MEASURABLE RIGIDITY FOR SOME AMALGAMATED FREE PRODUCTS

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1. Introduction

We mean by a *f.f.m.p.* action of a discrete countable group an essentially free, measure-preserving action of it on a standard Borel space with a finite positive measure. The research of orbit equivalence (OE) rigidity for f.f.m.p. actions is recently well developed, and many surprising results are established. Among other things, Ioana, Peterson and Popa [4], and Alvarez and Gaboriau [1] study f.f.m.p. actions of free products and they provide the following type of rigidity: If two f.f.m.p. actions $\Gamma_1 * \Gamma_2 \curvearrowright (X, \mu)$ and $\Lambda_1 * \Lambda_2 \curvearrowright (Y, \nu)$ of free products are OE (under ergodicity assumptions on actions of the factor groups if necessary), then the actions $\Gamma_i \curvearrowright (X, \mu)$ and $\Lambda_i \curvearrowright (Y, \nu)$ are OE for each $i = 1, 2$ up to the exchange of the indices.

In this article, we study f.f.m.p. actions of amalgamated free products $\Gamma = \Gamma_1 *_A \Gamma_2$, where A is always infinite and we impose algebraic assumptions on the pairs $A < \Gamma_1$, $A < \Gamma_2$ and ergodicity assumptions on actions of Γ_1, Γ_2 and A. We present superrigid f.f.m.p. actions $\Gamma \curvearrowright (X, \mu)$ of such a Γ, where superrigidity means that this action satisfies that if a f.f.m.p. action $\Lambda \curvearrowright (Y, \nu)$ of an arbitrary group is OE to the action $\Gamma \curvearrowright (X, \mu)$, then the cocycle associated with the OE is cohomologous to the constant cocycle. In particular, the two actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are conjugate (see Theorem 5.6).

Without the ergodicity assumptions on actions of Γ_1, Γ_2 and A, f.f.m.p. actions of Γ are not typically rigid in the above sense because of the universal property of amalgamated free products. If one considers f.f.m.p. actions of free products, then the superrigidiy in the above sense cannot occur. In spite of this nature, if one assumes very strong algebraic conditions on the pairs $A < \Gamma_1$, $A < \Gamma_2$, then any action of the amalgamated free product $\Gamma = \Gamma_1 *_A \Gamma_2$ can be superrigid. In particular, one sees that such a Γ is *ME rigid*, that is, if a discrete group Λ is measure equivalent (ME) to Γ, then they are virtually isomorphic. Recall that two discrete groups are said to be *virtually isomorphic* if they are isomorphic up to finitely many operations of taking finite index subgroups and taking the quotients by finite normal subgroups. An example of an amalgamated free product which is ME rigid is presented in Corollary 5.15.

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This article is organized as follows. Section 2 reviews the definition of ME and its relationships with OE and isomorphism of discrete measured groupoids. In Section 3, we define ME coupling rigidity for a pair of a discrete group and its representation into a standard Borel group. Thanks to this notion, one can apply Furman's method to diverse situations to deduce ME and OE rigidity results. Section 4 introduces several algebraic assumptions for amalgamated free products, and we provide rigidity for such groups in the next section. This section reviews the Bass-Serre trees for amalgamated free products, which are an important geometric tool to study such groups. Section 5 gives statements about rigidity for amalgamated free products.

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2. ME and isomorphism of groupoids

In this section, we shall review basic facts about ME and OE.

Definition 2.1. Two discrete groups Γ, Λ are said to be *measure equivalent (ME)* if there exist a standard Borel space (Σ, m) with a σ-finite positive measure and a measure-preserving action of $\Gamma \times \Lambda$ on Σ satisfying the following: There exist Borel subsets $X, Y \subset \Sigma$ such that $\Sigma = \bigsqcup_{\gamma \in \Gamma} \gamma Y = \bigsqcup_{\lambda \in \Lambda} \lambda X$ up to m-null sets. The space (Σ, m) equipped with the action of $\Gamma \times \Lambda$ as above is called an *ME coupling* of Γ and Λ. When $m(X) = m(Y)$, we say that the ME coupling has *coupling constant one*.

ME defines an equivalence relation among discrete groups (see [2]). A motivated example of an ME coupling is given as follows: Let G be a locally compact second countable group and Γ, Λ lattices in G (i.e., discrete subgroups of G with co-finite measure with respect to the Haar measure on G). Then the action of $\Gamma \times \Lambda$ on G given by

$$(\gamma, \lambda)g = \gamma g \lambda^{-1}, \quad g \in G,\ \gamma \in \Gamma,\ \lambda \in \Lambda$$

satisfies the axiom of ME. It is not difficult to see that two virtually isomorphic groups are ME. We refer to [2], [3] for fundamental discussion about ME and to the survey [9] for important results about ME.

