# THE MODULAR COCYCLE FROM COMMENSURATION AND ITS MACKEY RANGE

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## 1. Introduction

Let $\Gamma$ be a group. A subgroup $E$ of $\Gamma$ is called *quasi-normal* (or *commensurated*) in $\Gamma$ if for any $\gamma \in \Gamma$, the group $E \cap \gamma E \gamma^{-1}$ is of finite index in both $E$ and $\gamma E \gamma^{-1}$. In this case, the *modular homomorphism* $\mathfrak{m} \colon \Gamma \to \mathbb{R}_+^\times$ into the multiplicative group $\mathbb{R}_+^\times$ of positive real numbers is defined by the formula

$$\mathfrak{m}(\gamma) = [E : E \cap \gamma E \gamma^{-1}][\gamma E \gamma^{-1} : E \cap \gamma E \gamma^{-1}]^{-1}$$

for $\gamma \in \Gamma$. This $\mathfrak{m}$ depends only on the commensurability class of $E$, where two subgroups of $\Gamma$ are called *commensurable* if their intersection is of finite index in both of them. If that class is characteristic in $\Gamma$, then $\mathfrak{m}$ is invariant under any automorphism of $\Gamma$, and we can derive valuable information on $\Gamma$ from $\mathfrak{m}$. With regard to the *Baumslag-Solitar (BS) group* defined by the presentation

$$\mathrm{BS}(p, q) = \langle\, a,\, t \mid t a^p t^{-1} = a^q \,\rangle,$$

where $p$ and $q$ are integers with $2 \leq p \leq |q|$, the modular homomorphism $\mathfrak{m}$ is associated to the quasi-normal subgroup $\langle a \rangle$, and it turns out that the image of $\mathfrak{m}$ and hence the ratio $|q/p|$ is an isomorphism invariant among the BS groups. In [Ki2, Theorem 1.2], we realized this for transformation-groupoids from the BS groups. Namely, to the pair of a discrete measured groupoid and its quasi-normal subgroupoid, the modular cocycle is associated, and among transformation-groupoids from the BS groups, its Mackey range is shown to be an isomorphism invariant of the groupoid. This work was inspired by construction of the flow associated with type III transformations and invariance of its isomorphism class under orbit equivalence ([HOO], [Kr1] and [Kr2]). The aim of this note is to review the result in [Ki2] and provide new examples of groups to which it is applicable. We refer to [HR], [M] and [MV] for other related results on the BS groups and modular invariants.

Throughout this note, unless otherwise stated, all relations among measurable sets and maps are understood to hold up to sets of measure zero, and $(X, \mu)$ denotes a standard Borel space with a probability measure.

## 2. The modular cocycle

2.1. **Quasi-normal subgroupoids.** Normal subgroupoids generalize normal subgroups and normal sub-equivalence relations ([FSZ]). Quasi-normal subgroupoids further generalize quasi-normal subgroups. Let $\mathcal{G}$ be a p.m.p. discrete measured groupoid on $(X, \mu)$, where "p.m.p." stands for "probability-measure-preserving". Let $r, s \colon \mathcal{G} \to X$ be the range and source maps of $\mathcal{G}$, respectively. For $x, y \in X$, let $\mathcal{G}_x^y$ denote the set of elements of $\mathcal{G}$ whose range is $y$ and source is $x$. For a *non-negligible* subset $A$ of $X$, i.e., a measurable subset of $X$ with positive measure, we denote by $\mathcal{G}|_A := \{\, g \in \mathcal{G} \mid r(g),\, s(g) \in A \,\}$ the restriction of $\mathcal{G}$ to $A$, which is a discrete measured groupoid on $(A, \mu|_A)$. We define $[[\mathcal{G}]]$

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as the pseudo-group of partial isomorphisms from $\mathcal{G}$ into itself, that is, measurable maps $\phi\colon D_\phi \to \mathcal{G}$ from a measurable subset $D_\phi$ of $X$ such that $\phi$ is a section of the source map $s$ and the composed map $\phi^\circ := r \circ \phi\colon D_\phi \to X$ is injective. For $\phi \in [[\mathcal{G}]]$, let $R_\phi$ denote the range of the map $\phi^\circ$. The isomorphism $U_\phi\colon \mathcal{G}|_{D_\phi} \to \mathcal{G}|_{R_\phi}$ is defined by conjugating $\phi$, that is, defined by the formula $U_\phi(g) = \phi(y)g\phi(x)^{-1}$ for $g \in \mathcal{G}_x^y$.

Let $\mathcal{S}$ be a subgroupoid of $\mathcal{G}$. For $x \in X$, we define an equivalence relation on the set $s^{-1}(x)$ by declaring that two elements $g$, $h$ of $s^{-1}(x)$ are equivalent if $gh^{-1} \in \mathcal{S}$. We define the *index* of $\mathcal{S}$ in $\mathcal{G}$ at $x$, denoted by $[\,\mathcal{G} : \mathcal{S}\,]_x$, as the number of equivalence classes in $s^{-1}(x)$. We say that $\mathcal{S}$ is of *finite index* in $\mathcal{G}$ if the index $[\,\mathcal{G} : \mathcal{S}\,]_x$ is finite for any $x \in X$. We define $\mathrm{QN}_{\mathcal{G}}(\mathcal{S})$, the set of quasi-normalizers of $\mathcal{S}$ in $\mathcal{G}$, as the set of elements $\phi$ of $[[\mathcal{G}]]$ such that for any $x \in D_\phi$, the indexes

$$[\,\mathcal{S}|_{D_\phi} : \mathcal{S}|_{D_\phi} \cap U_\phi^{-1}(\mathcal{S}|_{R_\phi})\,]_x \quad \text{and} \quad [\,\mathcal{S}|_{R_\phi} : \mathcal{S}|_{R_\phi} \cap U_\phi(\mathcal{S}|_{D_\phi})\,]_{\phi^\circ(x)}$$

are both finite. We call $\mathcal{S}$ *quasi-normal* in $\mathcal{G}$ if $\mathrm{QN}_{\mathcal{G}}(\mathcal{S})$ generates $\mathcal{G}$. This property is stable under finite-index perturbation and restriction of the unit space. Namely,

(i) for two subgroupoids $\mathcal{S}$, $\mathcal{T}$ of $\mathcal{G}$ such that $\mathcal{S}$ is contained in $\mathcal{T}$ and of finite index in $\mathcal{T}$, $\mathcal{S}$ is quasi-normal in $\mathcal{G}$ if and only if so is $\mathcal{T}$ ([Ki2, Lemma 3.19]), and

(ii) for any non-negligible subset $A$ of $X$, if $\mathcal{S}$ is quasi-normal in $\mathcal{G}$, then $\mathcal{S}|_A$ is quasi-normal in $\mathcal{G}|_A$ ([Ki2, Lemma 3.18]).

