ON MY RESEARCH WORKS

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1. A summary

My research works are concerned with group actions on measure spaces and their orbit equivalence relations. Their studies stem from studies of *von Neumann algebras*, certain rings of bounded linear operators on a Hilbert space. In fact, many interesting examples of von Neumann algebras arise from group actions on measure spaces. Historically von Neumann algebras were introduced in the context of mathematical foundation of quantum mechanics and theory of unitary representations. We mean by an *orbit equivalence relation* of a group action the equivalence relation whose equivalence class is an orbit of the action. It reflects a certain aspect of the von Neumann algebra arising from the action.

I have studied several groups originating from geometry, topology and Lie group theory (e.g., the mapping class group of a surface), and discovered remarkable *rigidity* of their actions on probability measure spaces. Some of my results contribute to solving the rigidity problem on von Neumann algebras, which asks whether a group action can be recovered from the associated von Neumann algebra. This document briefly reviews my research field and my results on rigidity.

2. Background

Unitary representation theory studies representations of groups on a Hilbert space as unitary operators. Usually questions on such unitary representations are reduced to those for irreducible representations via irreducible decomposition. However this is not necessarily the case for representations of all topological groups. Indeed according to the works of Mackey, Glimm and Dixmier around 1960, there are groups whose representations are *not* uniquely decomposed into irreducible ones, and to make matters worse, it is impossible in a certain sense to classify all their irreducible representations. Such groups are called *wild*, and otherwise groups are called *tame*. It is known that a countable discrete group is wild unless it contains an abelian subgroup of finite index. Thus most of interesting countable groups fall into being wild.

Von Neumann algebras play an essential role in understanding representations of wild groups. They can measure complexity of representations, and we can find interesting and fruitful examples of von Neumann algebras in the wild circumstance rather than the tame circumstance. Connes [4] succeeded in describing representations of *amenable* groups from the standpoint of von Neumann algebras. He showed that among wild groups, amenable groups are exactly groups such that all their representations have the simplest complexity. This result reveals significance of amenability and is still now one of the most celebrated results on representations and operator algebras of wild groups. Typical examples of amenable groups are groups built up from finite or abelian groups, such as solvable groups.

Recent studies also target non-amenable groups which are much wilder than amenable groups. Non-abelian free groups and those containing them are examples of non-amenable groups. However

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it seems hopeless to establish general understanding for non-amenable groups and their representations even through von Neumann algebras, and our interests are often focused on unexpected and surprising aspects of specific groups.

Given a countable group G and its measure-preserving action on a standard probability space X, we can construct the von Neumann algebra $G \ltimes L^\infty(X)$. This construction is known as one of standard methods to construct von Neumann algebras. The action also yields the *orbit equivalence relation*

$$\mathcal{R}_{G\curvearrowright X} = \{\, (gx, x) \in X \times X \mid g \in G,\, x \in X \,\}.$$

This reflects some aspect of the von Neumann algebra $G \ltimes L^\infty(X)$. In fact, we can canonically construct the equivalence relation $\mathcal{R}_{G\curvearrowright X}$ from the pair $L^\infty(X) \subset G \ltimes L^\infty(X)$ of the von Neumann algebras, and vice versa. We say that two actions $G \curvearrowright X$, $H \curvearrowright Y$ are *orbit equivalent* if the orbit equivalence relations $\mathcal{R}_{G\curvearrowright X}$, $\mathcal{R}_{H\curvearrowright Y}$ are isomorphic.

The initial studies of orbit equivalence were focused on actions of particular amenable groups, and finally Ornstein-Weiss [24] concluded the following remarkable "uniqueness": All ergodic free probability-measure-preserving (p.m.p.) actions of infinite amenable groups are mutually orbit equivalent. Around the same time, Zimmer [29] applied Mackey's virtual group viewpoint to the Mostow-Margulis rigidity on lattices in simple Lie groups, and extended it to the framework of measured groupoids. The result implies the following strong rigidity in orbit equivalence: For any two ergodic free p.m.p. actions of (non-compact) simple higher-rank Lie groups, if they are orbit equivalent, then they are in fact conjugate and in particular the acting Lie groups are isomorphic. This is in sharp contrast to Ornstein-Weiss' uniqueness theorem because every such Lie group admits many ergodic free p.m.p. actions which are mutually non-conjugate and therefore mutually non-orbit equivalent by Zimmer's result.