Let us explain a reformulation of ME in terms of discrete measured groupoids associated with group actions. We mean by a *standard finite measure space* a standard Borel space with a finite positive measure. Given a measure-preserving action of a discrete group Γ on a standard finite measure space (X, μ), one can construct a discrete measured groupoid $\mathcal{G} = \Gamma \ltimes (X, \mu)$ on (X, μ) as follows: As a measure space, $\mathcal{G}$ is given by the Borel space $\Gamma \times X$ equipped with the product measure of the counting measure on Γ and μ. The range and source maps are defined by

$$r(\gamma, x) = \gamma x, \quad s(\gamma, x) = x,$$

respectively, and the product and inverse are defined by

$$(\gamma_1, \gamma_2 x)(\gamma_2, x) = (\gamma_1\gamma_2, x), \quad (\gamma, x)^{-1} = (\gamma^{-1}, \gamma x),$$

respectively, for $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $x \in X$. There is a close connection between orbit equivalence and isomorphism of two discrete measured groupoids associated with group actions. We first recall the definition of orbit equivalence.

Definition 2.2. Let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ be measure-preserving actions on standard finite measure spaces. The two actions are said to be *weakly orbit equivalent (WOE)* if there are Borel subsets $X' \subset X$, $Y' \subset Y$ with $\Gamma X' = X$, $\Lambda Y' = Y$ up to null sets, and a Borel isomorphism $f\colon X' \to Y'$ such that

- the two measures $f_*(\mu|_{X'})$ and $\nu|_{Y'}$ are equivalent; and
- $f(\Gamma x \cap X') = \Lambda f(x) \cap Y'$ for a.e. $x \in A_1$.

If we can take both X' and Y' to have full measure, then the two actions are said to be *orbit equivalent (OE)*.

Let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ be measure-preserving actions on standard finite measure spaces, and denote the associated groupoids by $\mathcal{G}$ and $\mathcal{H}$, respectively. It is easy to show that when the two actions are both essentially free, $\mathcal{G}$ and $\mathcal{H}$ are isomorphic as discrete measured groupoids if and only if the two actions are OE. It is clear that WOE is also formulated in terms of $\mathcal{G}$ and $\mathcal{H}$.

Given an ME coupling (Σ, m) of discrete groups Γ and Λ and Borel subsets $X, Y \subset \Sigma$ as in Definition 2.1, we can define actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ as follows: For $\gamma \in \Gamma$ and $x \in X$, we can find a unique $\alpha(\gamma, x) \in \Lambda$ such that $(\gamma, \alpha(\gamma, x))x \in X$ since X is a fundamental domain for the action $\Lambda \curvearrowright \Sigma$. It is easy to see that the map $(\gamma, x) \mapsto (\gamma, \alpha(\gamma, x))x$ defines an action of Γ on X which is measure-preserving with respect to the restriction of m to X. To distinguish this action and the original action of Γ on Σ, we use a dot for this new action, that is, we denote $(\gamma, \alpha(\gamma, x))x$ by $\gamma \cdot x$. The map $\alpha\colon \Gamma \times X \to \Lambda$ is called the *ME cocycle* (associated with X) and it satisfies the cocycle identity:

$$\alpha(\gamma_1\gamma_2, x) = \alpha(\gamma_1, \gamma_2 \cdot x)\alpha(\gamma_2, x), \quad \forall \gamma_1, \gamma_2 \in \Gamma, \text{ a.e. } x \in X.$$

We can define an action of Λ on Y in a similar way and denote the ME cocycle associated with Y by $\beta\colon \Lambda \times Y \to \Gamma$.

Let us construct an isomorphism between (the restrictions of) the two groupoids $\mathcal{G} = \Gamma \ltimes X$ and $\mathcal{H} = \Lambda \ltimes Y$.

Lemma 2.3 ([6, Lemma 2.27])**.** *In the above notation, one can choose X and Y so that $A = X \cap Y$ satisfies the following two conditions:*

- *$\Gamma \cdot A = X$ up to null sets when A is regarded as a subset of X;*
- *$\Lambda \cdot A = Y$ up to null sets when A is regarded as a subset of Y.*

Remark that replacing fundamental domains for the actions $\Gamma \curvearrowright \Sigma$, $\Lambda \curvearrowright \Sigma$ corresponds to exchanging the ME cocycles into cohomologous ones. Therefore, this replacing does not essentially affect the problem considering the ME coupling $\Gamma \times \Lambda \curvearrowright \Sigma$. Let us define groupoid homomorphisms

$$f\colon (\mathcal{G})_A \ni (\gamma, x) \mapsto (\alpha(\gamma, x), x) \in (\mathcal{H})_A,$$
$$g\colon (\mathcal{H})_A \ni (\lambda, y) \mapsto (\beta(\lambda, y), y) \in (\mathcal{G})_A,$$

where $(\mathcal{G})_A = \{g \in \mathcal{G} : r(g), s(g) \in A\}$ is the restriction of $\mathcal{G}$ to A, and $(\mathcal{H})_A$ is also defined similarly. Note that $\beta(\alpha(\gamma, x), x) = \gamma$ for any $\gamma \in \Gamma$ and a.e. $x \in A$ with $\gamma \cdot x \in A$ because $\gamma\alpha(\gamma, x)x = \gamma \cdot x \in A \subset Y$. Similarly, $\alpha(\beta(\lambda, y), y) = \lambda$ for any $\lambda \in \Lambda$ and a.e. $y \in A$ with $\lambda \cdot y \in A$. Therefore, we obtain the following.