In Subsections 2.2 and 2.3, to the pair of a groupoid $\mathcal{G}$ and its quasi-normal subgroupoid $\mathcal{S}$, the two cocycle $\mathfrak{D}, \mathfrak{I}\colon \mathcal{G} \to \mathbb{R}_+^\times$ are associated. For each $\phi \in [[\mathcal{G}]]$, setting $U = U_\phi$, $D = D_\phi$ and $R = R_\phi$ and taking an $x \in D_\phi$, the numbers $\mathfrak{D}(\phi(x))$, $\mathfrak{I}(\phi(x))$ are defined by measuring difference between the two groupoids $\mathcal{S}_- = \mathcal{S}|_D \cap U^{-1}(\mathcal{S}|_R)$ and $\mathcal{S}_+ = \mathcal{S} \cap U(\mathcal{S}|_D)$ in two different ways. The number $\mathfrak{D}(\phi(x))$ describes distortion under conjugating $\phi$ between the $\mathcal{S}_-$-ergodic probability measure on the component containing $x$ and the $\mathcal{S}_+$-ergodic probability measure on the component containing $\phi^\circ(x)$. The number $\mathfrak{I}(\phi(x))$ is defined as the ratio of indexes of some subgroupoids from $\mathcal{S}_-$ and $\mathcal{S}_+$, similarly to the modular homomorphism $\mathfrak{m}$. In fact, when $\mathcal{G}$ is a group, the cocycle $\mathfrak{I}$ coincides with $\mathfrak{m}$ (and the cocycle $\mathfrak{D}$ is trivial). Toward the definition of $\mathfrak{I}$, instead of the usual index, we introduce the local index of a subgroupoid, which holds stability under restriction of the unit space. The two cocycles $\mathfrak{D}$, $\mathfrak{I}$ are canonical, and their cohomology classes are invariant under finite-index perturbation of $\mathcal{S}$ and restriction of the unit space.

2.2. **The Radon-Nikodym cocycle.** Let $\mathcal{G}$ be a p.m.p. discrete measured groupoid on $(X,\mu)$ and $\mathcal{S}$ a quasi-normal subgroupoid of $\mathcal{G}$. We have the ergodic decomposition for $\mathcal{S}$ described by a measure-preserving map from $(X,\mu)$ into a standard probability space, $\pi\colon (X,\mu) \to (Z,\xi)$. Let $\mu = \int_Z \mu_z \, d\xi(z)$ be the disintegration of $\mu$ with respect to $\pi$. For $z \in Z$, we set $X_z = \pi^{-1}(z)$ and denote by $\mathcal{S}_z$ the restriction of $\mathcal{S}$ to the probability space $(X_z, \mu_z)$, which is an ergodic p.m.p. discrete measured groupoid.

To explain the idea of defining the Radon-Nikodym cocycle $\mathfrak{D} = \mathfrak{D}(\mathcal{G},\mathcal{S})\colon \mathcal{G} \to \mathbb{R}_+^\times$, let us start with an element $\phi$ of $[[\mathcal{G}]]$ whose domain is the whole space $X$. We set $U = U_\phi$, $\mathcal{S}_- = \mathcal{S} \cap U^{-1}(\mathcal{S})$ and $\mathcal{S}_+ = \mathcal{S} \cap U(\mathcal{S})$. We then have $U(\mathcal{S}_-) = \mathcal{S}_+$. Pick $x \in X$ and set $y = \phi^\circ(x)$. Because $\mathcal{S}_-$ is of finite index in $\mathcal{S}$, the $\mathcal{S}$-ergodic component $X_{\pi(x)}$ contains only finitely many $\mathcal{S}_-$-ergodic components. Let $\Sigma$ denote the $\mathcal{S}_-$-ergodic component containing $x$. The restriction $\mu_{\pi(x)}|_\Sigma$ is a constant multiple of the $\mathcal{S}_-$-ergodic probability measure on $\Sigma$, and the constant is equal to the measure $\mu_{\pi(x)}(\Sigma)$. The same thing holds for the $\mathcal{S}_+$-ergodic component containing $y$, denoted by $\Omega$. On the other hand, being an isomorphism from $\mathcal{S}_-$ onto $\mathcal{S}_+$, by uniqueness of the ergodic decomposition, $U$ sends the $\mathcal{S}_-$-ergodic probability measure on $\Sigma$ to the $\mathcal{S}_+$-ergodic probability measure on $\Omega$. It turns out that $U$ sends the measure $\mu_{\pi(x)}|_\Sigma$ to a constant multiple of the measure $\mu_{\pi(y)}|_\Omega$. The constant is equal to $\mu_{\pi(x)}(\Sigma)/\mu_{\pi(y)}(\Omega)$ and describes distortion between the ergodic measures for $\mathcal{S}$ induced by the conjugation of $\phi$. We want to define it as $\mathfrak{D}(\phi(x))$.

Here is a problem. We have to show that for almost every $x \in X$, this constant depends only on $\phi(x)$, not on $\phi$. To solve it, let us consider the same thing for a general element $\phi$ of $[[\mathcal{G}]]$. We fix the notation in the same manner: We set $D = D_\phi$, $R = R_\phi$, $U = U_\phi$, $\mathcal{S}_- = \mathcal{S}|_D \cap U^{-1}(\mathcal{S}|_R)$ and $\mathcal{S}_+ = \mathcal{S}_R \cap U(\mathcal{S}|_D)$. The rest of the process is the same as that in the previous paragraph. Thus for $x \in D$, setting $y = \phi^\circ(x)$ and denoting $\Sigma$ the $\mathcal{S}_-$-ergodic component containing $x$ and $\Omega$ the $\mathcal{S}_+$-ergodic component containing $y$, we obtain that $U$ sends the measure $\mu_{\pi(x)}|_\Sigma$ to a constant multiple of the measure $\mu_{\pi(y)}|_\Omega$, and the constant is equal to $\mu_{\pi(x)}(\Sigma)/\mu_{\pi(y)}(\Omega)$. Let us denote this constant by $\mathfrak{D}(\phi, x)$. Its crucial property is that for any $\phi \in [[\mathcal{G}]]$ and any non-negligible subset $A$ of $D_\phi$, the equation $\mathfrak{D}(\phi|_A, x) = \mathfrak{D}(\phi, x)$ holds for almost every $x \in A$. It turns out that for almost every $x \in D_\phi$, the constant $\mathfrak{D}(\phi, x)$ depends only on $\phi(x)$ and not on $\phi$. We hence obtain the map $\mathfrak{D}\colon \mathcal{G} \to \mathbb{R}_+^\times$ associated with the quasi-normal subgroupoid $\mathcal{S}$. The map $\mathfrak{D}$ is indeed a cocycle. We call this cocycle $\mathfrak{D} = \mathfrak{D}(\mathcal{G}, \mathcal{S})$ the *Radon-Nikodym cocycle* for the pair of $\mathcal{G}$ and $\mathcal{S}$. We refer to [Ki2, Subsection 6.1] for a more precise account and the proof of the above claims.