My results realize the same kind of strong rigidity for various countable discrete groups, e.g., the mapping class group of a surface. Details are discussed in the next section.

3. My results

3.1. **The mapping class group.** For a surface S, the homeomorphisms of S onto itself constitute a group under composition. Their isotopy classes also constitute a group, called the *mapping class group* of S and denoted by $\mathrm{Mod}(S)$. Remarkably this group can be identified with the (orbifold) fundamental group of the moduli space of hyperbolic structures on S, and has long been attracting researchers in geometry and topology.

The mapping class group shares various aspects with simple Lie groups and their lattices. Indeed the mapping class group of the torus is isomorphic to $SL_2(\mathbb{Z})$, and this analogy often suggests a successful approach to understanding the mapping class group. One of them is the Mostow-type rigidity due to Ivanov [10], which describes all isomorphisms between two finite index subgroups of $\mathrm{Mod}(S)$.

In [11, 12, 13], I extended this Mostow-type rigidity to the framework of measured groupoids. As well as in Zimmer's work, this implies the following strong rigidity in orbit equivalence: Suppose that the genus of S is more than or equal to 2. Then any two ergodic free p.m.p. actions of $\mathrm{Mod}(S)$ that are orbit equivalent are in fact conjugate (virtually, more precisely). This result gave the first example of a *countable* group with such rigidity in orbit equivalence.

Zimmer's rigidity also implies certain strong rigidity for actions of *lattices* in the Lie groups. This rigidity for lattices was further strengthened into "superrigidity" by Furman [5, 6]. Among others, he showed that the action $SL_3(\mathbb{Z}) \curvearrowright \mathbb{T}^3$ on the torus is *OE superrigid*, i.e., if this action is orbit equivalent to another ergodic free p.m.p. action of an *arbitrary* countable group, then those two actions are conjugate.

Applying Furman's method to my strong rigidity for $\mathrm{Mod}(S)$, I proved that every ergodic free p.m.p. action of $\mathrm{Mod}(S)$ is OE superrigid ([12, 13]). This was surprising because even higher rank lattices such as $SL_3(\mathbb{Z})$ admit actions which are *not* OE superrigid.

3.2. **Constructing groups with rigidity.** My next work was devoted to constructing countable groups satisfying the same rigidity as that for the mapping class group. In [14, 16], I showed that if we amalgamate two groups having rigidity in an appropriate way, then the resulting group also possesses rigidity. For example, let A be the subgroup of $SL_3(\mathbb{Z})$ consisting of matrices whose $(2,1)$ and $(3,1)$ entries are zero, and define the amalgamated free product $G = SL_3(\mathbb{Z}) *_A SL_3(\mathbb{Z})$. Then every ergodic free p.m.p. action of G is OE superrigid. This result enables us to produce many examples of groups with such rigidity. Moreover my proof of OE superrigidity (or rather, strong rigidity) for these amalgamated free products is much more accessible for operator algebraists than the proof for the mapping class group. This is because the former depends on Kazhdan's property of unitary representations and the Bass-Serre theory on groups acting on trees, and both topics are familiar to operator algebraists.

3.3. **Rigidity in von Neumann algebras.** For an ergodic free p.m.p. action $G \curvearrowright X$, we can construct the von Neumann algebra $G \ltimes L^\infty(X)$. The subalgebra $L^\infty(X)$ is called the *Cartan subalgebra* of $G \ltimes L^\infty(X)$. As mentioned above, there is one-to-one correspondence between the orbit equivalence relation $\mathcal{R}_{G \curvearrowright X}$ and the pair $L^\infty(X) \subset G \ltimes L^\infty(X)$. That is to say, two actions $G \curvearrowright X$, $H \curvearrowright Y$ are orbit equivalent if and only if there is an isomorphism from $G \ltimes L^\infty(X)$ onto $H \ltimes L^\infty(Y)$ preserving the Cartan subalgebras. It is classically known that an isomorphism from $G \ltimes L^\infty(X)$ onto $H \ltimes L^\infty(Y)$ does not necessarily preserve the Cartan subalgebras. However if we can show that any such isomorphism preserves the Cartan subalgebras for some OE superrigid action $G \curvearrowright X$, then isomorphism between the von Neumann algebras implies orbit equivalence between the two actions, and OE superrigidity implies conjugacy between the two actions. This means that we can "recover" the action $G \curvearrowright X$ from the von Neumann algebra $G \ltimes L^\infty(X)$. No such recovering result had been proved until recently.