Proposition 2.4 ([6, Proposition 2.29])**.** *In the above notation, the groupoid homomorphisms*

$$f\colon (\mathcal{G})_A \to (\mathcal{H})_A, \quad g\colon (\mathcal{H})_A \to (\mathcal{G})_A$$

satisfy $g \circ f = \mathrm{id}$ *and* $f \circ g = \mathrm{id}$.

Conversely, given an isomorphism between (the restrictions of) two discrete measured groupoids associated with measure-preserving actions of groups on standard finite measure spaces, one can construct the corresponding ME coupling (see Lemma 2.30 in [6]).

When both X and Y consist of a single point, the groupoids $\mathcal{G}$, $\mathcal{H}$ degenerate into groups Γ, Λ, respectively, and Proposition 2.4 gives an isomorphism between Γ and Λ. This simple observation often helps the study of self ME couplings (i.e., ME couplings of a discrete group and itself) as discussed in subsequent sections.

3. ME coupling rigidity

We introduce the notion of ME coupling rigidity for pairs of groups and their representations and give some consequences of the property. A few examples of such pairs are also presented.

3.1. **ME coupling rigidity and Furman's representation theorem.**

In [11], Zimmer considers the problem asking which groups are ME to $SL(n, \mathbb{Z})$ when $n \geq 3$ (or more generally, which groups are ME to a lattice in a connected simple Lie group with $\mathbb{R}$-rank at least two). He shows in the paper that if a discrete group Λ which is ME to $SL(n, \mathbb{Z})$ admits a linear representation over a finite dimensional space with infinite image, then Λ is virtually isomorphic to a lattice in $SL(n, \mathbb{R})$. Zimmer's cocycle superrigidity theorem ([10]) plays an indispensable role in his proof. As discussed below, Furman [2] introduces construction of a representation into $\mathrm{Aut}(PSL(n, \mathbb{R}))$ of an arbitrary group Λ which is ME to $SL(n, \mathbb{Z})$ by way of the cocycle superrigidity theorem, and gives a complete answer to the problem mentioned above. His construction of representations of unknown groups is applicable to a more general setting. ME coupling rigidity is a condition for a pair of a discrete group Γ and its representation into a standard Borel group G to which the construction is applicable. More precisely, for such a pair of Γ and G, if a discrete group Λ is ME to Γ, then a useful representation of Λ into G can be constructed.

Let G be a standard Borel group and Γ, Λ discrete groups. Given homomorphisms $\pi\colon \Gamma \to G$ and $\rho\colon \Lambda \to G$, we denote by (G, π, ρ) the Borel space G equipped with the action of $\Gamma \times \Lambda$ defined by

$$(\gamma, \lambda)g = \pi(\gamma)g\rho(\lambda)^{-1}, \quad g \in G,\ \gamma \in \Gamma,\ \lambda \in \Lambda.$$

Definition 3.1. Let Γ be a discrete group, G a standard Borel group and $\pi\colon \Gamma \to G$ a homomorphism. We say that Γ is *ME coupling rigid* with respect to (G, π) if the following conditions hold:

(i) For any self ME coupling Σ of Γ with coupling constant one, there exists an almost $(\Gamma \times \Gamma)$-equivariant Borel map $\Phi\colon \Sigma \to (G, \pi, \pi)$. This means that the equation
$$\Phi((\gamma_1, \gamma_2)g) = \pi(\gamma_1)g\pi(\gamma_2)^{-1}$$
holds for any $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $g \in G$.

(ii) The delta measure δ_e over the neutral element $e \in G$ is the only probability measure on G which is invariant under conjugation by each element of $\pi(\Gamma)$.

It is easy to see that the condition (ii) assures the essential uniqueness of the map Φ in the condition (i). A consequence of this property is stated in the following representation theorem. The proof follows Furman's argument in [2].

Theorem 3.2. *Let Γ be a discrete group and suppose that Γ is ME coupling rigid with respect to a pair (G, π). Let Σ be an ME coupling of Γ and a discrete group Λ. Then we can find the following:*

- *a homomorphism $\rho\colon \Lambda \to G$;*
- *an almost $(\Gamma \times \Lambda)$-equivariant Borel map $\Phi\colon \Sigma \to (G, \pi, \rho)$.*

In addition, if the kernel of the homomorphism $\pi\colon \Gamma \to G$ is finite and there is a Borel fundamental domain for the action of $\pi(\Gamma)$ on G given by left multiplication, then ρ can be chosen so that the kernel of ρ is finite.

Thanks to this theorem, one can study an unknown group Λ by using ρ and Φ.

3.2. **Examples.** We shall give examples of pairs of groups and their representations which are ME coupling rigid.