*Example* 2.1. Let $\Gamma = \mathrm{BS}(2,3) = \langle\, a,\, t \mid ta^2t^{-1} = a^3 \,\rangle$ be the BS group and set $E = \langle a \rangle$ and $E^n = \langle a^n \rangle$ for an integer $n$. Let $\Gamma \curvearrowright (X, \mu)$ be an ergodic p.m.p. action such that there are $E$-equivariant maps from $X$ into $E/E^2$ and from $X$ into $E/E^3$, or equivalently, each $E$-ergodic component contains exactly two $E^2$-ergodic components and exactly three $E^3$-ergodic components. We denote by $\mathcal{G} = \Gamma \ltimes X$ the *transformation-groupoid* associated with the action $\Gamma \curvearrowright (X, \mu)$. This is the discrete measured groupoid on $(X, \mu)$ consisting of the pair $(\gamma, x)$ of $\gamma \in \Gamma$ and $x \in X$ such that the range and source of $(\gamma, x)$ are $\gamma x$ and $x$, respectively, and the product of two elements $(\gamma_1, \gamma_2 x)$, $(\gamma_2, x)$ is $(\gamma_1\gamma_2, x)$. We set $\mathcal{E} = E \ltimes X$ and $\mathfrak{D} = \mathfrak{D}(\mathcal{G}, \mathcal{E})$. Then for any $\gamma \in \Gamma$ and $x \in X$, the equation $\mathfrak{D}(\gamma, x) = \mathfrak{m}(\gamma)$ holds, where $\mathfrak{m}\colon \Gamma \to \mathbb{R}_+^\times$ is the homomorphism defined by $\mathfrak{m}(a) = 1$ and $\mathfrak{m}(t) = 3/2$ and is in fact the modular homomorphism associated with $E$.

This claim is proved as follows: It suffices to show the desired equation only when $g = a$ and $t$. Let $\pi\colon (X, \mu) \to (Z, \xi)$ be the ergodic decomposition for $\mathcal{E}$ and let $\mu = \int_Z \mu_z \, d\xi(z)$ be the disintegration with respect to $\pi$. We set $\mathcal{E}_- = \mathcal{E} \cap U_g^{-1}(\mathcal{E})$ and $\mathcal{E}_+ = \mathcal{E} \cap U_g(\mathcal{E})$. Pick $x \in X$. Let $\Sigma$ be the $\mathcal{E}_-$-ergodic component containing $x$ and let $\Omega$ be the $\mathcal{E}_+$-ergodic component containing $gx$. If $g = a$, then $\mathcal{E} = \mathcal{E}_- = \mathcal{E}_+$ and $\mu_{\pi(x)}(\Sigma) = 1 = \mu_{\pi(ax)}(\Omega)$. The equation $\mathfrak{D}(a, x) = 1 = \mathfrak{m}(a)$ hence holds. If $g = t$, then $\mathcal{E}_- = E^2 \ltimes X$ and $\mathcal{E}_+ = E^3 \ltimes X$. By the assumption on the action $\Gamma \curvearrowright (X, \mu)$, we have $\mu_{\pi(x)}(\Sigma) = 1/2$ and $\mu_{\pi(tx)}(\Omega) = 1/3$. It follows that $\mathfrak{D}(t, x) = 3/2 = \mathfrak{m}(t)$.

The Radon-Nikodym cocycle for the pair of $\mathcal{G}$ and $\mathcal{S}$ is canonical, and its cohomology class is invariant under finite-index perturbation of $\mathcal{S}$ and restriction of the unit space:

**Proposition 2.2** ([Ki2, Lemmas 6.3 and 6.4])**.** *Let $\mathcal{G}$ be a p.m.p. discrete measured groupoid on $(X, \mu)$ and $\mathcal{S}$ a quasi-normal subgroupoid of $\mathcal{G}$. Then*

(i) *for any finite index subgroupoid $\mathcal{T}$ of $\mathcal{S}$, the two cocycles $\mathfrak{D}(\mathcal{G}, \mathcal{T})$, $\mathfrak{D}(\mathcal{G}, \mathcal{S})$ are equivalent, and*

(ii) *for any non-negligible subset $A$ of $X$, the cocycle $\mathfrak{D}(\mathcal{G}|_A, \mathcal{S}|_A)$ and the restriction of the cocycle $\mathfrak{D}(\mathcal{G}, \mathcal{S})$ to $\mathcal{G}|_A$ are equivalent.*

2.3. **The local-index cocycle.** Let $\mathcal{G}$ be a p.m.p. discrete measured groupoid on $(X, \mu)$. For a subgroupoid $\mathcal{H}$ of $\mathcal{G}$, we introduce its local index in $\mathcal{G}$ under the assumption that each $\mathcal{G}$-ergodic component contains only countably many $\mathcal{H}$-ergodic components. Let $\pi\colon X \to Z$ be the ergodic decomposition for $\mathcal{G}$ and let $\pi_{\mathcal{H}}\colon X \to W$ be the ergodic decomposition for $\mathcal{H}$. We then have the measure-preserving map $\sigma\colon W \to Z$ with $\pi = \sigma \circ \pi_{\mathcal{H}}$. The above assumption says that each fiber of the map $\sigma$ is at most countable. This is the case if $\mathcal{H}$ is of finite index in $\mathcal{G}$.

To explain what the local index is in a simpler case, we first suppose that $\mathcal{G}$ is ergodic and $\mathcal{H}$ is of finite index in $\mathcal{G}$. For any non-negligible subset $A$ of $X$, the index $[\,\mathcal{G}|_A : \mathcal{H}|_A\,]_x$ is then independent of $x$ and we denote it by $I(A)$. If we pass to a non-negligible subset of $X$, then the index does not increase and may strictly decrease. For example, if $\mathcal{H}$ is the disjoint union of $\mathcal{G}|_A$ and $\mathcal{G}|_{X\setminus A}$, then $I(A) = 1$ and $I(X) = 2$. On the contrary, if $\mathcal{H}$ is ergodic, then this decreasing does not occur, that is, for any non-negligible subset $A$ of $X$, we have $I(A) = I(X)$. It turns out that for a general finite-index subgroupoid $\mathcal{H}$, there is a partition of $X$ into finitely many non-negligible subsets such that on each piece, the index is stable under passing to its non-negligible subset. This stable value is defined as the local index of $\mathcal{H}$ in $\mathcal{G}$ at a point of the piece.

