is a subgroup of $\Gamma(M; m)$ for an integer $m \geq 3$ to prove the "if" part of Assertion (i). Recall that $\Gamma(M; m)$ consists of pure elements and is torsion-free (see Theorem 2.8). Let $\nu \in M(\mathcal{PMF})$ be an invariant measure for Γ such that $\nu(\mathcal{MIN}) = 1$.

Assume that Γ contains a reducible element g of infinite order. Let Δ^u and Ψ^s be the subsets of $\mathcal{PMF}$ associated with g as in the comment right before Theorem 2.6. We can choose a non-empty closed one-dimensional submanifold c of M which satisfies Property (P) for g and does not have superfluous components. Note that both Ψ^s and Δ^u are contained in $\mathcal{PMF} \setminus \mathcal{MIN}$. Let $\{U_n\}_n$ be a sequence of open subsets of $\mathcal{PMF}$ such that $U_n \supset U_{n+1}$ for each n, and $\Delta^u = \bigcap_n U_n$. It follows from $\nu(\Delta^u) = 0$ that $\nu(U_n) \searrow 0$. Let K be any compact subset of $\mathcal{PMF} \setminus \Psi^s$. By Theorem 2.6, for each n, there exists N such that $g^N K \subset U_n$, and thus $\nu(K) = \nu(g^N K) \leq \nu(U_n)$. Therefore, $\nu(K) = 0$. Since $\mathcal{PMF} \setminus \Psi^s$ can be expressed as a countable union of compact subsets, this implies that $\nu(\mathcal{PMF} \setminus \Psi^s) = 0$, which is a contradiction.

Thus, Γ does not contain a reducible element of infinite order, and it consists of pseudo-Anosov elements and the trivial element. If g is a pseudo-Anosov element of Γ, then by the dynamics of g on $\mathcal{PMF}$ (see Theorem 2.5), the support of ν is contained in $\{F_\pm(g)\}$. Since ν is invariant for Γ, this implies that Γ is an IA subgroup. □

Proposition 5.2. *Let M be a surface with $\kappa(M) \geq 0$ and let Γ be an infinite subgroup of $\Gamma(M)$. Then Γ is reducible if and only if there exists an invariant measure $\nu \in M(\mathcal{PMF})$ for Γ such that $\nu(\mathcal{PMF} \setminus \mathcal{MIN}) = 1$.*

Proof. The "only if" part has already been seen in Example 4.3. To prove the "if" part, let $\nu \in M(\mathcal{PMF})$ be an invariant measure for Γ such that $\nu(\mathcal{PMF} \setminus \mathcal{MIN}) = 1$. It follows from Theorem 2.7 that Γ is either IA, reducible or sufficiently large. By Proposition 5.1 (ii), Γ is not IA. If Γ were sufficiently large, then there exist pseudo-Anosov elements $g_1, g_2 \in \Gamma$ such that $\{F_\pm(g_1)\} \cap \{F_\pm(g_2)\} = \emptyset$. Theorem 2.5 implies that any sufficiently large subgroup admits no invariant probability measure on $\mathcal{PMF}$. Therefore, Γ is reducible. □

We next define IA and reducible subgroupoids, which is motivated by the above two propositions. We often use the following notation in what follows.

Notation. We refer to the following assumption as $(\star)$: Let M be a surface with $\kappa(M) > 0$ and let $m \geq 3$ be an integer. Let Γ be a finite index subgroup of $\Gamma(M; m)$. Suppose that Γ admits a measure-preserving action on a standard finite measure space (X, μ). We denote by $\mathcal{G}$ the associated groupoid $\Gamma \ltimes (X, \mu)$. Let $\rho\colon \mathcal{G} \to \Gamma$ be the groupoid homomorphism defined by $(g, x) \mapsto g$.

Propositions 5.1 and 5.2 imply that there exists no infinite subgroup of $\Gamma(M)$ which admits an invariant measure $\nu \in M(\mathcal{PMF})$ such that $0 < \nu(\mathcal{MIN}) < 1$. The following is a generalization of this fact.

Theorem 5.3. *Under Assumption $(\star)$, let Y be a Borel subset of X with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Suppose that there is an invariant Borel map $\varphi\colon Y \to M(\mathcal{PMF})$ for $\mathcal{S}$. Then there exists a Borel partition $Y = Y_1 \sqcup Y_2$ such that*

- *$\varphi(x)(\mathcal{MIN}) = 1$ for a.e. $x \in Y_1$;*

- $\varphi(x)(\mathcal{PMF} \setminus \mathcal{MIN}) = 1$ *for a.e.* $x \in Y_2$.

Recall that a Borel map $\varphi\colon Y \to M(\mathcal{PMF})$ is (ρ-)invariant for $\mathcal{S}$ if the equation $\rho(\gamma)\varphi(s(\gamma)) = \varphi(r(\gamma))$ holds for a.e. $\gamma \in \mathcal{S}$.

Remark 5.4. In Theorem 5.3, let us assume that there is another invariant Borel map $\psi\colon Y \to M(\mathcal{PMF})$ for $\mathcal{S}$. It is easy to check that ψ also satisfies

- $\psi(x)(\mathcal{MIN}) = 1$ for a.e. $x \in Y_1$;
- $\psi(x)(\mathcal{PMF} \setminus \mathcal{MIN}) = 1$ for a.e. $x \in Y_2$.

for the same Y_1 and Y_2 as in the theorem. (Consider the invariant Borel map $(\varphi + \psi)/2$ for $\mathcal{S}$ and apply the theorem.)

By this remark, the two subgroupoids $(\mathcal{S})_{Y_1}$ and $(\mathcal{S})_{Y_2}$ should be distinguished, and it is natural to define the following two classes of subgroupoids.

Definition 5.5. Under Assumption $(\star)$, let Y be a Borel subset of X with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type.

(i) We say that $\mathcal{S}$ is *IA* (= irreducible and amenable) if there is an invariant Borel map $\varphi\colon Y \to M(\mathcal{PMF})$ for $\mathcal{S}$ such that $\varphi(x)(\mathcal{MIN}) = 1$ for a.e. $x \in Y$.

(ii) We say that $\mathcal{S}$ is *reducible* if there is an invariant Borel map $\varphi\colon Y \to M(\mathcal{PMF})$ for $\mathcal{S}$ such that $\varphi(x)(\mathcal{PMF} \setminus \mathcal{MIN}) = 1$ for a.e. $x \in Y$.

It follows from Remark 5.4 that the classes of IA and reducible subgroupoids are mutually exclusive. The definition of reducible subgroupoids is also motivated by the following lemma.

Lemma 5.6. *Under Assumption* $(\star)$*, let* Y *be a Borel subset of* X *with positive measure and let* $\mathcal{S}$ *be a subgroupoid of* $(\mathcal{G})_Y$ *of infinite type. Then the following two conditions are equivalent:*

(i) $\mathcal{S}$ *is reducible.*

(ii) *There exists an invariant Borel map* $Y \to S(M)$ *for* $\mathcal{S}$.

It is clear that Assertion (ii) implies Assertion (i) because there is a $\Gamma(M)^{\diamond}$-equivariant embedding $\iota\colon S(M) \to \mathcal{PMF} \setminus \mathcal{MIN}$. To prove the converse implication, we construct a Borel map $H\colon \mathcal{PMF} \setminus \mathcal{MIN} \to S(M)$ which is equivariant for the action of $\Gamma(M)^{\diamond}$ and satisfies $H \circ \iota = \mathrm{id}$ (see Subsection 4.2 in [35]).

5.2 IA subgroupoids

The following lemma is the first important observation about invariant Borel maps for IA subgroupoids. It is known that there exists a natural $\Gamma(M)^{\diamond}$-equivariant map $\pi\colon \mathcal{MIN} \to \partial C$, which is continuous and surjective (see [39]). We can define a Borel structure on the set $M(\partial C)$ of all probability measures on ∂C by using a Borel section of $\pi\colon \mathcal{MIN} \to \partial C$, i.e., a Borel map $s\colon \partial C \to \mathcal{MIN}$ such that $\pi \circ s = \mathrm{id}$ (see the comment right before Proposition 4.30 in [35]). For a technical reason, we study invariant Borel maps into $M(\partial C)$ for IA subgroupoids instead of ones into $M(\mathcal{PMF})$.

Lemma 5.7. *Under Assumption* $(\star)$*, let* Y *be a Borel subset of* X *with positive measure and let* $\mathcal{S}$ *be a subgroupoid of* $(\mathcal{G})_Y$ *of infinite type. Then the following assertions hold:*

(i) *$\mathcal{S}$ is IA if and only if there exists an invariant Borel map $\varphi\colon Y \to M(\partial C)$ for $\mathcal{S}$.*

(ii) *If $\mathcal{S}$ is IA and $\varphi\colon Y \to M(\partial C)$ is an invariant Borel map for $\mathcal{S}$, then* $\mathrm{supp}(\varphi(x))$ *consists of at most two points.*

Here, for a measure ν, we denote by $\mathrm{supp}(\nu)$ the support of ν. It is easy to see the "only if" part of Assertion (i) by using the map $\pi\colon \mathcal{MIN} \to \partial C$.

We denote by $\partial_2 C$ the quotient space of $\partial C \times \partial C$ by the coordinate exchanging action of the symmetric group on two letters. Then $\partial_2 C$ can be viewed as a Borel subset of $M(\partial C)$ by regarding each element of $\partial_2 C$ as an atomic measure on ∂C such that each atom has measure 1 or 1/2. We denote by $M(\mathcal{MIN})$ the Borel subset of $M(\mathcal{PMF})$ consisting of all measures ν such that $\nu(\mathcal{MIN}) = 1$. We can prove the following lemma by using Lemma 5.7.

Lemma 5.8. *Under Assumption* $(\star)$*, let* Y *be a Borel subset of* X *with positive measure and let* $\mathcal{S}$ *be a subgroupoid of* $(\mathcal{G})_Y$ *of infinite type. If* $\mathcal{S}$ *is IA, then there exists an essentially unique invariant Borel map* $\varphi_0\colon Y \to \partial_2 C$ *for* $\mathcal{S}$ *satisfying the following condition: If* $Y' \subset Y$ *is a Borel subset with positive measure and* $\varphi\colon Y' \to M(\partial C)$ *is an invariant Borel map for* $\mathcal{S}$*, then*

$$\mathrm{supp}(\varphi(x)) \subset \mathrm{supp}(\varphi_0(x)) \quad \text{for a.e. } x \in Y'.$$

This unique invariant Borel map plays an important role when we study the normalizer of an IA subgroupoid (see Lemma 6.7).