Theorem 3.3. (i) ([2]) *When $n \geq 3$, $SL(n, \mathbb{Z})$ is ME coupling rigid with respect to $(\mathrm{Aut}(PSL(n, \mathbb{R})), \imath)$, where $\imath\colon SL(n, \mathbb{Z}) \to \mathrm{Aut}(PSL(n, \mathbb{R}))$ is the natural representation.*

(ii) ([5]) *Let M be a compact orientable surface of genus g and with p boundary components and let $\mathrm{Mod}^*(M)$ be the mapping class group of M. Suppose that $3g + p - 4 > 0$ and $(g, p) \neq (1, 2), (2, 0)$. Then $\mathrm{Mod}^*(M)$ is ME coupling rigid with respect to $(\mathrm{Mod}^*(M), \imath)$, where $\imath\colon \mathrm{Mod}^*(M) \to \mathrm{Mod}^*(M)$ is the identity.*

The first assertion is equivalent to the conclusion of Zimmer's cocycle superrigidity theorem for cocycles arising from self ME couplings of $SL(n, \mathbb{Z})$. Furman formulates the cocycle superrigidity theorem as above in terms of ME couplings, and this formulation is fit to apply Theorem 3.2. The mapping class group $\mathrm{Mod}^*(M)$ of M

is the group consisting of all diffeomorphisms of M up to isotopy which may move points of the boundary of M. Applying Theorem 3.2, one can deduce the following rigidity.

Theorem 3.4. (i) ([2]) *When $n \geq 3$, if a discrete group Λ is ME to $SL(n, \mathbb{Z})$, then Λ is virtually isomorphic to a lattice in $SL(n, \mathbb{R})$.*

(ii) ([5]) *Let M be the surface in Theorem 3.3 (ii). Then the mapping class group $\mathrm{Mod}^*(M)$ is ME rigid, that is, if a discrete group Λ is ME to $\mathrm{Mod}^*(M)$, then Λ is virtually isomorphic to $\mathrm{Mod}^*(M)$.*

Remark 3.5. In the assertion (i), combining the representation theorem 3.2 and Theorem 3.3 (i), one can construct a representation $\rho\colon \Lambda \to \mathrm{Aut}(SL(n, \mathbb{Z}))$ with finite kernel. Depending on Zimmer's argument in [11], Furman shows that the image $\rho(\Lambda)$ is a lattice in $\mathrm{Aut}(SL(n, \mathbb{Z}))$, that is a non-trivial fact. On the other hand, because of discreteness of $\mathrm{Mod}^*(M)$, it is not difficult to deduce Theorem 3.4 (ii) from Theorem 3.3 (ii) by way of the representation theorem 3.2.

4. Amalgamated free products and the Bass-Serre trees

The first subsection introduces a class of amalgamated free products $\Gamma = \Gamma_1 *_A \Gamma_2$ for which we prove OE and ME rigidity. We first define a class $\mathcal{C}$ of discrete groups to which the factor groups Γ_1, Γ_2 should belong, and define conditions which the pairs of subgroups and groups, $A < \Gamma_1$ and $A < \Gamma_2$, should satisfy. We call these conditions Assumption ($\star$).

The second subsection describes the Bass-Serre tree T associated with the amalgamated free product $\Gamma = \Gamma_1 *_A \Gamma_2$. The main reference of this subsection is Serre's book [8]. The natural action $\Gamma \curvearrowright T$ plays an important role in a subsequent discussion. This subsection states that under Assumption ($\star$), each automorphism of Γ naturally induces an automorphism of T, and therefore there exists an natural homomorphism from the automorphism group of Γ into the automorphism group of T. This fact inspires that Γ is ME coupling rigid with respect to the pair of the automorphism group of T and the natural action of Γ on T.

4.1. Assumption ($\star$).

Let $\mathcal{C}$ be the class of discrete groups consisting of infinite groups with Kazhdan's property (T) and the mapping class groups in Theorem 3.4 (ii). (Though we can prove the results discussed below for a broader class of discrete groups than this $\mathcal{C}$, we do not define the broader class because it is technical.)

Assumption 4.1. We denote the following assumption by ($\star$): Let $\Gamma_1, \Gamma_2 \in \mathcal{C}$ and let $A_i < \Gamma_i$ be proper infinite subgroups for $i = 1, 2$ and $\phi\colon A_1 \xrightarrow{\sim} A_2$ an isomorphism. Suppose one of the following two conditions:

(i) Both of the groups Γ_1, Γ_2 satisfy Kazhdan's property (T), and $LQN_{\Gamma_i}(A_i) = A_i$ for $i = 1, 2$.

(ii) A_i is almost malnormal in Γ_i for $i = 1, 2$.

Take the amalgamated free product $\Gamma = \langle \Gamma_1, \Gamma_2 | A_1 \simeq_\phi A_2 \rangle$ and denote by A the subgroup of Γ corresponding to $A_1 \simeq_\phi A_2$. Let T be the Bass-Serre tree associated with the decomposition of Γ and $\imath\colon \Gamma \to \mathrm{Aut}^*(T)$ be the homomorphism coming from the natural action $\Gamma \curvearrowright T$. Suppose that the kernel of $\imath$ is finite.