Localizing technique for Cartan subalgebras has surprisingly been advanced in this decade. The first groundbreaking result is due to Popa [26]. He showed that for the action $SL_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$, any automorphism of the von Neumann algebra $SL_2(\mathbb{Z}) \ltimes L^\infty(\mathbb{T}^2)$ preserves the Cartan subalgebra (up to unitary conjugacy). Afterward much progress was made on localization of Cartan subalgebras. Ozawa-Popa [25] discovered the first example of an ergodic free p.m.p. action whose von Neumann algebra admits the only one Cartan subalgebra. Popa-Vaes [27] showed that the same uniqueness holds for some action of some amalgamated free product dealt in [14]. Combining this with the OE superrigidity result in [14], we can conclude that the action is *W^*-superrigid*, i.e., if the associated von Neumann algebra is isomorphic to the von Neumann algebra of another ergodic free p.m.p. action of a countable group, then those two actions are conjugate. This is (one of) the first explicit examples of a W^*-superrigid action. Furthermore Houdayer-Popa-Vaes [8] proved that for the group $SL_3(\mathbb{Z}) *_A SL_3(\mathbb{Z})$ mentioned in the previous subsection, every its ergodic free p.m.p. action is W^*-superrigid. This gave the first example of a countable group with such rigidity.

In the work [2] collaborated with Ionut Chifan and Adrian Ioana, we studied this problem of Cartan subalgebras for actions of the mapping class group. We proved that if S is a surface of genus at most 2 with sufficiently many boundary components, then every ergodic free p.m.p. action of the mapping class group Mod(S) is W^*-superrigid (whether the same holds for a surface of genus more than 2 is unsolved). In [3], we obtained the same rigidity for several natural subgroups of the mapping class group, such as the Torelli group, the Johnson kernel and surface braid groups. The proof uses the computation results in [15, 17, 20, 21, 22] on automorphisms of certain simplicial complexes associated to a surface, where we proved the Mostow-type rigidity for those subgroups, following Ivanov [10].

Thus we have discovered quite a number of examples of W^*-superrigid actions in this decade, which provide various non-isomorphism results for the associated von Neumann algebras.

3.4. **Unsolved problems.** My strategy toward strong rigidity in orbit equivalence was to extend the Mostow-type rigidity to the framework of measured groupoids. This suggests that if a group satisfies some sort of the Mostow-type rigidity, we can expect the group to satisfy strong rigidity in orbit equivalence. There are several groups worth of consideration, e.g., rank-one simple Lie groups and their lattices, the semidirect product group $SL_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ for $n \geq 2$, and the outer automorphism group of the free group. It is unsolved whether these groups satisfy strong rigidity in orbit equivalence (see [1] for a partial answer for some rank-one simple Lie groups).

In [18, 19], I worked on the Baumslag-Solitar groups and constructed a new invariant under orbit equivalence for their ergodic free p.m.p. actions. The groups are defined by the presentation

$$\langle\, a, t \mid ta^p t^{-1} = a^q \,\rangle$$

with two parameterizing integers p, q with $2 \leq p \leq |q|$. This work was motivated from theory of orbit equivalence for non-singular actions on measure spaces (i.e., actions which do not necessarily preserve the measure itself, but preserve measurable subsets of measure zero) and also motivated from Whyte's result in [28] saying that all Baumslag-Solitar groups with $2 \leq p < |q|$ have quasi-isometric Cayley graphs. Two countable groups are called *measure equivalent* if they admit ergodic free p.m.p. actions which are orbit equivalent. Gromov [7] observed that measure equivalence and quasi-isometry have certain similarity. Thus Whyte's result makes us expect non-trivial measure equivalence among Baumslag-Solitar groups. In [18], I showed that whether $p = |q|$ or $p < |q|$ is an invariant under measure equivalence. It is unsolved whether p and q are also invariants (see [9, 23] for related results). A new method is necessary for solving this problem, and I hope it will lead to a new stage in studies of orbit equivalence.

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