Remark 5.9. In the proof of Propositions 5.1 and 5.2, it was important to observe the dynamics of each element of the mapping class group on $\mathcal{PMF}$. However, we cannot consider the dynamics of each element of a groupoid because powers γ^n of an element γ of a groupoid do not make sense in general. Hence, we cannot apply a similar argument in the proof of Theorem 5.3 and

Lemma 5.7. As a different approach, by using the finiteness properties of tight geodesics in the curve complex (see Theorem 4.21 and the comment around it), we construct the following natural Borel maps: Put

$$\delta C = \{(a, b, c) \in (\partial C)^3 : a \neq b \neq c \neq a\}$$

and define an action of $\Gamma(M)^\diamond$ on δC by $g(a, b, c) = (ga, gb, gc)$. Let $\mathcal{F}'(C)$ be the set of all non-empty finite subsets of $V(C)$ whose diameters are at least three, on which $\Gamma(M)^\diamond$ naturally acts. We can then construct a $\Gamma(M)^\diamond$-equivariant Borel map

$$MS\colon \delta C \to \mathcal{F}'(C)$$

(see Section 4.1 in [35]). A remarkable property of the set $\mathcal{F}'(C)$ is that the stabilizer of each element of $\mathcal{F}'(C)$ is finite (see Lemma 10 in [7]). Moreover, in the proof of Theorem 5.3, we construct a $\Gamma(M)^\diamond$-equivariant Borel map

$$G\colon \partial_2 C \times V(C) \to \mathcal{F}'(C),$$

where the action of $\Gamma(M)^\diamond$ on $\partial_2 C \times V(C)$ is given by $g(a, x) = (ga, gx)$ (see Lemma 4.40 in [35]). In this chapter, we do not further mention the proof of these facts.

As observed in Proposition 5.1, if an infinite subgroup Γ of $\Gamma(M; m)$ has an invariant measure $\nu \in M(\mathcal{PMF})$ such that $\nu(\mathcal{MIN}) = 1$, then Γ is IA and in particular, Γ is amenable. Hence, we can expect any IA subgroupoid $\mathcal{S}$ to be amenable, which is in fact shown in the following theorem. In the proof of this theorem, we use the amenability of the action $\Gamma(M)^\diamond$ on ∂C shown in Theorem 4.21 and Corollary 4.22. We give the proof of this theorem to show how to use this amenable action of the mapping class group.

Theorem 5.10. *Under Assumption ($\star$), let Y be a Borel subset of X with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. If $\mathcal{S}$ is IA, then $\mathcal{S}$ is amenable. Equivalently, if there is an invariant Borel map $Y \to \partial_2 C$ for $\mathcal{S}$, then $\mathcal{S}$ is amenable.*

Note that $\mathcal{S}$ is IA if and only if there exists an invariant Borel map $Y \to \partial_2 C$ for $\mathcal{S}$ (see Lemma 5.7 (ii)). Let $\varphi\colon Y \to \partial_2 C$ be an invariant Borel map for $\mathcal{S}$. An important point of the proof is to construct a standard Borel space S on which Γ acts so that

- $\mathcal{S}$ is isomorphic to a subgroupoid of $\Gamma \ltimes S$;
- we can construct a Γ-equivariant Borel map $S \to \partial_2 C$ by using φ.

If we can construct such a space S, then the theorem follows from Theorem 4.16 (ii), Theorem 4.18 (iii) and Corollary 4.22.

Proof of Theorem 5.10. We identify $\mathcal{S}$ with the groupoid on (X, μ) defined by the union $\{e_x \in \mathcal{G} : x \in X \setminus Y\} \cup \mathcal{S}$. Extend φ to the map from X defined by $\varphi(x) = a_0$ for $x \in X \setminus Y$, where $a_0 \in \partial_2 C$ is some fixed point. We denote by the same symbol φ the extended map. The extended map φ is then also invariant for $\mathcal{S}$.

Consider the action of Γ on $X \times \Gamma$ given by

$$g(x, g_1) = (x, g_1 g^{-1}) \ \text{ for } x \in X,\ g, g_1 \in \Gamma.$$

The equivalence relation $\mathcal{R}_1$ on $X \times \Gamma$ defined by

$$(s(\gamma), g) \sim (r(\gamma), \rho(\gamma)g) \ \text{ for } \gamma \in \mathcal{G},\ g \in \Gamma$$

admits a fundamental domain $F_1 = X \times \{e\}$, i.e., a Borel subset F_1 of the unit space $X \times \Gamma$ such that $\mathcal{R}_1 x \cap F_1$ consists of exactly one point for a.e. $x \in X \times \Gamma$, where $\mathcal{R}_1 x$ denotes the equivalence class containing x. Let $\mathcal{R}_2$ be the equivalence relation on $X \times \Gamma$ given by

$$(s(\gamma), g) \sim (r(\gamma), \rho(\gamma)g) \ \text{ for } \gamma \in \mathcal{S},\ g \in \Gamma.$$

Since $\mathcal{R}_2$ is a subrelation of $\mathcal{R}_1$, we can show that $\mathcal{R}_2$ also admits a fundamental domain $F_2 \subset X \times \Gamma$ (use Lemma 2.12 in [2]). Let S be the quotient space of $X \times \Gamma$ by $\mathcal{R}_2$, which is identified with F_2 as a measure space via the projection $X \times \Gamma \to S$. Note that the action of Γ on $X \times \Gamma$ induces an action of Γ on S. Denote the projection of $(x, g) \in X \times \Gamma$ onto S by $[x, g] \in S$. Then $\mathcal{S}$ can be identified with a Borel subgroupoid

$$\mathcal{H} = \{(\rho(\gamma), [s(\gamma), e]) \in \Gamma \ltimes S : \gamma \in \mathcal{S}\}$$

of $\Gamma \ltimes S$ via an isomorphism

$$\mathcal{S} \ni \gamma \mapsto (\rho(\gamma), [s(\gamma), e]) \in \mathcal{H}.$$

Using the invariant Borel map $\varphi\colon X \to \partial_2 C$ for $\mathcal{S}$, we construct a Borel map $\varphi'\colon S \to \partial_2 C$ by the formula $[x, g] \mapsto g^{-1}\varphi(x)$. Then φ' is well-defined and Γ-equivariant. By Theorem 4.18 (iii) and Corollary 4.22, the groupoid $\Gamma \ltimes S$ is amenable. Since $\mathcal{H}$ is a subgroupoid of $\Gamma \ltimes S$, it is also amenable. □

5.3 Reducible subgroupoids

We shall recall Assumption $(\star)$: Let M be a surface with $\kappa(M) > 0$ and let $m \geq 3$ be an integer. Let Γ be a finite index subgroup of $\Gamma(M; m)$. Suppose that Γ admits a measure-preserving action on a standard finite measure space (X, μ). We denote by $\mathcal{G}$ the associated groupoid $\Gamma \ltimes (X, \mu)$. Let $\rho\colon \mathcal{G} \to \Gamma$ be the groupoid homomorphism defined by $(g, x) \mapsto g$.

Let Y be a Borel subset of X with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Suppose that $\mathcal{S}$ is reducible. By Lemma 5.6, we can construct an invariant Borel map $\varphi\colon Y \to S(M)$ for $\mathcal{S}$, i.e., $\rho(\gamma)\varphi(s(\gamma)) = \varphi(r(\gamma))$ for a.e. $\gamma \in \mathcal{S}$. In general, there are many such maps φ.

The aim of this subsection is to construct an invariant Borel map $Y \to S(M)$ for $\mathcal{S}$ with some nice properties, called the canonical reduction system (CRS) for $\mathcal{S}$. This is a generalization of the canonical reduction system (CRS) for a reducible subgroup, introduced by Birman, Lubotzky, and McCarthy [8], and Ivanov [30]. It is shown that the CRS exists essentially uniquely for each reducible subgroupoid. This uniqueness will be useful when we study the normalizer of a reducible subgroupoid (see Lemma 6.8).

We shall recall the definition and some fundamental facts of the CRS for a subgroup of the mapping class group. Let M be a surface with $\kappa(M) \geq 0$ and let $m \geq 3$ be an integer. We first define the CRS for a subgroup of $\Gamma(M; m)$.

Definition 5.11. Let M be a surface with $\kappa(M) \geq 0$ and let $m \geq 3$ be an integer. Let Γ be a subgroup of $\Gamma(M; m)$.

(i) An element $\alpha \in V(C)$ is called an *essential reduction class* for Γ if the following two conditions are satisfied:

- $g\alpha = \alpha$ for any $g \in \Gamma$;
- If $\beta \in V(C)$ satisfies $i(\alpha, \beta) \neq 0$, then there exists $g \in \Gamma$ such that $g\beta \neq \beta$.

(ii) The *canonical reduction system (CRS)* for Γ is defined to be the set of all essential reduction classes for Γ. We denote by $\sigma(\Gamma)$ the CRS for Γ.

It is easy to check that $\sigma(\Gamma) \in S(M) \cup \{\emptyset\}$. It can be shown that if Λ is a finite index subgroup of Γ, then $\sigma(\Lambda) = \sigma(\Gamma)$. Therefore, we can define the CRS for a general subgroup Γ of $\Gamma(M)$ as the CRS for $\Gamma \cap \Gamma(M; m)$, which is independent of m. We refer to Chapter 7 in [30] for more details. Note that if Γ is finite, then $\sigma(\Gamma) = \emptyset$ because $\sigma(\{e\}) = \emptyset$. The following is a fundamental fact on the CRS for an infinite subgroup of $\Gamma(M)$.

Theorem 5.12 ([30, Corollary 7.17]). *An infinite subgroup Γ of $\Gamma(M)$ is reducible if and only if $\sigma(\Gamma)$ is non-empty.*

In the next theorem, we give a geometric meaning of CRS's. We introduce the following notation.

Notation. Let M be a surface with $\kappa(M) \geq 0$ and let $m \geq 3$ be an integer. Let Γ be a subgroup of $\Gamma(M; m)$ and assume that each element of Γ fixes $\sigma \in S(M)$. By Theorem 2.8 (iii), there is a natural homomorphism

$$p_\sigma \colon \Gamma \to \prod_Q \Gamma(Q),$$

where Q runs through all components of M_σ, the surface obtained by cutting M along a realization of σ. For each component Q of M_σ, let $p_Q \colon \Gamma \to \Gamma(Q)$ be the composition of p_σ and the projection onto $\Gamma(Q)$.

Theorem 5.13 ([30, Theorem 7.16]). *Let M be a surface with $\kappa(M) \geq 0$ and let Γ be a subgroup of $\Gamma(M)$. Then there exists a unique $\sigma \in S(M) \cup \{\emptyset\}$ satisfying the following three conditions, and then σ is in fact the CRS for Γ:*

(i) *All elements of Γ fix σ.*

(ii) *Let $m \geq 3$ be an integer and put $\Gamma_0 = \Gamma \cap \Gamma(M; m)$. For each component Q of M_σ, the quotient group $p_Q(\Gamma_0)$ cannot be infinite reducible.*

(iii) *$\sigma \in S(M)$ is the minimal one satisfying the above Conditions* (i), (ii) *(for some/any m).*

Example 5.14. We present some examples of reducible subgroups whose CRS can be computed. Let M be a surface with $\kappa(M) \geq 0$.

(i) Let $\sigma \in S(M)$ and let D_σ be the subgroup of $\Gamma(M)$ generated by all Dehn twists about curves in σ, which is isomorphic to a free abelian group of rank $|\sigma|$. Then $\sigma(D_\sigma) = \sigma$.

(ii) Let $g \in \Gamma(M)$ be a pure element and take a closed one-dimensional submanifold c (may be empty) of M such that Condition (P) is satisfied for c and some representative of g (see the comment right after Theorem 2.5). If we denote by $\sigma \in S(M) \cup \{\emptyset\}$ the isotopy class of c, then σ is the CRS for the cyclic subgroup of $\Gamma(M)$ generated by g.

(iii) Take $\sigma \in S(M)$. If we denote by $\Gamma_\sigma = \{g \in \Gamma(M) : g\sigma = \sigma\}$ its stabilizer, then $\sigma(\Gamma_\sigma) = \sigma$.