We collect here the new notation used in the above definition. For a pair of a group Γ and a subgroup A, we put

$$LQN_\Gamma(A) = \{\gamma \in \Gamma : [A : \gamma A \gamma^{-1} \cap A] < \infty\},$$

called the *left quasi-normalizer* of A in Γ, which is a subsemigroup of Γ containing A. We say that A is *almost malnormal* in Γ if $\gamma A \gamma^{-1} \cap A$ is finite for each $\gamma \in \Gamma \setminus A$. It is clear that if A is almost malnormal in Γ, then $LQN_\Gamma(A) = A$. Given a simplicial tree T with at most countable simplices, we denote by $\mathrm{Aut}^*(T)$ the automophism group of T equipped with the standard Borel structure associated with the pointwise convergence topology. In the next subsection, we shall recall the Bass-Serre tree associated with amalgamated free products.

4.2. **The Bass-Serre trees.** Given an amalgamated free product $\Gamma = \Gamma_1 *_A \Gamma_2$, one can construct a tree T as follows. Let $V(T) = \Gamma/\Gamma_1 \sqcup \Gamma/\Gamma_2$ be the set of vertices of T, and let $E(T) = \Gamma/A$ be the set of edges of T. For each $\gamma \in \Gamma$, the two vertices of the edge $\gamma A \in E(T)$ is given by $\gamma\Gamma_1, \gamma\Gamma_2 \in V(T)$. It is an excercise to show that this indeed defines a connected tree and Γ acts on T as simplicial automorphisms by left multiplication. It is easy to see the following properties:

- Let $v_i \in V(T)$ be the vertex corresponding to $\Gamma_i \in \Gamma/\Gamma_i$ for $i = 1, 2$. The stabilizer of v_i in Γ is equal to Γ_i, and there is a Γ_i-equivariant one-to-one correspondence between Γ_i/A and the link of v_i. In particular, if A is of infinite index in Γ_i, then the link of v_i consists of infinitely many vertices.
- Take two distinct edges e_1, e_2 having a common vertex. The stabilizer of e_1 and e_2 in Γ is conjugate in Γ to $\gamma A \gamma^{-1} \cap A$ for some $\gamma \in \Gamma_1 \cup \Gamma_2$.
- We introduce an orientation on T as follows: For each $\gamma \in \Gamma$, let $\gamma\Gamma_1, \gamma\Gamma_2 \in V(T)$ be the origin and terminal of the edge $\gamma A \in E(T)$, respectively. Let $\mathrm{Aut}(T)$ be the group consisting of all elements in $\mathrm{Aut}^*(T)$ preserving this orientation. Then $\mathrm{Aut}(T)$ is a subgroup of $\mathrm{Aut}^*(T)$ of index two, and it consists of automorphisms of T without inversions.

Under Assumption $(\star)$, one can show that the amalgamated free product $\Gamma = \Gamma_1 *_A \Gamma_2$ satisfies the following remarkable property.

Theorem 4.2. *Under Assumption $(\star)$, for each $f \in \mathrm{Aut}(\Gamma)$, there exists a unique $\varphi \in \mathrm{Aut}^*(T)$ such that $\imath(f(\gamma)) = \varphi\imath(\gamma)\varphi^{-1}$ for any $\gamma \in \Gamma$. This correspondence $f \mapsto \varphi$ defines a natural homomorphism $\imath\colon \mathrm{Aut}(\Gamma) \to \mathrm{Aut}^*(T)$.*

Remark 4.3. The *(abstract) commensurator* $\mathrm{Comm}(\Gamma)$ of Γ is the group of all isomorphisms between finite index subgroups of Γ up to the equivalence relation so that two such isomorphisms are equivalent if there exists a finite index subgroup of Γ on which they are equal. There is a natural homomorphism from $\mathrm{Aut}(\Gamma)$ into $\mathrm{Comm}(\Gamma)$. Under Assumption $(\star)$, one can show along the same idea that there is a natural homomorphism $\mathrm{Comm}(\Gamma) \to \mathrm{Aut}^*(T)$, which is a stronger statement than Theorem 4.2.

As discussed in Section 2, considering a self ME coupling of Γ with coupling constant one is equivalent to considering an isomorphism between discrete measured groupoids arising from two measure-preserving actions of Γ on standard finite measure spaces. The latter situation can be seen as a generalization of considering an automorphism of Γ. This observation motivates us to show the following theorem along the same idea as in the proof of Theorem 4.2.

Theorem 4.4. *Under Assumption* $(\star)$*,* Γ *is ME coupling rigid with respect to* $(\mathrm{Aut}^*(T), \imath)$.

Combining the representation theorem 3.2, we obtain the following.

Corollary 4.5. *Under Assumption* $(\star)$*, let* Σ *be an ME coupling of* Γ *and a discrete group* Λ*. Then there exist a homomorphism* $\rho\colon \Lambda \to \mathrm{Aut}^*(T)$ *with finite kernel and an almost* $(\Gamma \times \Lambda)$*-equivariant Borel map* $\Phi\colon \Sigma \to (\mathrm{Aut}^*(T), \imath, \rho)$.

5. Rigidity

Under Assumption $(\star)$, let Σ be an ME coupling of Γ and a discrete group Λ. This section discusses the structure of Λ and the ME coupling. This is closely linked with the study of ergodic f.f.m.p. actions of Γ from the viewpoint of orbit equivalence.