Let us define the local index for a general $\mathcal{G}$ and its subgroupoid $\mathcal{H}$ under the assumption in the first paragraph of this subsection. Recall that we have the maps $\pi$, $\pi_{\mathcal{H}}$ and $\sigma$. For $z \in Z$, let $\mathcal{G}_z$ be the ergodic groupoid $\mathcal{G}$ restricted to the fiber $X_z := \pi^{-1}(z)$. Similarly, for $w \in W$, let $\mathcal{H}_w$ be the ergodic groupoid $\mathcal{H}$ restricted to the fiber $X_w := \pi_{\mathcal{H}}^{-1}(w)$. For each $z \in Z$, the fiber $X_z$ is decomposed into countably many fibers of $\pi_{\mathcal{H}}$, and each of them is non-negligible with respect to the measure $\mu_z$. As observed in the previous paragraph, for each $w \in W$, the index of $\mathcal{H}_w$ in $\mathcal{G}_{\sigma(w)}|_{X_w}$ is stable under passing to a non-negligible subset of $X_w$. This stable value is defined as the *local index* of $\mathcal{H}$ in $\mathcal{G}$ at a point of $X_w$. For any $x \in X_w$, the value is denoted by $[[\,\mathcal{G} : \mathcal{H}\,]]_x$, while it depends only on the set $X_w$ and is independent of a point $x \in X_w$.

The local index has the advantage of satisfying the following formulas, which do not hold for the index in general:

(i) ([Ki2, Lemma 3.13]). The local index is stable under passing to a non-negligible subset of $X$. Namely, for any non-negligible subset $A$ of $X$ and any $x \in A$,
$$[[\,\mathcal{G}|_A : \mathcal{H}|_A\,]]_x = [[\,\mathcal{G} : \mathcal{H}\,]]_x.$$

(ii) ([Ki2, Lemma 3.12]). If $\mathcal{K}$ is an intermediate subgroupoid between $\mathcal{H}$ and $\mathcal{G}$, then for any $x \in X$,
$$[[\,\mathcal{G} : \mathcal{H}\,]]_x = [[\,\mathcal{G} : \mathcal{K}\,]]_x [[\,\mathcal{K} : \mathcal{H}\,]]_x.$$

The following computes the local index for transformation-groupoids, refining [Ki2, Lemma 3.14]:

**Proposition 2.3.** *Let $\Gamma$ be a countable group and $\Lambda$ a finite index subgroup of $\Gamma$. Let $\Gamma \curvearrowright (X,\mu)$ be a p.m.p. action and set $\mathcal{G} = \Gamma \ltimes X$ and $\mathcal{H} = \Lambda \ltimes X$. Let $\pi\colon X \to Z$ be the ergodic decomposition for $\mathcal{G}$, and for $z \in Z$, let $\mu_z$ be the $\mathcal{G}$-ergodic probability measure on the fiber $X_z := \pi^{-1}(z)$. Let $\pi_\Lambda\colon X \to W$ be the ergodic decomposition for $\mathcal{H}$. Then for any $x \in X$,*
$$[[\,\mathcal{G} : \mathcal{H}\,]]_x = [\Gamma : \Lambda]\mu_{\pi(x)}(X_{\pi_\Lambda(x)}).$$

*Proof.* Let $N$ be a finite-index normal subgroup of $\Gamma$ contained in $\Lambda$. For example, let $N$ be the intersection $\bigcap_{\gamma\in\Gamma} \gamma\Lambda\gamma^{-1}$. Set $\mathcal{N} = N \ltimes X$ and let $\pi_N\colon X \to V$ be the ergodic decomposition for $\mathcal{N}$. We have the measure-preserving maps $\theta\colon V \to W$ and $\sigma\colon W \to Z$ such that $\pi_\Lambda = \theta \circ \pi_N$ and $\pi = \sigma \circ \pi_\Lambda$. Because $N$ is normal in $\Gamma$, for any $z \in Z$, the group $\Gamma/N$ transitively acts on the fiber $V_z := (\sigma\circ\theta)^{-1}(z)$, and for any $x \in X_z$, the local index of $\mathcal{N}$ in $\mathcal{G}$ at $x$ is equal to the index of $N$ in the stabilizer of a point of the fiber $V_z$. It follows that for any $x \in X_z$, we have $[[\,\mathcal{G} : \mathcal{N}\,]]_x = [\Gamma : N]|V_z|^{-1}$. By the same reason, for any $w \in W$ and $x \in X_w$, setting $V_w := \theta^{-1}(w)$, we have $[[\,\mathcal{H} : \mathcal{N}\,]]_x = [\Lambda : N]|V_w|^{-1}$.

Combining the obtained equations, for any $x \in X$, we have

$$[[\mathcal{G} : \mathcal{H}]]_x = [[\mathcal{G} : \mathcal{N}]]_x [[\mathcal{H} : \mathcal{N}]]_x^{-1} = \frac{[\Gamma : N]}{|V_{\pi(x)}|} \frac{|V_{\pi_\Lambda(x)}|}{[\Lambda : N]}$$
$$= [\Gamma : \Lambda] \frac{\text{the number of } N\text{-ergodic components in } \pi_\Lambda(x)}{\text{the number of } N\text{-ergodic components in } \pi(x)},$$

and this concludes the proposition. □

We are now in a position to introduce the local-index cocycle. Let $\mathcal{G}$ be a p.m.p. discrete measured groupoid on $(X, \mu)$ and $\mathcal{S}$ a quasi-normal subgroupoid of $\mathcal{G}$. For $g \in \mathcal{G}_x^y$ with $x, y \in X$, taking an element $\phi$ of $[[\mathcal{G}]]$ with $\phi(x) = g$, we define a value $\mathfrak{I}(g) \in \mathbb{R}_+^\times$ by

$$\mathfrak{I}(g) = [[\mathcal{S}|_{R_\phi} : \mathcal{S} \cap U_\phi(\mathcal{S})]]_y \, [[\mathcal{S}|_{D_\phi} : \mathcal{S} \cap U_\phi^{-1}(\mathcal{S})]]_x^{-1}.$$