In the same manner, we can define the canonical reduction system for a reducible subgroupoid as an invariant Borel map into $S(M)$ satisfying some special properties. In the following definition, a purely ρ-invariant pair corresponds to an essential reduction class. We shall recall Assumption $(\star)$: Let M be a surface with $\kappa(M) > 0$ and let $m \geq 3$ be an integer. Let Γ be a finite index subgroup of $\Gamma(M; m)$. Suppose that Γ admits a measure-preserving action on a standard finite measure space (X, μ). We denote by $\mathcal{G}$ the associated groupoid $\Gamma \ltimes (X, \mu)$. Let $\rho\colon \mathcal{G} \to \Gamma$ be the groupoid homomorphism defined by $(g, x) \mapsto g$.

Definition 5.15. Under Assumption $(\star)$, let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Let A be a Borel subset of Y with positive measure and let $\alpha \in V(C)$.

(i) We say that the pair (α, A) is *ρ-invariant* for $\mathcal{S}$ if there exists a countable Borel partition $A = \bigsqcup_n A_n$ of A such that for each n, the constant map $A_n \to \{\alpha\}$ is invariant for $\mathcal{S}$, i.e., $\rho(\gamma)\alpha = \alpha$ for a.e. $\gamma \in (\mathcal{S})_{A_n}$.

(ii) Suppose that (α, A) is ρ-invariant for $\mathcal{S}$. The pair (α, A) is said to be *purely ρ-invariant* for $\mathcal{S}$ if (β, B) is not ρ-invariant for $\mathcal{S}$ for any Borel subset B of A with positive measure and any $\beta \in V(C)$ with $i(\alpha, \beta) \neq 0$. (In [35], we refer to such a pair as an essential ρ-invariant one for $\mathcal{S}$.)

Remark 5.16. In the notation of Definition 5.15, it is easy to see the following:

(i) If (α, A) is a ρ-invariant pair for $\mathcal{S}$, then so is the pair (α, B) for any Borel subset B of A with positive measure. The same statement is true for purely ρ-invariant pairs for $\mathcal{S}$.

(ii) For each $n \in \mathbb{N}$, let A_n be a Borel subset of Y with positive measure. If (α, A_n) is a ρ-invariant pair for $\mathcal{S}$, then so is the pair $(\alpha, \bigcup_{n\in\mathbb{N}} A_n)$. The same statement is true for purely ρ-invariant pairs for $\mathcal{S}$.

Theorem 5.17 ([35, Theorem 4.50]). *Under Assumption $(\star)$, let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. If $\mathcal{S}$ is reducible, then there exists a purely ρ-invariant pair for $\mathcal{S}$.*

In the notation of Theorem 5.17, for $\alpha \in V(C)$, let $\mathcal{M}_\alpha$ be the set of all Borel subsets A of Y such that either $\mu(A) = 0$ or the pair (α, A) is purely ρ-invariant for $\mathcal{S}$. Put $m_\alpha = \sup_{A\in\mathcal{M}_\alpha} \mu(A)$. By Remark 5.16 (ii), there exists an essentially unique Borel subset Y_α of Y such that $\mu(Y_\alpha) = m_\alpha$. Theorem 5.17 implies the equation $Y = \bigcup_{\alpha\in V(C)} Y_\alpha$ up to null sets if $\mathcal{S}$ is reducible. By the definition of purely ρ-invariant pairs, if $\alpha, \beta \in V(C)$ satisfy $\mu(Y_\alpha \cap Y_\beta) > 0$, then $i(\alpha, \beta) = 0$. We then define a Borel map $\varphi\colon Y \to S(M)$ by the formula

$$\varphi(x) = \{\alpha \in V(C) : x \in Y_\alpha\}$$

for x in a conull Borel subset of Y.

Definition 5.18. The map $\varphi\colon Y \to S(M)$ constructed above is called the *canonical reduction system (CRS)* for a reducible subgroupoid $\mathcal{S}$.

The following theorem states that the invariance and the uniqueness of the CRS for a reducible subgroupoid.

Theorem 5.19 ([35, Lemma 4.53]). *Under Assumption $(\star)$, let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Suppose that $\mathcal{S}$ is reducible. Then the CRS $\varphi\colon Y \to S(M)$ for $\mathcal{S}$ is an essentially unique invariant Borel map for $\mathcal{S}$ such that*

- *if $\sigma \in S(M)$ satisfies $\mu(\varphi^{-1}(\sigma)) > 0$ and if $\alpha \in \sigma$, then $(\alpha, \varphi^{-1}(\sigma))$ is a purely ρ-invariant pair for $\mathcal{S}$;*
- *if (α, A) is a purely ρ-invariant pair for $\mathcal{S}$, then*

$$\mu(A \setminus \varphi^{-1}(\{\sigma \in S(M) : \alpha \in \sigma\})) = 0.$$

6 Normal subgroupoids

Feldman, Sutherland, and Zimmer [14] introduced the notion of normal subrelations of discrete measured equivalence relations. We define the notion of normal subgroupoids as its generalization, which is also a generalization of the notion of normal subgroups. It will be shown that if a subgroup of the mapping class group is IA (resp. infinite and reducible), then so is its normalizer. We prove a similar statement in the setting of groupoids. These facts will be used repeatedly in Section 7.

6.1 Generalities

Let G be a discrete group and let H be a subgroup of G. We refer to the subgroup $N_G(H) = \{g \in G : gHg^{-1} = H\}$ as the normalizer of H in G. If $N_G(H) = G$, then H is called a normal subgroup of G.

Let $\mathcal{G}$ be a discrete measured groupoid on a standard measure space (X, μ) and let $r, s\colon \mathcal{G} \to X$ be the range, source maps, respectively. Let $\mathcal{S}$ be a subgroupoid of $\mathcal{G}$. (We mean by a subgroupoid of $\mathcal{G}$ a Borel subgroupoid of $\mathcal{G}$ whose unit space is the same as the one for $\mathcal{G}$.) We denote by $\mathrm{End}_{\mathcal{G}}(\mathcal{S})$ the set of all Borel maps $\phi\colon \mathrm{dom}(\phi) \to \mathcal{G}$ from a Borel subset $\mathrm{dom}(\phi)$ of X such that

- $s(\phi(x)) = x$ for a.e. $x \in \mathrm{dom}(\phi)$;
- for a.e. $\gamma \in (\mathcal{G})_{\mathrm{dom}(\phi)}$, the following equivalence holds: $\gamma \in \mathcal{S}$ if and only if $\phi(r(\gamma))\gamma\phi(s(\gamma))^{-1} \in \mathcal{S}$.

If X consists of a single atom, i.e., if $\mathcal{G}$ and $\mathcal{S}$ are groups, then $\mathrm{End}_{\mathcal{G}}(\mathcal{S})$ is equal to the normalizer $N_{\mathcal{G}}(\mathcal{S})$ of $\mathcal{S}$ in $\mathcal{G}$.

Remark 6.1. Let $\phi \in \mathrm{End}_{\mathcal{G}}(\mathcal{S})$. Note that the groupoid homomorphism $(\mathcal{S})_{\mathrm{dom}(\phi)} \ni \gamma \mapsto \phi(r(\gamma))\gamma\phi(s(\gamma))^{-1} \in \mathcal{G}$ does not define an isomorphism onto its image when the map $\mathrm{dom}(\phi) \ni x \mapsto r(\phi(x)) \in X$ is not injective. Hence, we use the symbol "End".

Definition 6.2. Let $\mathcal{G}$ be a discrete measured groupoid on a standard measure space (X, μ). A subgroupoid $\mathcal{S}$ of $\mathcal{G}$ is said to be *normal* in $\mathcal{G}$ if the following condition is satisfied: There exists a countable family $\{\phi_n\}$ of elements of $\mathrm{End}_{\mathcal{G}}(\mathcal{S})$ such that for a.e. $\gamma \in \mathcal{G}$, we can find ϕ_n in the family such that $r(\gamma) \in \mathrm{dom}(\phi_n)$ and $\phi_n(r(\gamma))\gamma \in \mathcal{S}$. In this case, we write $\mathcal{S} \lhd \mathcal{G}$, and we call $\{\phi_n\}$ a family of normal choice functions for the pair $(\mathcal{G}, \mathcal{S})$.

Example 6.3. *Normal subgroups.* Let G be a discrete group and let H be a subgroup of G. When we regard G as a groupoid, $\mathrm{End}_G(H) = N_G(H)$ as noted above. It is easy to see that H is normal in G in the sense of Definition 6.2 if and only if we can choose all representatives of G/H from $N_G(H)$, that is, $G = N_G(H)$. This means that H is a normal subgroup of G.

Lemma 6.4. *Suppose that we are given a non-singular action of a discrete group G on a standard measure space (X,μ). We denote by $\mathcal{G}$ the associated groupoid. Let H be a normal subgroup of G and let $\mathcal{S}$ be the subgroupoid of $\mathcal{G}$ associated with the action $H \curvearrowright (X,\mu)$. Then $\mathcal{S}$ is normal in $\mathcal{G}$.*

Proof. For $g \in G$, let $\phi_g\colon X \to \mathcal{G}$ be the Borel map defined by $\phi_g(x) = (g,x)$. We show that $\phi_g \in \mathrm{End}_{\mathcal{G}}(\mathcal{S})$. Let $(h,x) \in \mathcal{G}$. If $(h,x) \in \mathcal{S}$, then

$$\phi_g(hx)(h,x)\phi_g(x)^{-1} = (g,hx)(h,x)(g,g^{-1}x) = (ghg^{-1}, g^{-1}x) \in \mathcal{S}$$

since H is a normal subgroup of G. Conversely, if $(ghg^{-1}, g^{-1}x) \in \mathcal{S}$, then $ghg^{-1} \in H$, which implies that $h \in H$ and $(h,x) \in \mathcal{S}$. Thus, $\phi_g \in \mathrm{End}_{\mathcal{G}}(\mathcal{S})$.

Since $\phi_{g^{-1}}(gx)(g,x) = e_x \in \mathcal{S}$ for $(g,x) \in \mathcal{G}$, $\{\phi_g\}_{g\in G}$ is a family of normal choice functions for the pair $(\mathcal{G},\mathcal{S})$. □

We omit the proof of the following lemma.