By Corollary 4.5, there exist a homomorphism $\rho\colon \Lambda \to \mathrm{Aut}^*(T)$ with finite kernel and an almost $(\Gamma \times \Lambda)$-equivariant Borel map $\Phi\colon \Sigma \to (\mathrm{Aut}^*(T), \imath, \rho)$. As already noted, the group $\mathrm{Aut}(T)$ of automorphisms of T is a subgroup of $\mathrm{Aut}^*(T)$ of index two. Therefore, we may assume that ρ and Φ are both valued in $\mathrm{Aut}(T)$ to understand the structure of Λ and the ME coupling Σ. In what follows, we always assume this condition.

5.1. Fundamental facts.

For each $s \in V(T) \cup E(T)$, we put

$$\mathrm{Stab}(s) = \{\varphi \in \mathrm{Aut}(T) : \varphi(s) = s\},$$

$$\Sigma_s = \Phi^{-1}(\mathrm{Stab}(s)),\ \Gamma_s = \imath^{-1}(\imath(\Gamma) \cap \mathrm{Stab}(s)),\ \Lambda_s = \rho^{-1}(\rho(\Lambda) \cap \mathrm{Stab}(s)).$$

The following lemma can be shown by observing that $\mathrm{Stab}(s)$ is not only a $(\Gamma_s \times \Lambda_s)$-invariant Borel subset of $\mathrm{Aut}(T)$ but also a group.

Lemma 5.1. *For each* $s \in V(T) \cup E(T)$*, the Borel subset* Σ_s *of* Σ *is an ME coupling of* Γ_s *and* Λ_s*. That is,* Σ_s *has positive measure and the action* $\Gamma_s \times \Lambda_s \curvearrowright \Sigma_s$ *satisfies the axiom of ME.*

Since Λ acts on T through ρ without inversions, by the Bass-Serre theory, one can see the structure of Λ if Λ_s for $s \in V(T) \cup E(T)$ and the quotient graph $T/\rho(\Lambda)$ are understood. This lemma gives nice information about Λ_s. In particular, Λ_s is ME to one of the subgroups Γ_1, Γ_2, A of Γ because Γ_s is conjugate with one of them. The next lemma gives information about the quotient graph $T/\rho(\Lambda)$ when we assume some ergodicity assumptions on the action $\Gamma \curvearrowright \Sigma/\Lambda$. For $i = 1, 2$, let $V_i(T) = \Gamma/\Gamma_i$ be the subset of $V(T) = \Gamma/\Gamma_1 \sqcup \Gamma/\Gamma_2$.

Lemma 5.2. *Let S be one of $V_1(T)$, $V_2(T)$ and $E(T)$, and take $s \in S$. Then the action $\rho(\Lambda) \curvearrowright S$ is transitive if and only if there exists a fundamental domain for the action $\Lambda \curvearrowright \Sigma$ contained in Σ_s. This is the case if the action $\Gamma_s \curvearrowright \Sigma/\Lambda$ is ergodic.*

In particular, if the action $A \curvearrowright \Sigma/\Lambda$ is ergodic, then the above equivalence holds for any $s \in S$ and any S. Combining these two lemmas, we obtain the following description of the structure of $\rho(\Lambda)$.

Corollary 5.3. *Let $v_i \in V_i(T)$, $e \in E(T)$ be the simplices of T corresponding to the cosets containing the trivial element. Put $\Lambda_i = \Lambda_{v_i}$ for $i = 1, 2$ and $B = \Lambda_e$. If the action $A \curvearrowright \Sigma/\Lambda$ is ergodic, then $\rho(\Lambda)$ is isomorphic to the amalgamated free product $\Lambda_1 *_B \Lambda_2$ such that $\Gamma_i \sim_{ME} \Lambda_i$ for $i = 1, 2$ and $A \sim_{ME} B$. Moreover, their ME couplings have all the same ratio of measures of fundamental domains for the actions of two groups.*

5.2. **OE rigidity.** The aim of this subsection is to conclude that the two actions $\Gamma \curvearrowright \Sigma/\Lambda$ and $\Lambda \curvearrowright \Sigma/\Gamma$ are virtually conjugate when we impose stronger assumptions on the groups Γ_1, Γ_2, A and their actions on Σ/Λ. We shall collect the assumptions on the groups here.

Assumption 5.4. We denote the following assumption by (†): Under Assumption ($\star$), suppose that for each $i = 1, 2$, Γ_i is ME coupling rigid with respect to a pair (G_i, π_i) such that the kernel of π_i is trivial and one of the following two conditions is satisfied:

(i) G_i is a discrete countable group;
(ii) $\Gamma_i = PSL(n, \mathbb{Z})$ for some $n \geq 3$ and $(G_i, \pi_i) = (\mathrm{Aut}(PSL(n, \mathbb{R})), \imath)$, where $\imath \colon PSL(n, \mathbb{Z}) \to \mathrm{Aut}(PSL(n, \mathbb{R}))$ is the natural homomorphism.