It follows from assertion (i) right before Proposition 2.3 that this value does not depend on the choice of $\phi$ and is hence well-defined. Thanks to assertion (ii), we can show that the map $\mathfrak{I}\colon \mathcal{G} \to \mathbb{R}_+^\times$ is a cocycle, following the proof that the modular homomorphism for a quasi-normal subgroup is in fact a homomorphism ([Ki2, Lemma 6.5]). We call this cocycle $\mathfrak{I} = \mathfrak{I}(\mathcal{G}, \mathcal{S})$ the *local-index cocycle* for the pair of $\mathcal{G}$ and $\mathcal{S}$. The stability of its cohomology class holds as well as the Radon-Nikodym cocycle:

**Proposition 2.4** ([Ki2, Lemma 6.6])**.** *Let $\mathcal{G}$ be a p.m.p. discrete measured groupoid on $(X, \mu)$ and $\mathcal{S}$ a quasi-normal subgroupoid of $\mathcal{G}$. Then*

(i) *for any finite index subgroupoid $\mathcal{T}$ of $\mathcal{S}$, the two cocycles $\mathfrak{I}(\mathcal{G}, \mathcal{T})$, $\mathfrak{I}(\mathcal{G}, \mathcal{S})$ are equivalent, and*

(ii) *for any non-negligible subset $A$ of $X$, the cocycle $\mathfrak{I}(\mathcal{G}|_A, \mathcal{S}|_A)$ coincides with the restriction of the cocycle $\mathfrak{I}(\mathcal{G}, \mathcal{S})$ to $\mathcal{G}|_A$.*

2.4. **Computation.** Let $\mathcal{G}$ be a p.m.p. discrete measured groupoid on $(X, \mu)$ and $\mathcal{S}$ a quasi-normal subgroupoid of $\mathcal{G}$. We define the *modular cocycle* $\delta = \delta(\mathcal{G}, \mathcal{S})\colon \mathcal{G} \to \mathbb{R}_+^\times$ for the pair of $\mathcal{G}$ and $\mathcal{S}$ by setting $\delta(g) = \mathfrak{D}(g)\mathfrak{I}(g)$ for $g \in \mathcal{G}$. We compute this when $\mathcal{G}$ and $\mathcal{S}$ arise from transformation-groupoids, refining [Ki2, Lemma 6.7]:

**Proposition 2.5.** *Let $\Gamma$ be a countable group and $E$ a quasi-normal subgroup of $\Gamma$. Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action and set $\mathcal{G} = \Gamma \ltimes X$ and $\mathcal{E} = E \ltimes X$. Then the modular cocycle $\delta\colon \mathcal{G} \to \mathbb{R}_+^\times$ associated with $\mathcal{E}$ is equal to the modular homomorphism $\mathfrak{m}\colon G \to \mathbb{R}_+^\times$ associated with $E$ in the sense that the equation $\delta(\gamma, x) = \mathfrak{m}(\gamma)$ holds for any $\gamma \in \Gamma$ and $x \in X$.*

*Proof.* Pick $\gamma \in \Gamma$ and set $E_- = E \cap \gamma^{-1} E \gamma$ and $E_+ = E \cap \gamma E \gamma^{-1}$. We identify $\gamma$ with the element of $[[\mathcal{G}]]$ defined by the map sending $x \in X$ to $(\gamma, x) \in \mathcal{G}$. Set $\mathcal{E}_- = \mathcal{E} \cap U_\gamma^{-1}(\mathcal{E})$ and $\mathcal{E}_+ = \mathcal{E} \cap U_\gamma(\mathcal{E})$. We then have $\mathcal{E}_- = E_- \ltimes X$ and $\mathcal{E}_+ = E_+ \ltimes X$. Let $\pi$, $\pi_-$ and $\pi_+$ be the ergodic decompositions for $\mathcal{E}$, $\mathcal{E}_-$ and $\mathcal{E}_+$, respectively. Applying Proposition 2.3, we obtain the desired equation as follows: For any $x \in X$,

$$\begin{aligned} \mathfrak{I}(\gamma, x) &= [[\mathcal{E} : \mathcal{E}_+]]_{\gamma x} \, [[\mathcal{E} : \mathcal{E}_-]]_x^{-1} \\ &= [E : E_+] \mu_{\pi(\gamma x)}(X_{\pi_+(\gamma x)}) ([E : E_-] \mu_{\pi(x)}(X_{\pi_-(x)}))^{-1} \\ &= \mathfrak{m}(\gamma) \mathfrak{D}(\gamma, x)^{-1}. \end{aligned}$$

□

## 3. Localizing quasi-normal subgroupoids

In this section, we utilize the modular cocycle along the following scheme: Suppose that we aim to distinguish discrete measured groupoids in a certain class $\mathcal{C}$, and to each of them, its quasi-normal subgroupoid is attached. Suppose also that the attached subgroupoid is characteristic in the following sense: For any isomorphism $f$ between two groupoids $\mathcal{G}$, $\mathcal{H}$ in $\mathcal{C}$, if we denote by $\mathcal{E}$ and $\mathcal{F}$ the subgroupoids attached to $\mathcal{G}$ and $\mathcal{H}$, respectively, then $\mathcal{E}$ and $f^{-1}(\mathcal{F})$ are commensurable, i.e., $\mathcal{E} \cap f^{-1}(\mathcal{F})$ is of finite index in both $\mathcal{E}$ and $f^{-1}(\mathcal{F})$. Let $\delta_{\mathcal{G}}$ and $\delta_{\mathcal{H}}$ be the modular cocycles associated with the attached subgroupoids $\mathcal{E}$, $\mathcal{F}$, respectively. The cocycle $\delta_{\mathcal{H}} \circ f$ then coincides with the modular cocycle for the pair of $\mathcal{G}$ and $f^{-1}(\mathcal{F})$. By Propositions 2.2 and 2.4, the cocycles $\delta_{\mathcal{G}}$ and $\delta_{\mathcal{H}} \circ f$ are equivalent and their Mackey ranges are isomorphic. It turns out that the Mackey ranges of $\delta_{\mathcal{G}}$ and $\delta_{\mathcal{H}}$ are isomorphic and therefore the Mackey range of $\delta_{\mathcal{G}}$ is an invariant under isomorphism among groupoids in $\mathcal{C}$. In [Ki2], we carried out this scheme for transformation-groupoids from the BS groups. We here introduce a broader class of groups, including generalized BS groups, and apply the scheme to those groups. The main task is to localize the attached quasi-normal subgroupoid under isomorphism of groupoids (Theorem 3.2).