Lemma 6.5 ([36, Lemma 2.13]). *Let $\mathcal{G}$ be a discrete measured groupoid on a standard measure space (X,μ). Let $\mathcal{S}$ be a normal subgroupoid of $\mathcal{G}$. If A is a Borel subset of X with positive measure, then $(\mathcal{S})_A$ is normal in $(\mathcal{G})_A$.*

6.2 Normalizers of IA and reducible subgroupoids

Let M be a surface with $\kappa(M) \geq 0$ and let N be an infinite subgroup of $\Gamma(M)$. If N is IA, then there exists a pseudo-Anosov element $g \in N$ such that $\{F_\pm(g)\}$, the set of pseudo-Anosov foliations of g, is fixed by all elements of N. If N is reducible, then N fixes an element of $S(M)$. The CRS $\sigma(N)$ for N is a special element of $S(M)$ fixed by N (see Subsection 5.3). By using these special fixed elements of N, we show the following

Proposition 6.6. *Let M be a surface with $\kappa(M) \geq 0$ and let Γ be an infinite subgroup of $\Gamma(M)$. Suppose that Γ contains an infinite normal subgroup N.*

(i) *If N is IA (resp. reducible), then so is Γ.*

(ii) *If N is reducible, then $\sigma(N) \subset \sigma(\Gamma)$.*

Proof. We first assume that N is IA. By Theorem 2.7, there exists a pseudo-Anosov element $g \in N$ such that $h\{F_\pm(g)\} = \{F_\pm(g)\}$ for any $h \in N$. Let $\gamma \in \Gamma$. Then $\gamma^{-1}g\gamma\{F_\pm(g)\} = \{F_\pm(g)\}$ since $\gamma^{-1}g\gamma \in N$. Thus, $g\gamma\{F_\pm(g)\} = \gamma\{F_\pm(g)\}$. On the other hand, the fixed point set on $\mathcal{PMF}$ for g consists of exactly the two points $F_\pm(g)$. Hence, $\gamma\{F_\pm(g)\} = \{F_\pm(g)\}$. This means that every $\gamma \in \Gamma$ fixes $\{F_\pm(g)\}$ and that Γ is IA by Theorem 2.7.

We next assume that N is reducible. For $\gamma \in \Gamma$, the equation $\gamma\sigma(N) = \sigma(\gamma N\gamma^{-1}) = \sigma(N)$ holds. The first equation follows by definition. Thus, Γ is reducible. By the definition of essential reduction classes for Γ, we see that $\sigma(N) \subset \sigma(\Gamma)$. $\square$

In this subsection, we prove a result similar to the above proposition in the framework of groupoids. We recall Assumption ($\star$): Let M be a surface with $\kappa(M) > 0$ and let $m \geq 3$ be an integer. Let Γ be an infinite subgroup of $\Gamma(M; m)$. Suppose that Γ admits a measure-preserving action on a standard finite measure space (X, μ). Let $\mathcal{G}$ be the associated groupoid $\Gamma \ltimes (X, \mu)$. Define a cocycle $\rho\colon \mathcal{G} \to \Gamma$ by $(g, x) \mapsto g$.

Using the uniqueness of the invariant Borel maps for IA and reducible subgroupoids constructed in Subsections 5.2 and 5.3, we show that if a subgroupoid $\mathcal{T}$ of $\mathcal{G}$ contains an IA (resp. reducible) subgroupoid as a normal one, then $\mathcal{T}$ is also IA (resp. reducible). We give only the proof of Lemma 6.7, where IA subgroupoids are dealt with. The proof of Lemma 6.8 for reducible ones is not given here. We refer to Lemma 4.60 in [35] for the proof, in which we assume that the action $\Gamma \curvearrowright (X, \mu)$ is essentially free. However, one can show Lemma 6.8 along the same line as in Lemma 4.60 in [35].

Lemma 6.7. *Under Assumption ($\star$), let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Suppose that $\mathcal{S}$ is IA. Let $\mathcal{T}$ be a subgroupoid of $(\mathcal{G})_Y$ with $\mathcal{S} \lhd \mathcal{T}$. Let $\varphi_0\colon Y \to \partial_2 C$ be the essentially unique Borel map constructed in Lemma 5.8. Then φ_0 is invariant for $\mathcal{T}$. In particular, $\mathcal{T}$ is IA by Lemma 5.7 (i).*

Proof. Let $r\colon \mathcal{T} \to Y$ be the range map. Take $g \in \mathrm{End}_{\mathcal{T}}(\mathcal{S})$. Recall that g is a Borel map from a Borel subset $\mathrm{dom}(g)$ of Y into $\mathcal{T}$ such that

- $s(g(x)) = x$ for a.e. $x \in \mathrm{dom}(g)$;
- for a.e. $\gamma \in (\mathcal{T})_{\mathrm{dom}(g)}$, the following equivalence holds: $\gamma \in \mathcal{S}$ if and only if $g(r(\gamma))\gamma g(s(\gamma))^{-1} \in \mathcal{S}$.

It is enough to show that $\rho(g(x)^{-1})\varphi_0(r(g(x))) = \varphi_0(x)$ for a.e. $x \in \mathrm{dom}(g)$. By applying Theorem 2.9 (iv) to the composition $r \circ g\colon \mathrm{dom}(g) \to Y$, we get a countable Borel partition $\mathrm{dom}(g) = \bigsqcup_n Y_n$ satisfying the following: Let g_n denote the restriction of g to Y_n. The image $r \circ g_n(Y_n)$ is a Borel subset of Y, and the map $r \circ g_n\colon Y_n \to r \circ g_n(Y_n)$ is a Borel isomorphism. Moreover, each g_n is an element of $\mathrm{End}_{\mathcal{T}}(\mathcal{S})$. We may therefore assume that $g \in \mathrm{End}_{\mathcal{T}}(\mathcal{S})$ satisfies that $r \circ g(\mathrm{dom}(g))$ is a Borel subset of Y and the map $r \circ g\colon \mathrm{dom}(g) \to r \circ g(\mathrm{dom}(g))$ is a Borel isomorphism.

We define a Borel map h from $\mathrm{dom}(h) = r \circ g(\mathrm{dom}(g))$ into $\mathcal{T}$ by $h(y) = g((r \circ g)^{-1}(y))^{-1}$ for $y \in \mathrm{dom}(h)$. It is easy to see that $h \in \mathrm{End}_{\mathcal{T}}(\mathcal{S})$.

We put $\psi_g(x) = \rho(g(x)^{-1})\varphi_0(r(g(x)))$ for $x \in \mathrm{dom}(g)$. Take $\gamma \in (\mathcal{S})_{\mathrm{dom}(g)}$ and put $x = r(\gamma)$ and $y = s(\gamma)$. Then the equation

$$\begin{aligned}\rho(\gamma)\psi_g(y) &= \rho(\gamma)\rho(g(y)^{-1})\varphi_0(r(g(y)))\\ &= \rho(g(x)^{-1})\rho(g(x))\rho(\gamma)\rho(g(y)^{-1})\varphi_0(r(g(y)))\\ &= \rho(g(x)^{-1})\varphi_0(r(g(x))) = \psi_g(x)\end{aligned}$$

holds since $g(x)\gamma g(y)^{-1} \in \mathcal{S}$ and φ_0 is invariant for $\mathcal{S}$. The map ψ_g is thus invariant for $(\mathcal{S})_{\mathrm{dom}(g)}$. By Lemma 5.8, we have

$$\mathrm{supp}(\psi_g(x)) \subset \mathrm{supp}(\varphi_0(x)) \quad \text{for a.e. } x \in \mathrm{dom}(g).$$

By considering h instead of g, we have

$$\mathrm{supp}(\rho(h(y)^{-1})\varphi_0(r(h(y)))) \subset \mathrm{supp}(\varphi_0(y)) \quad \text{for a.e. } y \in \mathrm{dom}(h).$$

By putting $y = r \circ g(x)$ in the above two inclusions, we get the equation $\psi_g(x) = \varphi_0(x)$ for a.e. $x \in \mathrm{dom}(g)$. Therefore, φ_0 is invariant for $\mathcal{T}$. □

Lemma 6.8. *Under Assumption $(\star)$, let $Y \subset X$ be a Borel subset with positive measure and let $\mathcal{S}$ be a subgroupoid of $(\mathcal{G})_Y$ of infinite type. Suppose that $\mathcal{S}$ is reducible. Let $\mathcal{T}$ be a subgroupoid of $(\mathcal{G})_Y$ with $\mathcal{S} \lhd \mathcal{T}$. Let $\varphi_0\colon Y \to S(M)$ be the CRS for $\mathcal{S}$ (see Definition 5.18). Then φ_0 is invariant for $\mathcal{T}$. In particular, $\mathcal{T}$ is reducible.*

7 Characterization of reducible subgroupoids

In this section, we prove Theorem 3.6. This theorem states that any isomorphism between groupoids associated with measure-preserving actions of mapping class groups preserves subgroupoids generated by Dehn twists. To prove it, we characterize such subgroupoids algebraically in terms of discrete measured groupoids. As in the previous sections, we first investigate the case of groups. We give a complete proof in the case of groups, and give only some comments about the case of groupoids. Most theorems in the case of groupoids can be shown by an idea similar to the one in the case of groups.

Classification of components into three types. We first consider the action of a reducible subgroup on each component of the surface obtained by cutting along the CRS for the subgroup. We recall the following notation.

Notation. Let M be a surface with $\kappa(M) \geq 0$ and let $m \geq 3$ be an integer. Let Γ be a reducible subgroup of $\Gamma(M; m)$ and assume that each element of Γ

fixes $\sigma \in S(M)$. By Theorem 2.8 (iii), there is a natural homomorphism

$$p_\sigma\colon \Gamma \to \prod_Q \Gamma(Q),$$

where Q runs through all components of M_σ, the surface obtained by cutting M along a realization of σ. For each component Q of M_σ, let $p_Q\colon \Gamma \to \Gamma(Q)$ be the composition of p_σ and the projection onto $\Gamma(Q)$.

In the following theorem, we consider the quotient groups $p_Q(\Gamma)$ when σ is the CRS for Γ (see also Theorem 5.13).

Theorem 7.1 ([30, Lemma 1.6, Corollary 7.18]). *Let M be a surface with $\kappa(M) \geq 0$ and let $m \geq 3$ be an integer. Let Γ be an infinite reducible subgroup of $\Gamma(M; m)$ and let $\sigma \in S(M)$ be the CRS for Γ. If Q is a component of M_σ, then the following assertions hold:*

(i) $p_Q(\Gamma)$ *is torsion-free.*

(ii) $p_Q(\Gamma)$ *either is trivial or contains a pseudo-Anosov element of* $\Gamma(Q)$.

If $p_Q(\Gamma)$ is trivial, infinite amenable or non-amenable, then we say that Q is *T*, *IA* or *IN* for Γ, respectively. Theorem 7.1 implies that any component Q of M_σ is either T, IA or IN, and the following assertions hold:

(A) Q is T for Γ if and only if $p_Q(\Gamma)$ is trivial.

(B) Q is IA for Γ if and only if $p_Q(\Gamma)$ is an IA subgroup of $\Gamma(Q)$.

(C) Q is IN for Γ if and only if $p_Q(\Gamma)$ is a sufficiently large subgroup of $\Gamma(Q)$.

Remark 7.2. These three types of Q can be characterized in terms of fixed points for the action of $p_Q(\Gamma)$ on the space $M(\mathcal{PMF}(Q))$ of probability measures on $\mathcal{PMF}(Q)$ as follows:

(a) Q is T for Γ if and only if either Q is a pair of pants ($= M_{0,3}$) or $p_Q(g)\alpha = \alpha$ for any $g \in \Gamma$ and for any/some $\alpha \in V(C(Q))$.

(b) Q is IA for Γ if and only if the following three conditions are satisfied:

- Q is not a pair of pants;
- $p_Q(g)\alpha \neq \alpha$ for any $g \in \Gamma \setminus \{e\}$ and for any/some $\alpha \in V(C(Q))$;
- There exists $\mu \in M(\mathcal{PMF}(Q))$ such that $p_Q(g)\mu = \mu$ for any $g \in \Gamma$ and $\mu(\mathcal{MIN}(Q)) = 1$.