Let $v_i \in V_i(T)$, $e \in E(T)$ be the simplices of T corresponding to the cosets containing the trivial element. For $i = 1, 2$, since Σ_{v_i} is an ME coupling of Γ_i and $\Lambda_i = \rho^{-1}(\rho(\Lambda) \cap \mathrm{Stab}(v_i))$ and since Γ_i is ME coupling rigid with respect to (G_i, π_i), there exist a homomorphism $\rho_i \colon \Lambda_i \to G_i$ with finite kernel and an almost $(\Gamma_i \times \Lambda_i)$-equivariant Borel map $\Phi_i \colon \Sigma_{v_i} \to (G_i, \pi_i, \rho_i)$. Therefore, we have the following diagram:

$$(\diamond) \qquad \begin{array}{ccccc} \Sigma_{v_1} & \supset & \Sigma_e & \subset & \Sigma_{v_2} \\ {\scriptstyle \Phi_1}\downarrow & & & & \downarrow{\scriptstyle \Phi_2} \\ (G_1, \pi_1, \rho_1) & & & & (G_2, \pi_2, \rho_2) \end{array}$$

Let us give an important remark about the map Φ_i when Γ_i satisfies the condition (ii) in Assumption (†). Furman [2] shows that the image of the measure on Σ_i via Φ_i is a linear combination of the Haar measure on G_i and atomic measures on G_i. In this case, the Haar measure can not be involved as shown in the following.

Lemma 5.5. *The image of the measure on Σ_{v_i} via Φ_i is an atomic measure on G_i, that is, it is supported on a countable subset of G_i.*

The existence of the smaller ME coupling Σ_e of infinite subgroups of infinite index in Γ_i and Λ_i plays an important role in the proof of this lemma. If we assume that the image of the measure on Σ_{v_i} is the Haar measure on G_i, then we can deduce a contradiction by using Moore's theorem about unitary representations of Lie groups (see Theorem 2.2.19 in [10]). Thanks to this lemma, we may assume that the image of Φ_i is contained in the *commensurator* $\mathrm{Comm}_{G_i}(\pi_i(\Gamma_i))$ of $\pi_i(\Gamma_i)$ in G_i by replacing Φ_i and ρ_i if necessary. The group $\mathrm{Comm}_{G_i}(\pi_i(\Gamma_i))$ by definition consists of all elements $g \in G_i$ such that $[\pi_i(\Gamma_i) : g^{-1}\pi_i(\Gamma_i)g \cap \pi_i(\Gamma_i)] < \infty$ and $[\pi_i(\Gamma_i) : g\pi_i(\Gamma_i)g^{-1} \cap \pi_i(\Gamma_i)] < \infty$.

To prove rigidity results discussed below, it is important to understand the two maps Φ_1, Φ_2 defined on the common subset Σ_e. If the "difference" between them is not so big, then one can deduce OE or ME rigidity results. This difference can be small if we impose ergodicity assumptions on the actions of the groups Γ_1, Γ_2, A on Σ/Λ. The following is one consequence of the above argument stated in terms of OE.

Theorem 5.6. *Under Assumption (†), let Λ be a discrete group and suppose that two ergodic, essentially free and measure-preserving actions $\Gamma \curvearrowright (X, \mu)$, $\Lambda \curvearrowright (Y, \nu)$ on standard finite measure spaces are WOE. We assume the following two conditions:*

- (i) *Let $X' \subset X$ and $Y' \subset Y$ be Borel subsets of positive measure on which there exists a Borel isomorphism preserving the class of measures and the orbits of the actions of Γ and Λ. Then $\mu(X')/\mu(X) \leq \nu(Y')/\nu(Y)$; and*
- (ii) *Either the action $A \curvearrowright X$ is aperiodic or the actions $\Gamma_1 \curvearrowright X$ and $\Gamma_2 \curvearrowright X$ are both aperiodic, the action $A \curvearrowright X$ is ergodic and A is ICC.*

Then the cocycle arising from the WOE is cohomologous to the constant cocycle. In particular, the two actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are conjugate.

We say that an action of a discrete group Γ on a measure space is *aperiodic* if any finite index subgroup of Γ acts ergodically. It seems that one can construct counterexamples by using the universal property of amalgamated free products if one of the assumptions in Theorem 5.6 is dropped.

5.3. **Examples.** We shall give several examples satsifying Assumption (†).

Example 5.7. Let M be the surface in Theorem 3.3 (ii) and let $\{F_\pm\}$ be a pair of points in the Thurston boundary for M which is fixed by a pseudo-Anosov element in $\mathrm{Mod}^*(M)$. Let A be the stabilizer of the pair $\{F_\pm\}$. Then A is virtually isomorphic to $\mathbb{Z}$ and is almost malnormal in $\mathrm{Mod}^*(M)$.

Example 5.8. Let $n \geq 2$ be an integer and take a flag F of the vector space $\mathbb{R}^n$ such that it is a sequence of subspaces all of whose bases can be chosen as a subset of the standard basis of $\mathbb{R}^n$. Let $\Gamma = SL(n, \mathbb{Z})$ and let A be the stabilizer of F in Γ. Then $LQN_\Gamma(A) = A$. When $n \geq 3$, $SL(n, \mathbb{Z})$ satisfies Kazhdan's property (T).

In [7], many examples of subgroups of $SL(n, \mathbb{Z})$ which are almost malnormal in $SL(n, \mathbb{Z})$ are presented.