**The class of groups under consideration.** Let us consider a graph of groups satisfying the following four conditions:

(1) the numbers of vertices and edges are finite,
(2) for any two (possibly the same) vertices $u$, $v$ connected by an edge, the edge group is of finite index in the vertex group of $u$ and in that of $v$,
(3) any vertex group is amenable, and
(4) its fundamental group is non-amenable.

Let $\Gamma$ be the fundamental group. Let $T$ be the associated Bass-Serre tree, on which $\Gamma$ acts by simplicial automorphisms, and let $V(T)$ be the set of vertices of $T$. Any vertex group is quasi-normal in $\Gamma$ and any two of them are commensurable in $\Gamma$. We hence have the modular homomorphism $\mathfrak{m}\colon \Gamma \to \mathbb{R}_+^\times$ associated to any or some vertex group.

We mean by generalized BS groups the fundamental group of a graph of groups such that the numbers of vertices and edges are finite, and any vertex group and any edge group are isomorphic to $\mathbb{Z}$. Any non-amenable, generalized BS group is an example of the group $\Gamma$.

Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action and set $\mathcal{G} = \Gamma \ltimes X$. Let $A$ be a non-negligible subset of $X$. We say that a subgroupoid $\mathcal{E}$ of $\mathcal{G}|_A$ is *elliptic* if there exists an $\mathcal{E}$-invariant map into $V(T)$, i.e., a measurable map $\varphi\colon A \to V(T)$ with $\varphi(\gamma x) = \gamma\varphi(x)$ for any $(\gamma, x) \in \mathcal{E}$. We say that a discrete measured groupoid on $(X, \mu)$ is *nowhere amenable* if its restriction to any non-negligible subset of $X$ is not amenable.

**Lemma 3.1.** *With the above notation, let $\mathcal{S}$ and $\mathcal{T}$ be subgroupoids of $\mathcal{G}|_A$ such that $\mathcal{S}$ is amenable and quasi-normal in $\mathcal{T}$, and $\mathcal{T}$ is nowhere amenable. Then $\mathcal{S}$ is elliptic.*

*Proof.* We give only a sketch and refer to [Ki2, Theorem 5.1] for a precise account. Suppose that $\mathcal{S}$ is not elliptic. We can then find a non-negligible subset $B$ of $X$ such that there is no $\mathcal{S}$-invariant map from any non-negligible subset of $B$ into $V(T)$. Because $\mathcal{S}$ is amenable, there exists an $\mathcal{S}$-invariant map from $A$ into $M(\partial T)$, the space of probability measures on the boundary $\partial T$ of $T$. It follows from nowhere ellipticity of $\mathcal{S}$ on $B$ and hyperbolicity of $T$ that for any $\mathcal{S}$-invariant map $\varphi\colon B_1 \to M(\partial T)$, where $B_1$ is a non-negligible subset of $B$, for almost every $x \in B_1$, the support of the measure $\varphi(x)$ consists of at most two points. Let $\partial_2 T$ denote the quotient space of $\partial T \times \partial T$ with respect to the action of the symmetric group of two letters that exchanges the two coordinates. This space is naturally identified with the set of non-empty subsets of $\partial T$ consisting of at most two points. It turns out that there exists an $\mathcal{S}$-invariant map $\varphi_0\colon B \to \partial_2 T$ which is maximal in the sense that the measure of the set $\{\, x \in B \mid |\operatorname{supp}\varphi_0(x)| = 2 \,\}$ is maximal among $\mathcal{S}$-invariant maps from $B$ into $\partial_2 T$. The map $\varphi_0$ is therefore canonical and is in fact shown to be invariant under any element of $\mathrm{QN}_{\mathcal{G}}(\mathcal{S})$. In particular, $\varphi_0$ is $\mathcal{T}$-invariant. On the other hand, if the space

$\partial_2 T$ is equipped with any probability measure quasi-invariant under the action $\Gamma \curvearrowright \partial_2 T$, then the action is amenable in the sense of Zimmer ([Ki1, Corollary 3.4]). This implies that the restriction $\mathcal{T}|_B$ is amenable, which contradicts our assumption. □

Let $\Lambda$ be another group of the same kind as $\Gamma$. Namely, let $\Lambda$ be the fundamental group of a graph of groups satisfying conditions (1)–(4) required for the graph of groups for $\Gamma$. Let $\mathfrak{n}\colon \Lambda \to \mathbb{R}_+^\times$ be the modular homomorphism associated to a vertex group of $\Lambda$.

**Theorem 3.2.** *Let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ be p.m.p. actions and set $\mathcal{G} = \Gamma \ltimes X$ and $\mathcal{H} = \Lambda \ltimes Y$. Suppose that we have an isomorphism $f\colon \mathcal{G}|_Z \to \mathcal{H}|_{f(Z)}$, where $Z$ is a measurable subset of $X$ with $\Gamma Z = X$ and $\Lambda f(Z) = Y$. Then the two cocycles $\mathfrak{m}$ and $\mathfrak{n} \circ f$ from $\mathcal{G}|_Z$ into $\mathbb{R}_+^\times$ are equivalent, where $\mathfrak{m}$ is identified with its composition with the projection from $\mathcal{G}$ onto $\Gamma$, and the same identification is performed for $\mathfrak{n}$.*

*Proof.* We follow the proof of [Ki2, Theorem 7.3]. Let $T_\Gamma$ and $T_\Lambda$ be the Bass-Serre trees associated to $\Gamma$ and $\Lambda$, respectively. For a vertex $u$ of $T_\Gamma$, we denote by $\Gamma_u$ the stabilizer of $u$ in $\Gamma$ and set $\mathcal{G}_u = \Gamma_u \ltimes X$. We similarly define $\Lambda_v$ and $\mathcal{H}_v$ for a vertex $v$ of $T_\Lambda$. Fix a vertex $u$ of $T_\Gamma$. Because $\Gamma_u$ is quasi-normal in $\Gamma$, the subgroupoid $\mathcal{G}_u|_Z$ is quasi-normal in $\mathcal{G}|_Z$ and hence its image $f(\mathcal{G}_u|_Z)$ is quasi-normal in $\mathcal{H}|_{f(Z)}$. Because $\Lambda$ is non-amenable, it follows from Lemma 3.1 that $f(\mathcal{G}_u|_Z)$ is elliptic in $\mathcal{H}|_{f(Z)}$. There exists a non-negligible subset $Z_1$ of $Z$ such that the inclusion $f(\mathcal{G}_u|_{Z_1}) < \mathcal{H}_v|_{f(Z_1)}$ holds for some vertex $v$ of $T_\Lambda$. Applying this procedure for $f^{-1}$ and $\mathcal{H}_v|_{f(Z_1)}$, we obtain a non-negligible subset $A$ of $Z_1$ such that $\mathcal{H}_v|_{f(A)} < f(\mathcal{G}_{u'}|_A)$ for some vertex $u'$ of $T_\Gamma$. If we set $\mathcal{E} = f^{-1}(\mathcal{H}_v|_{f(A)})$, then the inclusion

$$((\Gamma_u \cap \Gamma_{u'}) \ltimes X)|_A < \mathcal{E} < (\Gamma_{u'} \ltimes X)|_A$$

holds. The left hand side is of finite index in the right hand side by condition (2) for $\Gamma$, and hence so is $\mathcal{E}$. We define the modular cocycles

$$\delta_\Gamma = \delta(\mathcal{G}|_A, \mathcal{E}), \quad \delta_\Lambda = \delta(\mathcal{H}|_{f(A)}, \mathcal{H}_v|_{f(A)}).$$