(c) Q is IN for Γ if and only if the following two conditions are satisfied:

- Q is not a pair of pants;
- There exists no fixed point for the action of $p_Q(\Gamma)$ on $M(\mathcal{PMF}(Q))$.

In the setting of groupoids, motivated by the above characterization, we can define three types of components of the surface obtained by cutting M along the CRS for a reducible subgroupoid $\mathcal{S}$. They are defined in terms of invariant Borel maps into $M(\mathcal{PMF}(Q))$, etc. for $\mathcal{S}$. We refer to Theorems 5.6, 5.9 and Section 5.2 in [35] for a precise definition of them.

As an application of Theorem 7.1, we give a criterion for amenability of reducible subgroups.

Proposition 7.3. *Let M be a surface with $\kappa(M) \geq 0$ and let $m \geq 3$ be an integer. Let Γ be a reducible subgroup of $\Gamma(M; m)$ and let $\sigma \in S(M)$ be the CRS for Γ. Then Γ is amenable if and only if each component of M_σ is either T or IA for Γ.*

Theorem 7.1 (ii) implies the "only if" part because the quotient group $p_Q(\Gamma)$ is amenable for each component Q of M_σ if Γ is amenable. The "if" part follows since the intersection of the kernels of p_Q for all components Q of M_σ is amenable by the following proposition (see Lemma 2.1 (1) in [8] or Corollary 4.1.B, Lemma 4.1.C in [32]).

Proposition 7.4. *Let M be a surface with $\kappa(M) \geq 0$. Let G be a reducible subgroup of $\Gamma(M)$ and let $\sigma \in S(M)$ be an element such that $g\sigma = \sigma$ for any $g \in G$. Let $p\colon G \to \Gamma(M_c)$ be the natural homomorphism into the mapping class group of the disconnected surface M_c obtained by cutting M along a realization c of σ. Then the following assertions hold:*

(i) $\ker p$ *is contained in the subgroup D_σ of $\Gamma(M)$ generated by Dehn twists about curves in σ.*

(ii) *All elements of D_σ belong to the center of* $\ker p$.

Characterization of some subgroups. Let M be a surface with $\kappa(M) > 0$. The following is our plan to characterize subgroups of $\Gamma(M)$ generated by Dehn twists.

(I) Characterize reducible subgroups of $\Gamma(M)$ in terms of amenability and normal subgroups.

(II) Describe maximal reducible subgroups of $\Gamma(M; m)$ explicitly, where $m \geq 3$ is an integer.

(III) Describe an infinite amenable normal subgroup N of a maximal reducible subgroup in Step (II). In fact, such a subgroup N is contained in the subgroup generated by the Dehn twist about some element of $V(C)$.

One important observation for Step (I) is the following lemma. This gives a sufficient condition for a subgroup to be reducible.

Lemma 7.5. *Let M be a surface with $\kappa(M) > 0$. Let G be a non-amenable subgroup of $\Gamma(M)$ and let N be an infinite normal subgroup of G. If N is amenable, then G is reducible.*

Proof. It follows from Theorem 2.7 that N is either IA or reducible. By Proposition 6.6 (i), if N is IA (resp. reducible), then so is G. Since G is non-amenable, G must be reducible. □

Remark 7.6. When $\kappa(M) = 0$, there exists no non-amenable reducible subgroup of $\Gamma(M)$. This fact implies that we cannot characterize reducible subgroups of $\Gamma(M)$ as in Propositions 7.7 and 7.8.

We characterize infinite reducible subgroups in the next two propositions. Although it is not necessary to characterize infinite amenable reducible subgroups for our purpose (because maximal reducible subgroups in Step (II) are always non-amenable), we give it for completeness.

Proposition 7.7. *Let M be a surface with $\kappa(M) > 0$ and let Γ be an infinite amenable subgroup of $\Gamma(M)$. Then the following two assertions are equivalent:*

(i) *Γ is reducible.*

(ii) *There exist four subgroups Γ_0, Γ', Γ'' and Λ of Γ satisfying the following:*

  (a) *Γ_0 is a finite index subgroup of Γ;*

  (b) *Γ' is amenable and $\Gamma_0 < \Gamma'$;*

  (c) *Γ'' is infinite and $\Gamma'' < \Gamma'$;*

  (d) *Λ is non-amenable and $\Gamma'' \lhd \Lambda$.*

Proof. We first show that Assertion (ii) implies Assertion (i). It follows from Lemma 7.5 that Γ'' and Λ are both reducible. By Theorem 2.7, Γ' must be either IA or reducible since Γ' is amenable. If Γ' were IA, then there would exist a finite index subgroup of Γ' which is cyclic and generated by a pseudo-Anosov element. This contradicts the assumption that Γ' contains the infinite reducible subgroup Γ''. Thus, Γ' is reducible and so are both Γ_0 and Γ.

We next show that the converse holds. Put $\Gamma_0 = \Gamma \cap \Gamma(M; 3)$ and let $\sigma \in S(M)$ be the CRS for Γ_0. Note that for each T component Q of M_σ for Γ_0, the quotient $p_Q(\Gamma_0)$ is trivial by Theorem 7.1 (ii). For each IA component R of M_σ for Γ_0, let $\{F^R_\pm\}$ be the pair of pseudo-Anosov foliations in $\mathcal{PMF}(R)$ such that

$$p_R(g)\{F^R_\pm\} = \{F^R_\pm\} \ \text{ for any } g \in \Gamma_0.$$

Let Γ' be the subgroup of $\Gamma(M; 3)$ consisting of all $g \in \Gamma(M; 3)$ satisfying the following three conditions:

- $g\sigma = \sigma$;
- $p_Q(g)\alpha = \alpha$ for all $\alpha \in V(C(Q))$ and all T components Q for Γ_0 which is not a pair of pants;
- $p_R(g)\{F_\pm^R\} = \{F_\pm^R\}$ for all IA components R for Γ_0.

Then $\Gamma_0 < \Gamma'$. Moreover, the CRS for Γ' is σ and a component of M_σ is T (resp. IA) for Γ' if and only if so is for Γ_0. In particular, Γ' is amenable since there is no IN component for Γ_0 and thus for Γ' (see Lemma 7.3).

In general, given $\tau \in S(M)$, we denote by D_τ the subgroup of $\Gamma(M)$ generated by Dehn twists about curves in τ. If $|\sigma| < \kappa(M) + 1$, then put $\Gamma'' = D_\sigma \cap \Gamma(M;3)$. This subgroup is infinite. Let Λ be the stabilizer of σ in $\Gamma(M;3)$, i.e., $\Lambda = \{g \in \Gamma(M;3) : g\sigma = \sigma\}$. Since there is a component of M_σ which is not a pair of pants, Λ is non-amenable. (Recall that when we cut M along curves in $\tau \in S(M)$ and get the surface M_τ, all components of M_τ are pairs of pants if and only if $|\tau| = \kappa(M) + 1$.) Moreover, Γ'' is a normal subgroup of Λ by Proposition 7.4 (ii). These Γ'', Λ satisfy the conditions of Assertion (ii).

If $|\sigma| = \kappa(M) + 1$, then we see that $\Gamma' = D_\sigma \cap \Gamma(M;3)$. Choose $\alpha_0 \in \sigma$. Let $\sigma' = \sigma \setminus \{\alpha_0\}$ and put $\Gamma'' = D_{\sigma'} \cap \Gamma(M;3)$. This subgroup is infinite and satisfies $\Gamma'' < \Gamma'$. If we define Λ to be the stabilizer of σ' in $\Gamma(M;3)$, then these subgroups satisfy the conditions of Assertion (ii). □

Proposition 7.8. *Let M be a surface with $\kappa(M) > 0$ and let Γ be a non-amenable subgroup of $\Gamma(M)$. Then the following two assertions are equivalent:*

(i) Γ *is reducible.*

(ii) *There exist two subgroups Γ', Γ'' of $\Gamma(M)$ satisfying the following:*

   (a) $\Gamma < \Gamma'$*;*

   (b) Γ'' *is infinite amenable and* $\Gamma'' \lhd \Gamma'$.

Proof. We first show that Assertion (ii) implies Assertion (i). It follows from Lemma 7.5 that Γ' and Γ'' are both reducible. Thus, so is Γ.

We next show that the converse holds. Let $\sigma \in S(M)$ be the CRS for Γ. Let Γ' be the stabilizer of σ, i.e., $\Gamma' = \{g \in \Gamma(M) : g\sigma = \sigma\}$. Then Γ' contains Γ. Let Γ'' be the subgroup of $\Gamma(M)$ generated by Dehn twists about curves in σ. By Proposition 7.4 (ii), we see that $\Gamma'' \lhd \Gamma'$. □

Corollary 7.9. *Let M be a surface with $\kappa(M) > 0$. Let Γ_1, Γ_2 be finite index subgroups of $\Gamma(M)$. If $f\colon \Gamma_1 \to \Gamma_2$ is an isomorphism and Λ is an infinite reducible subgroup of Γ_1, then $f(\Lambda)$ is an infinite reducible subgroup of Γ_2.*

Notation. Let M be a surface with $\kappa(M) \geq 0$. Given a subgroup Γ of $\Gamma(M)$ and $\sigma \in S(M)$, we denote by

$$\Gamma_\sigma = \{g \in \Gamma : g\sigma = \sigma\}$$

the stabilizer of σ in Γ. When σ consists of only one element $\alpha \in V(C)$, we denote Γ_σ by Γ_α for simplicity.

In the next lemma, we explicitly describe maximal reducible subgroups.

Lemma 7.10. *Let M be a surface with $\kappa(M) \geq 0$ and let $m \geq 3$ be an integer. Let Γ be a finite index subgroup of $\Gamma(M; m)$ and let $\alpha \in V(C)$. Then the following assertions hold:*

(i) *Γ_α is a maximal reducible subgroup of Γ, that is, if Λ is a reducible subgroup of Γ with $\Gamma_\alpha < \Lambda$, then $\Lambda = \Gamma_\alpha$.*

(ii) *Conversely, any reducible subgroup of Γ is contained in Γ_α for some $\alpha \in V(C)$.*

Proof. Assertion (ii) follows from Theorem 2.8 (iii). We prove Assertion (i). One can show that α is the only class in $V(C)$ fixed by all elements of Γ_α. In fact, if $\beta \in V(C)$ satisfies $i(\alpha, \beta) \neq 0$, then some power of the Dehn twist about α is in Γ_α and does not fix β (see Theorem 2.6). Suppose that $\beta \in V(C)$ satisfies $\alpha \neq \beta$ and $i(\alpha, \beta) = 0$. Let M_α be the surface obtained by cutting M along a realization of α. Let Q be a component of M_α such that $\beta \in V(C(Q))$. Since Γ is a finite index subgroup of $\Gamma(M; m)$, the component Q is IN for Γ_α. Hence, $p_Q(\Gamma_\alpha)$ does not fix β. This proves the claim. Assertion (i) then follows because Λ fixes some curve in $V(C)$, which has to be α by this claim. □

Finally, we give an algebraic characterization of subgroups generated by Dehn twists.

Lemma 7.11. *Let M be a surface with $\kappa(M) \geq 0$ and let $m \geq 3$ be an integer. Let Γ be a subgroup of finite index in $\Gamma(M; m)$ and let $\alpha \in V(C)$. We denote by D_α the intersection of Γ with the subgroup of $\Gamma(M)$ generated by the Dehn twist about α.*

(i) *Let N be an infinite amenable subgroup of Γ_α with $N \lhd \Gamma_\alpha$. Then N is contained in D_α.*

(ii) *Conversely, any subgroup of D_α is amenable and is a normal one of Γ_α.*

Proof. Assertion (ii) follows from Proposition 7.4 (ii). We show Assertion (i). When $\kappa(M) = 0$, Proposition 7.4 (i) implies that $\Gamma_\alpha = D_\alpha$, and Assertion (i) follows. We assume that $\kappa(M) > 0$. Let $\sigma \in S(M)$ be the CRS for N. Note that the CRS for Γ_α is $\{\alpha\}$ (see Example 5.14 (iii)). By Proposition 6.6 (ii), we see that $\sigma \subset \{\alpha\}$, which means that $\sigma = \{\alpha\}$. By Proposition 7.4 (i), it is enough to show that each component of M_α is T for N, which follows from the next Lemma 7.12. □

Lemma 7.12. *Let M be a surface with $\kappa(M) > 0$ and let $m \geq 3$ be an integer. Let Γ be a subgroup of $\Gamma(M;m)$ and let N be an infinite normal subgroup of Γ. Suppose that N is reducible and let $\sigma \in S(M)$ be the CRS for N. (Note that $\sigma \subset \sigma(\Gamma)$ by Proposition 6.6 (ii).) If a component Q of M_σ is IA for N, then Q is a component of $M_{\sigma(\Gamma)}$ and it is IA for Γ.*

Proof. Recall that σ is fixed by Γ by Proposition 6.6 (ii). If Q were not a component of $M_{\sigma(\Gamma)}$, then there would exist $\alpha \in \sigma(\Gamma) \setminus \sigma$ such that $\alpha \in V(C(Q))$ since $\sigma \subset \sigma(\Gamma)$. Then α is fixed by all elements of N. This contradicts the assumption that Q is IA for N. Thus, Q is a component of $M_{\sigma(\Gamma)}$. Since $p_Q(N)$ is IA and is a normal subgroup of $p_Q(\Gamma)$, we see that $p_Q(\Gamma)$ is also IA by Proposition 6.6 (i). □

Corollary 7.13. *Let M be a surface with $\kappa(M) > 0$ and let $m \geq 3$ be an integer. Let Γ_1, Γ_2 be finite index subgroups of $\Gamma(M;m)$ and let $f\colon \Gamma_1 \to \Gamma_2$ be an isomorphism. For $i \in \{1,2\}$ and $\alpha \in V(C)$, let D_α^i be the intersection of Γ_i with the cyclic subgroup of $\Gamma(M)$ generated by the Dehn twist about α. Then for each $\alpha \in V(C)$, there exists $\beta \in V(C)$ such that $f(D_\alpha^1) = D_\beta^2$.*

Proof. For $i \in \{1,2\}$ and $\alpha \in V(C)$, we denote by Γ_α^i the stabilizer of α in Γ_i. Let $\alpha \in V(C)$. By Lemma 7.10 (i), Γ_α^1 is a maximal reducible subgroup of Γ_1. It follows from Corollary 7.9 that $f(\Gamma_\alpha^1)$ is also a maximal reducible subgroup of Γ_2. Thus, there exists $\beta \in V(C)$ such that $f(\Gamma_\alpha^1) = \Gamma_\beta^2$ by Lemma 7.10 (ii). Since D_α^1 is a normal subgroup of Γ_α^1, we see that $f(D_\alpha^1)$ is also a normal subgroup of Γ_β^2. By Lemma 7.11 (i), $f(D_\alpha^1) < D_\beta^2$. Considering f^{-1}, we see that there exists $\alpha' \in V(C)$ such that $D_\beta^2 < f(D_{\alpha'}^1)$. Since D_α^1 and $D_{\alpha'}^1$ has non-trivial intersection, we obtain the equality $\alpha = \alpha'$ by Lemma 3.4. □

By using this corollary in place of Theorem 3.3 in Ivanov's argument in Subsection 3.1, we can show Theorem 3.2, which states that any isomorphism between finite index subgroups of the extended mapping class group $\Gamma(M)^\diamond$ with $\kappa(M) > 0$ and $M \neq M_{1,2}, M_{2,0}$ is equal to the inner conjugation by a unique element of $\Gamma(M)^\diamond$.

The case of groupoids. We first restate Theorem 3.6. Recall the following notation.

Notation. We refer to the following assumption as (•):

- Let M be a surface with $\kappa(M) > 0$ and let $m \geq 3$ be an integer. Let Γ_1 and Γ_2 be finite index subgroups of $\Gamma(M;m)$. Let (Ω,ω) be an ME coupling of Γ_1 and Γ_2
- Take fundamental domains $X_1 \subset \Omega$ for the Γ_2-action on Ω, and $X_2 \subset \Omega$ for the Γ_1-action on Ω. Recall that the natural actions $\Gamma_1 \curvearrowright X_1$ and

$\Gamma_2 \curvearrowright X_2$ are denoted by $(\gamma, x) \mapsto \gamma \cdot x$ by using a dot. By Lemma 2.27, we can choose X_1, X_2 so that $Y = X_1 \cap X_2$ satisfies that for $i \in \{1, 2\}$, $\Gamma_i \cdot Y = X_i$ up to null sets when Y is regarded as a subset of X_i.

- For $i \in \{1, 2\}$, set $\mathcal{G}^i = \Gamma_i \ltimes X_i$ and let $\rho_i\colon \mathcal{G}^i \to \Gamma_i$ be the projection, which is a groupoid homomorphism. By Proposition 2.29, there exists a groupoid isomorphism
$$f\colon (\mathcal{G}^1)_Y \to (\mathcal{G}^2)_Y.$$
Note that f is the identity on the unit space Y.
- For $i \in \{1, 2\}$ and $\alpha \in V(C)$, let D^i_α be the intersection of Γ_i with the subgroup of $\Gamma(M)$ generated by the Dehn twist $t_\alpha \in \Gamma(M)$ about α. Let $\mathcal{G}^i_\alpha$ be the subgroupoid of $\mathcal{G}^i$ generated by the action of D^i_α, i.e.,
$$\mathcal{G}^i_\alpha = \{(\gamma, x) \in \mathcal{G}^i : \gamma \in D^i_\alpha,\ x \in X_i\}.$$

Theorem 3.6. *Under Assumption* ($\bullet$), *for each* $\alpha \in V(C)$, *there exist a countable Borel partition* $Y = \bigsqcup Y_n$ *and* $\beta_n \in V(C)$ *such that*
$$f((\mathcal{G}^1_\alpha)_{Y_n}) = (\mathcal{G}^2_{\beta_n})_{f(Y_n)} \quad \textit{for each } n.$$

This theorem states that f preserves subgroupoids generated by Dehn twists up to a countable Borel partition. When each of X_1 and X_2 consists of a single atom, this theorem reduces to Corollary 7.13. To prove Theorem 3.6, we characterize subgroupoids generated by Dehn twists algebraically in terms of discrete measured groupoids. Our plan is the following:

(1) We characterize reducible subgroupoids algebraically. It follows that the isomorphism f in Assumption ($\bullet$) preserves reducible subgroupoids.
(2) By Step (1), f preserves maximal reducible subgroupoids. We explicitly describe such subgroupoids.
(3) A subgroupoid generated by a Dehn twist can be characterized algebraically as an amenable normal subgroupoid of infinite type of some maximal reducible subgroupoid. This implies Theorem 3.6.

Note that the above steps correspond to the ones for the proof of Corollary 7.13 given right before Lemma 7.5. In this final part of Section 7, we give only precise statements and some comments for the above steps. Most statements can be proved along the same line as in the case of groups. We refer to Section 4 in [36] for the proof of them.

About Step (1). We shall recall Assumption ($\star$): Let M be a surface with $\kappa(M) > 0$ and let $m \geq 3$ be an integer. Let Γ be a finite index subgroup of $\Gamma(M; m)$. Suppose that Γ admits a measure-preserving action on a standard finite measure space (X, μ). Let $\mathcal{G}$ be the associated groupoid $\Gamma \ltimes (X, \mu)$. Define a groupoid homomorphism $\rho\colon \mathcal{G} \to \Gamma$ by $(g, x) \mapsto g$.

Proposition 7.14 ([36, Proposition 4.1]). *Under Assumption* ($\star$), *let* $Y \subset X$ *be a Borel subset with positive measure and let* $\mathcal{S}$ *be a subgroupoid of* $(\mathcal{G})_Y$ *of infinite type. Suppose that* $\mathcal{S}$ *is amenable. Then the following two assertions are equivalent:*

(i) $\mathcal{S}$ *is reducible.*

(ii) *For any Borel subset* A *of* Y *with positive measure, there exist a Borel subset* B *of* A *with positive measure and the following three subgroupoids* $\mathcal{S}'$, $\mathcal{S}''$ *and* $\mathcal{T}$ *of* $(\mathcal{G})_B$*:*

  (a) $\mathcal{S}'$ *is amenable and* $(\mathcal{S})_B < \mathcal{S}'$;

  (b) $\mathcal{S}''$ *is of infinite type and* $\mathcal{S}'' < \mathcal{S}'$;

  (c) $\mathcal{T}$ *is non-amenable and* $\mathcal{S}'' \lhd \mathcal{T}$.

Proposition 7.15 ([36, Proposition 4.2]). *Under Assumption* ($\star$), *let* $Y \subset X$ *be a Borel subset with positive measure and let* $\mathcal{S}$ *be a subgroupoid of* $(\mathcal{G})_Y$ *of infinite type. Suppose that* $(\mathcal{S})_{Y'}$ *is not amenable for any Borel subset* Y' *of* Y *with positive measure. Then the following two assertions are equivalent:*

(i) $\mathcal{S}$ *is reducible.*

(ii) *For any Borel subset* A *of* Y *with positive measure, there exist a Borel subset* B *of* A *with positive measure and the following two subgroupoids* $\mathcal{S}'$ *and* $\mathcal{S}''$ *of* $(\mathcal{G})_B$*:*

  (a) $(\mathcal{S})_B < \mathcal{S}'$;

  (b) $\mathcal{S}''$ *is an amenable subgroupoid of infinite type and* $\mathcal{S}'' \lhd \mathcal{S}'$.

Along the same line as in the proof of Propositions 7.7 and 7.8, these propositions are proved by using invariant Borel maps developed in Section 5. Thanks to these algebraic characterizations, we obtain the following corollary.