We have $\delta_\Gamma = \delta_\Lambda \circ f$ because $\mathcal{H}|_{f(A)} = f(\mathcal{G}|_A)$ and $\mathcal{H}_v|_{f(A)} = f(\mathcal{E})$. By Propositions 2.2 and 2.4, $\delta_\Gamma$ is equivalent to $\delta(\mathcal{G}, \mathcal{G}_{u'})$ as a cocycle from $\mathcal{G}|_A$ into $\mathbb{R}_+^\times$, and by Proposition 2.5, the latter cocycle is equivalent to $\mathfrak{m}$. By Propositions 2.2 and 2.4, $\delta_\Lambda$ is equivalent to $\mathfrak{n}$ as a cocycle from $\mathcal{H}|_{f(A)}$ into $\mathbb{R}_+^\times$. Hence $\mathfrak{m}$ and $\mathfrak{n} \circ f$ are equivalent as cocycles from $\mathcal{G}|_A$ into $\mathbb{R}_+^\times$. This proof indeed shows that there exists a partition of $Z$ into countably many non-negligible subsets such that for any piece $A$ from it, $\mathfrak{m}$ and $\mathfrak{n} \circ f$ are equivalent as cocycles from $\mathcal{G}|_A$ into $\mathbb{R}_+^\times$. The conclusion of the theorem then follows. □

*Remark* 3.3. The above proof also shows that if $\Gamma_u$ is infinite, then so is $\Lambda_v$. Hence, if $\mathcal{C}$ denotes the collection of the fundamental group of a graph of groups satisfying conditions (1)–(4), then whether a vertex group is finite or infinite is invariant under stable isomorphism among the groupoids $\Gamma \ltimes X$ associated with a p.m.p. action of a group from $\mathcal{C}$. We note that if a vertex group is finite, then the associated modular homomorphism is trivial.

Let $\mathcal{G}$ be a discrete measured groupoid on $(X, \mu)$ and $\alpha\colon \mathcal{G} \to H$ a cocycle into a locally compact second countable group $H$. The *Mackey range* of $\alpha$ is defined as follows: Let $\mathcal{R}$ be the discrete measured equivalence relation on $X \times H$ such that for any $g \in \mathcal{G}_x^y$ and $h \in H$, the two points $(x, h)$ and $(y, \alpha(g)h)$ are equivalent. Let $Z$ be the space of ergodic components of $\mathcal{R}$. The action of $H$ on $X \times H$ by right-multiplication on the second variable then induces the action of $H$ on $Z$. The action $H \curvearrowright Z$ is exactly the Mackey range of $\alpha$. The isomorphism class of the Mackey range of $\alpha$ depends only on the equivalence class of $\alpha$ and also coincides with that of the restriction of $\alpha$ to $\mathcal{G}|_A$ for any measurable subset $A$ with $\mathcal{G}A = X$. We therefore obtain:

**Corollary 3.4.** *Under the assumption in Theorem 3.2, the Mackey range of the cocycle* $\mathfrak{m}\colon \mathcal{G} \to \mathbb{R}_+^\times$ *and that of the cocycle* $\mathfrak{n}\colon \mathcal{H} \to \mathbb{R}_+^\times$ *are isomorphic.*

Let us consider the Mackey range of the cocycle $\log \circ \mathfrak{m}\colon \mathcal{G} \to \mathbb{R}$ instead of $\mathfrak{m}$. If $\mathfrak{m}(\Gamma)$ is closed in $\mathbb{R}_+^\times$, then it is described as follows: Let $Z$ be the space of ergodic components for the action $\ker \mathfrak{m} \curvearrowright X$. The action of $\mathfrak{m}(\Gamma)$ on $Z$ is induced, and the action $\log(\mathfrak{m}(\Gamma))$ on $Z$ is obtained through the isomorphism log between $\mathbb{R}_+^\times$ and $\mathbb{R}$. If $\mathfrak{m}(\Gamma)$ is closed in $\mathbb{R}_+^\times$, then the Mackey range of the cocycle $\log \circ \mathfrak{m}$ coincides with the action of $\mathbb{R}$ induced from the action $\log(\mathfrak{m}(\Gamma)) \curvearrowright Z$.

We itemize immediate consequences of Theorem 3.2 and Corollary 3.4: Let $\Gamma$ be the group as above and let $\Gamma \curvearrowright (X, \mu)$ be an ergodic p.m.p. action.

(i) If $\mathfrak{m}(\Gamma)$ is trivial, then the Mackey range of $\log \circ \mathfrak{m}$ is the translation flow on $\mathbb{R}$. Otherwise the Mackey range is a p.m.p. flow. Hence, if $\mathcal{C}$ denotes the collection in Remark 3.3, then whether $\mathfrak{m}(\Gamma)$ is trivial or not is an invariant under stable orbit equivalence among free, ergodic and p.m.p. actions of groups in $\mathcal{C}$.

(ii) If $\mathfrak{m}(\Gamma)$ is a lattice in $\mathbb{R}_+^\times$, then the Mackey range of $\log \circ \mathfrak{m}$ is a non-trivial flow. If $\mathfrak{m}(\Gamma)$ is dense in $\mathbb{R}_+^\times$ and the action $\Gamma \curvearrowright (X, \mu)$ is mildly mixing ([S]), then the Mackey range of $\log \circ \mathfrak{m}$ is the trivial flow. Hence the closure of $\mathfrak{m}(\Gamma)$ in $\mathbb{R}_+^\times$ is an invariant under stable orbit equivalence among free, mildly mixing and p.m.p. actions of groups in $\mathcal{C}$.