Corollary 7.16. *Under Assumption* ($\bullet$), *let* A *be a Borel subset of* Y *with positive measure and let* $\mathcal{S}^1$ *be a subgroupoid of* $(\mathcal{G}^1)_A$ *of infinite type. Then* $\mathcal{S}^1$ *is reducible if and only if the image* $f(\mathcal{S}^1)$ *is reducible.*

About Step (2). Under Assumption ($\star$), let $Y \subset X$ be a Borel subset with positive measure. For a Borel map $\varphi\colon Y \to V(C)$, we write

$$\mathcal{S}_\varphi = \{\gamma \in (\mathcal{G})_Y : \rho(\gamma)\varphi(s(\gamma)) = \varphi(r(\gamma))\}.$$

This subgroupoid can be viewed as the stabilizer of φ in $(\mathcal{G})_Y$. As in the case of groups, we can show that $\mathcal{S}_\varphi$ is a maximal reducible subgroupoid in $(\mathcal{G})_Y$, and conversely that any reducible subgroupoid is contained in $\mathcal{S}_\varphi$ for some φ. Thus, we obtain the following

Corollary 7.17. *Under Assumption* ($\bullet$)*, let* A_1 *be a Borel subset of* Y_1 *with positive measure and let* $\varphi_1\colon A_1 \to V(C)$ *be a Borel map. Put* $A_2 = f(A_1)$*. Then there exists a Borel map* $\varphi_2\colon A_2 \to V(C)$ *such that* $f(\mathcal{S}^1_{\varphi_1}) = \mathcal{S}^2_{\varphi_2}$*, where* $\mathcal{S}^i_{\varphi_i} = \{\gamma \in (\mathcal{G}^i)_{A_i} : \rho_i(\gamma)\varphi_i(s(\gamma)) = \varphi_i(r(\gamma))\}$ *for* $i \in \{1,2\}$*.*

About Step (3). Under Assumption ($\star$), let $Y \subset X$ be a Borel subset with positive measure and let $\varphi\colon Y \to V(C)$ be a Borel map. As in Lemma 7.11, if $\mathcal{S}$ is an amenable subgroupoid of $\mathcal{S}_\varphi$ of infinite type with $\mathcal{S} \lhd \mathcal{S}_\varphi$, then we can show that there exists a countable Borel partition $Y = \bigsqcup Y_n$ of Y satisfying the following two conditions:

(i) The map φ is constant a.e. on Y_n. Let $\alpha_n \in V(C)$ be its value on Y_n.

(ii) $(\mathcal{S})_{Y_n} < (\mathcal{G}_{\alpha_n})_{Y_n} < (\mathcal{S}_\varphi)_{Y_n}$ for each n.

Here, for $\alpha \in V(C)$, we denote by $\mathcal{G}_\alpha$ the subgroupoid of $\mathcal{G}$ generated by the intersection D_α of Γ with the cyclic subgroup of $\Gamma(M)$ generated by the Dehn twist about α.

In what follows, we prove Theorem 3.6. Under Assumption ($\bullet$), let $\alpha \in V(C)$ and let $\varphi_1\colon Y \to V(C)$ be the constant map with value α. Since $\mathcal{S}^1_{\varphi_1}$ is the subgroupoid generated by the action of the stabilizer of α in Γ_1, we see that $(\mathcal{G}^1_\alpha)_Y \lhd (\mathcal{S}^1_{\varphi_1})_Y$ by Lemma 6.5. By Corollary 7.17 and the above fact, there exist a Borel map $\varphi_2\colon Y \to V(C)$ and a countable Borel partition $Y = \bigsqcup Y_n$ of Y satisfying the following two conditions:

(i) The map φ_2 is constant a.e. on Y_n. Let $\beta_n \in V(C)$ be its value on Y_n.

(ii) $f((\mathcal{G}^1_\alpha)_{Y_n}) < (\mathcal{G}^2_{\beta_n})_{f(Y_n)} < (\mathcal{S}^2_{\varphi_2})_{f(Y_n)}$ for each n.

By considering f^{-1}, we can show that $f((\mathcal{G}^1_\alpha)_{Y_n}) = (\mathcal{G}^2_{\beta_n})_{f(Y_n)}$ for each n. This proves Theorem 3.6.

8 Concluding remarks

We present some comments about other related results shown in the series of papers [35], [36], [37], [38].

1. Classification of mapping class groups up to ME. For $i \in \{1,2\}$, let M_i be a surface of type (g_i, p_i), that is, of genus g_i and with p_i boundary components. When are the mapping class groups $\Gamma(M_1)$ and $\Gamma(M_2)$ ME? Note that for a surface M of type (g,p), the mapping class group $\Gamma(M)$ is finite if and only if $\kappa(M) = 3g + p - 4 < 0$ and $(g,p) \neq (1,0)$. We may exclude these cases. If $(g,p) = (1,0), (1,1)$, then $\Gamma(M)$ is isomorphic to $SL(2,\mathbb{Z})$. If $(g,p) = (0,4)$, then there exists a finite index subgroup of $\Gamma(M)$ isomorphic to $PSL(2,\mathbb{Z})$ (see Section 7 in [29]). In particular, if $(g,p) = (1,0), (0,4), (1,1)$, then $\Gamma(M)$ is ME to $SL(2,\mathbb{Z})$ and is hyperbolic in the sense of Gromov. It follows from Theorem 1.5 that if $\kappa(M) > 0$ and if a discrete group Λ is ME to $\Gamma(M)$, then they are commensurable up to finite kernels. Thus, if $\kappa(M) > 0$, then $\Gamma(M)$ and $SL(2,\mathbb{Z})$ are not ME since $\Gamma(M)$ is not hyperbolic. Hence, if we classify the case $\kappa(M) > 0$, then we obtain a complete classification. Thanks to Theorem 1.5, this remaining problem is reduced to a simple algebraic problem of mapping class groups. By Theorem 2 in [57], we obtain the following

Theorem 8.1 ([36, Theorem 1.2]). *Let M^1 and M^2 be distinct surfaces of type (g_1, p_1), (g_2, p_2), respectively, such that $\kappa(M^1), \kappa(M^2) > 0$ and $g_1 \leq g_2$. Suppose that $\Gamma(M^1)$ and $\Gamma(M^2)$ are ME. Then we have the following only two possibilities:* $((g_1, p_1), (g_2, p_2)) = ((0,5),(1,2)), ((0,6),(2,0))$.

In Chapters 5 and 6 in [35], we obtain a weaker classification result by a complete different approach, using tools developed in Sections 5 and 6.

Gaboriau [20] proved that the sequence $\{\beta_n(\Gamma)\}_{n\in\mathbb{N}}$ of ℓ^2-Betti numbers for a discrete group Γ is an invariant for ME in the following sense: If two discrete groups Γ and Λ are ME, then there exists a positive real number c such that $\beta_n(\Gamma) = c\beta_n(\Lambda)$ for all n. Combining this with results due to Gromov [22] and McMullen [46], we can calculate the ℓ^2-Betti numbers of mapping class groups as follows: If M is a surface with $\kappa(M) \geq 0$, then

$$\beta_{\kappa(M)+1}(\Gamma(M)) > 0 \quad \text{and} \quad \beta_n(\Gamma(M)) = 0 \quad \text{for} \quad n \neq \kappa(M)+1.$$

Therefore, the value $\kappa(M)$ is invariant under ME. The reader is referred to Appendix D in [35] for more details, in which explicit values of $\beta_{\kappa(M)+1}(\Gamma(M))$ are also discussed.

2. Exactness of mapping class groups. We defined amenability of a group action in a measurable sense in Section 4. We can also define amenability of a group action in a topological sense. A discrete group is said to be *exact* if it admits an amenable action on some compact Hausdorff space in a topological sense (see [4], [54] for the definition). It is widely expected that the class of exact groups is huge. Indeed, all amenable groups, hyperbolic ones and linear ones are exact. Exactness is closed under taking subgroups, extensions, direct unions and amalgamated free products. Moreover, exactness has many equivalent conditions in terms of geometry of Cayley graphs and operator algebras, and has many applications to various research fields, the study of the Baum-Connes conjecture and the classification of group von Neumann algebras. We recommend the reader to consult [54], [61] and the references therein for more details.

As a byproduct of Theorem 4.21, we can show that if M is a surface with $\kappa(M) > 0$, then the action of $\Gamma(M)^\diamond$ on its Stone-Čech compactification is amenable in a topological sense (see Theorem C.5 in [35]). Hence, $\Gamma(M)^\diamond$ and all its subgroups are exact. Note that the action of $\Gamma(M)^\diamond$ on $\mathcal{PMF}$ is not amenable because there exist non-amenable stabilizers and note that ∂C is not compact (see Proposition 3.8 in [35]). Hamenstädt [26] also proved that $\Gamma(M)^\diamond$ is exact by constructing an explicit compact space on which $\Gamma(M)^\diamond$ admits an amenable action in a topological sense.

3. Direct products of mapping class groups. We can also prove an ME rigidity result for finite direct products of mapping class groups.

Theorem 8.2 ([36, Theorem 1.3]). *Let n be a positive integer and let M_i be a surface with $\kappa(M_i) > 0$ for all $i \in \{1, \ldots, n\}$. If a discrete group Λ is ME to the direct product $\Gamma(M_1) \times \cdots \times \Gamma(M_n)$, then there exists a homomorphism $\rho\colon \Lambda \to G = \mathrm{Aut}(C(M_1)) \times \cdots \times \mathrm{Aut}(C(M_n))$ such that the kernel of ρ and the index $[G : \rho(\Lambda)]$ are both finite.*

Let (Σ, m) be an ME coupling of $\Gamma = \Gamma(M_1) \times \cdots \times \Gamma(M_n)$ and an unknown group Λ. For the proof of Theorem 8.2, we first consider a self ME coupling $\Omega = \Sigma \times_\Lambda \Lambda \times_\Lambda \check{\Sigma}$ of Γ as in the proof of Theorem 1.5. We then construct an almost $(\Gamma \times \Gamma)$-equivariant Borel map

$$\Omega \to G = \mathrm{Aut}(C(M_1)) \times \cdots \times \mathrm{Aut}(C(M_n)) \tag{8.1}$$

for some $(\Gamma \times \Gamma)$-action on G for which G is a self ME coupling of Γ (see Theorem 7.1 and Corollary 7.2 in [36] for a more explicit statement).

Monod and Shalom [48] introduced the class $\mathcal{C}$ consisting of discrete groups Δ which admit a mixing unitary representation π on a Hilbert space such that the second bounded cohomology $H^2_b(\Delta, \pi)$ of Δ with coefficient π does not vanish. They studied self ME couplings of discrete groups of the form $\Delta_1 \times \cdots \times \Delta_n$ with $\Delta_i \in \mathcal{C}$ and $n \geq 2$ via the theory of bounded cohomology. They obtained many interesting measurable rigidity results on ergodic standard actions of such product groups. Whether a discrete group is in the class $\mathcal{C}$ or not is invariant under ME, and $\mathcal{C}$ contains all non-elementary hyperbolic groups in the sense of Gromov. Hamenstädt [25] proved that the mapping class group $\Gamma(M)$ with $\kappa(M) > 0$ is in $\mathcal{C}$. We apply these results to our situation, and construct the map in (8.1).

We note that a theorem similar to Theorem 1.6 can also be shown for direct products of mapping class groups (see Theorem 1.4 in [36]).

4. Construction of non-OE actions. Let M be a surface with $\kappa(M) > 0$ and put $\Gamma = \Gamma(M)^\diamond$. In Corollary 3.12, we proved that if two ergodic standard (i.e., measure-preserving and essentially free) actions of Γ are OE, then they are conjugate. In the theory of OE, it is an interesting problem to construct (continuously) many ergodic standard actions of one specified group which are mutually non-OE. Thanks to Corollary 3.12, if we construct non-conjugate actions, then they are non-OE. In [38], we give a family of non-OE actions of Γ as shown in the following: Let $\alpha \in V(C)$ and consider its Γ-orbit $K = \Gamma\alpha$, on which Γ naturally acts. Let (X_0, μ_0) be a standard probability space, i.e., a standard Borel space with a probability measure. We assume that (X_0, μ_0) may contain atoms, whereas (X_0, μ_0) is non-trivial, i.e., it does not consist of a single atom. The generalized Bernoulli action of Γ on $(X_0, \mu_0)^K = \prod_K (X_0, \mu_0)$ is defined by

$$g(x_\beta)_{\beta \in K} = (x_{g^{-1}\beta})_{\beta \in K}, \quad (x_\beta)_{\beta \in K} \in X_0^K, \; g \in \Gamma.$$

This action is ergodic and standard. We can show that for two non-trivial standard probability spaces (X_0, μ_0) and (Y_0, ν_0), the two generalized Bernoulli actions of Γ on $(X_0, \mu_0)^K$ and $(Y_0, \nu_0)^K$ are conjugate if and only if (X_0, μ_0) and (Y_0, ν_0) are isomorphic, i.e., there exists a Borel isomorphism $f \colon X_0' \to Y_0'$ between conull Borel subsets $X_0' \subset X_0$ and $Y_0' \subset Y_0$ such that $f_*\mu_0 = \nu_0$. Hence, this example gives a family of continuously many ergodic standard actions of Γ which are mutually non-OE.

References

[1] S. Adams, Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups, *Topology* **33** (1994), 765–783.

[2] S. Adams, Indecomposability of equivalence relations generated by word hyperbolic groups, *Topology* **33** (1994), 785–798.

[3] S. Adams, G. A. Elliott, and T. Giordano, Amenable actions of groups, *Trans. Amer. Math. Soc.* **344** (1994), 803–822.

[4] C. Anantharaman-Delaroche and J. Renault, *Amenable groupoids*, Monogr. Enseign. Math., 36. Enseignement Math., Geneva, 2000.

[5] C. Anantharaman-Delaroche and J. Renault, Amenable groupoids, in *Groupoids in analysis, geometry, and physics* (Boulder, CO, 1999), 35–46, Contemp. Math., 282, Amer. Math. Soc., Providence, RI, 2001.

[6] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's property (T)*, New Math. Monogr. 11, Cambridge Univ. Press, Cambridge 2008.

[7] M. Bestvina and K. Fujiwara, Bounded cohomology of subgroups of mapping class groups, *Geom. Topol.* **6** (2002), 69–89.

[8] J. S. Birman, A. Lubotzky, and J. McCarthy, Abelian and solvable subgroups of the mapping class groups, *Duke Math. J.* **50** (1983), 1107–1120.

[9] B. H. Bowditch, Tight geodesics in the curve complexes, *Invent. Math.* **171** (2008), 281–300.

[10] A. Connes, J. Feldman, and B. Weiss, An amenable equivalence relation is generated by a single transformation, *Ergodic Theory Dynam. Systems* **1** (1981), 431–450 (1982).

[11] H. A. Dye, On groups of measure preserving transformation. I, *Amer. J. Math.* **81** (1959), 119–159.

[12] H. A. Dye, On groups of measure preserving transformations. II, *Amer. J. Math.* **85** (1963), 551–576.

[13] A. Fathi, F. Laudenbach, and V. Poénaru, *Travaux de Thurston sur les surfaces.* Séminaire Orsay, Astérisque, 66–67. Soc. Math. France, Paris, 1979.

[14] J. Feldman, C. E. Sutherland, and R. J. Zimmer, Subrelations of ergodic equivalence relations, *Ergodic Theory Dynam. Systems* **9** (1989), 239–269.

[15] J. M. G. Fell and R. S. Doran, *Representations of ∗-algebras, locally compact groups, and Banach ∗-algebraic bundles. Vol. 1. Basic representation theory of groups and algebras.* Pure and Appl. Math., 125. Academic Press, Inc., Boston, MA, 1988.

[16] A. Furman, Gromov's measure equivalence and rigidity of higher rank lattices, *Ann. of Math. (2)* **150** (1999), 1059–1081.

[17] A. Furman, Orbit equivalence rigidity, *Ann. of Math. (2)* **150** (1999), 1083–1108.

[18] A. Furman, Mostow-Margulis rigidity with locally compact targets, *Geom. Funct. Anal.* **11** (2001), 30–59.

[19] A. Furman, On Popa's cocycle superrigidity theorem, *Int. Math. Res. Not. IMRN* **2007**, no. 19, Art. ID mm073.

[20] D. Gaboriau, Invariants ℓ^2 de relations d'équivalence et de groupes, *Publ. Math. Inst. Hautes Études Sci.* No. **95** (2002), 93–150.

[21] D. Gaboriau, Examples of groups that are ME to the free group, *Ergodic Theory Dynam. Systems* **25** (2005), 1809–1827.

[22] M. Gromov, Kähler hyperbolicity and L_2-Hodge theory, *J. Differential Geom.* **33** (1991), 263–292.

[23] M. Gromov, Asymptotic invariants of infinite groups, in *Geometric group theory, Vol. 2* (Sussex, 1991), 1–295, London Math. Soc. Lecture Note Ser., 182, Cambridge Univ. Press, Cambridge, 1993.

[24] U. Hamenstädt, Train tracks and the Gromov boundary of the complex of curves, in *Spaces of Kleinian groups*, 187–207, London Math. Soc. Lecture Note Ser., 329, Cambridge Univ. Press, Cambridge, 2006.

[25] U. Hamenstädt, Bounded cohomology and isometry groups of hyperbolic spaces, *J. Eur. Math. Soc.* **10** (2008), 315–349.

[26] U. Hamenstädt, Geometry of the mapping class groups I: Boundary amenability, preprint, `math.GR/0510116`, to appear in *Invent. Math.*

[27] U. Hamenstädt, Geometry of the complex of curves and of Teichmüller space, in *Handbook of Teichmüller theory* (A. Papadopoulos, ed.), Volume I, 447–467, EMS publishing house, Zürich, 2007.

[28] W. J. Harvey, Boundary structure of the modular group, in *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference*, 245–251,

Ann. of Math. Stud., 97, Princeton Univ. Press, Princeton, N.J., 1981.

[29] N. V. Ivanov, Automorphisms of Teichmüller modular groups, in *Topology and geometry—Rohlin Seminar*, 199–270, Lecture Notes in Math., 1346, Springer, Berlin, 1988.

[30] N. V. Ivanov, *Subgroups of Teichmüller modular groups*, Transl. of Math. Monogr., 115. Amer. Math. Soc., Providence, RI, 1992.

[31] N. V. Ivanov, Automorphism of complexes of curves and of Teichmüller spaces, *Internat. Math. Res. Notices* **1997**, no. 14, 651–666.

[32] N. V. Ivanov, Mapping class groups, in *Handbook of geometric topology*, 523–633, North-Holland, Amsterdam, 2002.

[33] V. A. Kaimanovich and H. Masur, The Poisson boundary of the mapping class group, *Invent. Math.* **125** (1996), 221–264.

[34] A. S. Kechris, *Classical descriptive set theory*, Grad. Texts in Math., 156. Springer-Verlag, New York, 1995.

[35] Y. Kida, The mapping class group from the viewpoint of measure equivalence theory, *Mem. Amer. Math. Soc.* **196** (2008), no. 916.

[36] Y. Kida, Measure equivalence rigidity of the mapping class group, preprint, `math.GR/0607600`, to appear in *Ann. of Math.*

[37] Y. Kida, Orbit equivalence rigidity for ergodic actions of the mapping class group, *Geom. Dedicata* **131** (2008), 99–109.

[38] Y. Kida, Classification of certain generalized Bernoulli actions of mapping class groups, preprint.

[39] E. Klarreich, The boundary at infinity of the curve complex and the relative Teichmüller space, preprint, available at `http://nasw.org/users/klarreich/`.

[40] M. Korkmaz, Automorphisms of complexes of curves on punctured spheres and on punctured tori, *Topology Appl.* **95** (1999), 85–111.

[41] A. A. Kosinski, *Differential manifolds*, Pure Appl. Math., 138. Academic Press, Inc., Boston, MA, 1993.

[42] F. Luo, Automorphisms of the complex of curves, *Topology* **39** (2000), 283–298.

[43] H. A. Masur and Y. N. Minsky, Geometry of the complex of curves. I. Hyperbolicity, *Invent. Math.* **138** (1999), 103–149.

[44] H. A. Masur and Y. N. Minsky, Geometry of the complex of curves. II. Hierarchical structure, *Geom. Funct. Anal.* **10** (2000), 902–974.

[45] J. McCarthy and A. Papadopoulos, Dynamics on Thurston's sphere of projective measured foliations, *Comment. Math. Helv.* **64** (1989), 133–166.

[46] C. T. McMullen, The moduli space of Riemann surfaces is Kähler hyperbolic, *Ann. of Math. (2)* **151** (2000), 327–357.

[47] Y. N. Minsky, A geometric approach to the complex of curves on a surface, in *Topology and Teichmüller spaces* (Katinkulta, 1995), 149–158, World Sci. Publishing, River Edge, NJ, 1996.

[48] N. Monod and Y. Shalom, Orbit equivalence rigidity and bounded cohomology, *Ann. of Math. (2)* **164** (2006), 825–878.

[49] L. Mosher, Geometric survey of subgroups of mapping class groups, in *Handbook of Teichmüller theory* (A. Papadopoulos, ed.), Volume I, 387–410, EMS publishing house, Zürich, 2007.

[50] D. S. Ornstein, Bernoulli shifts with the same entropy are isomorphic, *Adv. Math.* **4** (1970), 337–352.

[51] D. S. Ornstein, Two Bernoulli shifts with infinite entropy are isomorphic, *Adv. Math.* **5** (1970), 339–348.

[52] D. S. Ornstein and B. Weiss, Ergodic theory of amenable group actions. I. The Rohlin lemma, *Bull. Amer. Math. Soc. (N.S.)* **2** (1980), 161–164.

[53] D. S. Ornstein and B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, *J. Analyse Math.* **48** (1987), 1–141.

[54] N. Ozawa, Amenable actions and applications, in *International Congress of Mathematicians. Vol. II*, 1563–1580, Eur. Math. Soc., Zürich, 2006.

[55] S. Popa, Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, II, *Invent. Math.* **165** (2006), 409–451.

[56] S. Popa, Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups, *Invent. Math.* **170** (2007), 243–295.

[57] K. J. Shackleton, Combinatorial rigidity in curve complexes and mapping class groups, *Pacific J. Math.* **230** (2007), 217–232.

[58] Y. Shalom, Measurable group theory, in *European Congress of Mathematics*, 391–423, Eur. Math. Soc., Zürich, 2005.

[59] Y. Takesaki, *Theory of Operator algebras. III*, Encyclopaedia of Mathematical Sciences, 127. Operator Algebras and Non-commutative Geometry, 8. Springer-Verlag, Berlin, 2003.

[60] S. Vaes, Rigidity results for Bernoulli actions and their von Neumann algebras (after Sorin Popa), *Astérisque* **311** (2007), 237–294.

[61] G. Yu, Higher index theory of elliptic operators and geometry of groups, in *International Congress of Mathematicians. Vol. II*, 1623–1639, Eur. Math. Soc., Zürich, 2006.

[62] R. J. Zimmer, Amenable ergodic group actions and an application to Poisson boundaries of random walks, *J. Funct. Anal.* **27** (1978), 350–372.

[63] R. J. Zimmer, *Ergodic theory and semisimple groups*, Monogr. Math., 81. Birkhäuser Verlag, Basel, 1984.

Index