Let $\Lambda$ be the group as above and let $\Lambda \curvearrowright (Y, \nu)$ be an ergodic p.m.p. action. Suppose that the two groupoids $\Gamma \ltimes X$ and $\Lambda \ltimes Y$ are stably isomorphic.

(iii) If $\mathfrak{n}(\Lambda)$ is a lattice in $\mathbb{R}_+^\times$, then by Theorem 3.2, a $\Gamma$-equivariant map from $X$ into $\mathbb{R}_+^\times/\mathfrak{n}(\Lambda)$ is induced, where $\Gamma$ acts on $\mathbb{R}_+^\times/\mathfrak{n}(\Lambda)$ by translation through $\mathfrak{m}$. If $\mathfrak{m}(\Gamma)$ is further dense in $\mathbb{R}_+^\times/\mathfrak{n}(\Lambda)$, then the image of $\mu$ under the induced map is the Lebesgue measure and hence the action $\Gamma \curvearrowright (X, \mu)$ is not weakly mixing.

(iv) If both $\mathfrak{m}(\Gamma)$ and $\mathfrak{n}(\Lambda)$ are lattices in $\mathbb{R}_+^\times$, then $\ker \mathfrak{m}$ and $\ker \mathfrak{n}$ are *measure equivalent*, that is, there are ergodic p.m.p. actions of $\ker \mathfrak{m}$ and of $\ker \mathfrak{n}$ such that the associated transformation-groupoids are stably isomorphic. If $\mathfrak{m}(\Gamma) = \mathfrak{n}(\Lambda)$ further, then the action of $\mathfrak{m}(\Gamma)$ on the space of $\ker \mathfrak{m}$-ergodic components and the action of $\mathfrak{n}(\Lambda)$ on the space of $\ker \mathfrak{n}$-ergodic components are conjugate. The proof of these assertions are obtained as well as [Ki2, Corollaries 7.5 and 7.6].

For countable groups $G$ and $H$, we mean by a $(G, H)$-*coupling* a standard Borel space with a $\sigma$-finite measure, $(\Sigma, m)$, on which the group $G \times H$ acts by measure-preserving automorphisms such that each of the restrictions to $G \times \{e\}$ and to $\{e\} \times H$ admits a measurable fundamental domain of finite measure ([F]). This $(G, H)$-coupling naturally gives rise to a stable isomorphism between the transformation-groupoids from some p.m.p. actions of $G$ and of $H$, and vice versa. As its consequence, two countable groups $G$ and $H$ are measure equivalent in the sense mentioned above if and only if there exists a $(G, H)$-coupling. The following is a translation of Theorem 3.2 in terms of a coupling (see [Ki2, Theorem 7.3]):

**Corollary 3.5.** *Let* $\Gamma$ *and* $\Lambda$ *be the groups in Theorem 3.2. Then for any* $(\Gamma, \Lambda)$*-coupling* $(\Sigma, m)$*, there exists a* $(\Gamma \times \Lambda)$*-equivariant map from* $\Sigma$ *into* $\mathbb{R}$*, where the group* $\Gamma \times \Lambda$ *acts on* $\mathbb{R}$ *by the formula* $(\gamma, \lambda)t = \log \mathfrak{m}(\gamma) + t - \log \mathfrak{n}(\lambda)$ *for* $\gamma \in \Gamma$*,* $\lambda \in \Lambda$ *and* $t \in \mathbb{R}$*.*

Combining this corollary and a variant of Furman's machinery [F] constructing a homomorphism from equivariant maps of the above kind, we obtain the following:

**Theorem 3.6.** *Let* $\Gamma$ *be the group as above and suppose further that* $\mathfrak{m}$ *is non-trivial. Let* $\Gamma \curvearrowright (X, \mu)$ *be a free, ergodic and p.m.p. action. Let* $\Delta$ *be an arbitrary countable group* *and $\Delta \curvearrowright (Y, \nu)$ a free, weakly mixing and p.m.p. action. If those two actions are stably orbit equivalent, then there exists a non-trivial homomorphism $\rho\colon \Delta \to \mathbb{R}$. If $\log(\mathfrak{m}(\Gamma))$ is further a lattice in $\mathbb{R}$, then the image of the obtained $\rho$ is also a lattice in $\mathbb{R}$.*

*Proof.* We follow the proof of [Ki2, Theorem 1.3]. Thanks to Corollary 3.5, we can apply [Ki2, Theorem 4.3] to the coupling $\Sigma$ associated with the stable orbit equivalence in our assumption. We then obtain a homomorphism $\rho\colon \Delta \to \mathbb{R}$ and a $(\Gamma \times \Delta)$-equivariant map $\Phi\colon \Sigma \to \mathbb{R}$, where the equivariance condition means that the following equation holds:

$$\Phi((\gamma, g)x) = \log(\mathfrak{m}(\gamma)) + \Phi(x) - \rho(g) \quad \text{for } \gamma \in \Gamma,\ g \in \Delta \text{ and } x \in \Sigma.$$

If $\rho$ were trivial, then $\Phi$ would induce a $\Gamma$-equivariant map from $X = \Sigma/\Delta$ into $\mathbb{R}$, which contradicts that $\mathfrak{m}$ is non-trivial. The latter assertion of the theorem is obtained along the third paragraph in the proof of [Ki2, Theorem 1.3]. □

*Remark* 3.7. Theorem 3.2 holds as well for the fundamental group of a graph of groups satisfying conditions (1) and (2) and the condition that any vertex group has property (T). In fact, let $\Gamma$ and $\Lambda$ be the fundamental groups of such graphs of groups and let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ be p.m.p. actions. Suppose that the groupoids $\Gamma \ltimes X$, $\Lambda \ltimes Y$ are stably isomorphic. Fix a vertex $u$ of the tree for $\Gamma$. For localizing the image of $\Gamma_u \ltimes X$ under the isomorphism, we appeal to the fact that any simplicial action of a discrete measured groupoid with property (T) on a tree admits a fixed point ([AS], [A]). The image of $\Gamma_u \ltimes X$ is then contained in $\Lambda_v \ltimes Y$ for some vertex $v$ of the tree for $\Lambda$ after restricting it to some non-negligible subset of $Y$. For the rest, the same proof works. The corollaries of Theorem 3.2 are also available for those groups.

The group $\mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ with $n \geq 3$ has property (T), and its subgroups $\mathrm{SL}_n(\mathbb{Z}) \ltimes k\mathbb{Z}^n$ with $k$ a positive integer are of finite index and mutually isomorphic. The HNN extension of $\mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ relative to an isomorphism between those subgroups is an example under this remark such that the modular homomorphism can be non-trivial.

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