5.4. **ME rigidity.** Theorem 5.6 assumes strong conditions on actions of subgroups of Γ. The goal of this subsection is to deduce rigidity by adding a stronger assumption on the pairs of groups $A_i < \Gamma_i$ for $i = 1, 2$ instead of the ergodicity assumptions on their actions. This leads to a new example of ME rigid groups. Recall that a discrete group Γ is said to be *ME rigid* if any discrete group which is ME to Γ is virtually isomorphic to Γ.

Assumption 5.9. We denote the following assumption by (‡): Under Assumption (†), suppose the following two conditions: Put $\overline{\Gamma_i} = \pi_i(\Gamma_i)$, $\overline{A_i} = \pi_i(A_i)$, $C_i = \mathrm{Comm}_{G_i}(\overline{\Gamma_i})$ and $C(\overline{A_i}) = \mathrm{Comm}_{C_i}(\overline{A_i})$ for $i = 1, 2$. Recall that we assumed that π_i is injective.

- (i) The isomorphism $\pi_2 \circ \phi \circ \pi_1^{-1} \colon \overline{A_1} \to \overline{A_2}$ can be extended to an isomorphism $\overline{\phi} \colon C(\overline{A_1}) \to C(\overline{A_2})$.
- (ii) For $i = 1, 2$, the delta measure δ_e on the neutral element of C_i is the only probability measure on C_i which is invariant under conjugation by each element of $\overline{A_i}$.

Take the amalgamated free product $G = \langle C_1, C_2 | C(\overline{A_1}) \simeq_{\overline{\phi}} C(\overline{A_2}) \rangle$. Note that π_1 and π_2 induce an injective homomorphism $\pi \colon \Gamma \to G$.

It is not difficult to see that the second condition assures the uniqueness of the extension of the isomorphism $\pi_2 \circ \phi \circ \pi_1^{-1}$ in the first condition. In the diagram ($\diamond$), we may assume that the map Φ_i is valued in C_i for $i = 1, 2$, and therefore in the sole group G. The condition (ii) is so strong that one can prove that the two maps Φ_1, Φ_2 are equal on Σ_e up to multiplication of an element of G. This fact helps to show the following.

Theorem 5.10. *Under Assumption (‡), let Σ be an ME coupling of Γ and a discrete group Λ. Suppose that there are a homomorphism $\rho \colon \Lambda \to \mathrm{Aut}(T)$ with finite kernel and an almost $(\Gamma \times \Lambda)$-equivariant Borel map $\Phi \colon \Sigma \to (\mathrm{Aut}(T), \imath, \rho)$. Then there exist a homomorphism $\rho_0 \colon \Lambda \to G$ with finite kernel and an almost $(\Gamma \times \Lambda)$-equivariant Borel map $\Phi_0 \colon \Sigma \to (G, \pi, \rho_0)$.*

Since G is countable, $\pi(\Gamma)$ and $\rho_0(\Lambda)$ are commensurable, and this theorem implies the following.

Corollary 5.11. *Under Assumption (‡), Γ is ME rigid.*

Remark 5.12. One can show the following stronger theorem than Theorem 5.10 along the same idea:

Theorem 5.13. *Under Assumption (‡), let Σ be a self ME coupling of Γ such that there is an almost $(\Gamma \times \Gamma)$-equivariant Borel map $\Phi \colon \Sigma \to (\mathrm{Aut}(T), \imath, \imath)$. Then there exists an almost $(\Gamma \times \Gamma)$-equivariant Borel map $\Phi_0 \colon \Sigma \to (G, \pi, \pi)$.*

Theorem 5.10 follows from this theorem by (the proof of) the representation theorem 3.2. By using this theorem and the countability of G, we can conclude the following.

Theorem 5.14. *Under Assumption* (‡), *if* $\mathrm{Comm}(\Gamma)$ *is countable (in particular, if* Γ *is finitely generated), then* Γ *is ME coupling rigid with respect to* $(\mathrm{Comm}(\Gamma), \imath)$, *where* $\mathrm{Comm}(\Gamma)$ *is equipped with the discrete Borel structure, and* $\imath\colon \Gamma \to \mathrm{Comm}(\Gamma)$ *is the natural homomorphism.*

Note that under Assumption (†), Γ is ICC, and therefore the homomorphism $\imath\colon \Gamma \to \mathrm{Comm}(\Gamma)$ is injective.

This article ends with presenting an example satisfying Assumption (‡). Let $\{e_1, e_2, e_3\}$ be the standard basis for the vector space $\mathbb{R}^3$. Let F be the flag consisting of the subspaces $\{0\} \subset \langle e_1 \rangle \subset \mathbb{R}^3$. Let A be the stabilizer of $SL(3, \mathbb{Z})$, which consists of all matrices in $SL(3, \mathbb{Z})$ whose $(2, 1)$-, $(3, 1)$-entries are both 0. Then we can prove that the amalgamated free product $SL(3, \mathbb{Z}) *_A SL(3, \mathbb{Z})$ satisfies Assumption (‡).

Corollary 5.15. *The amalgamated free product* $SL(3, \mathbb{Z}) *_A SL(3, \mathbb{Z})$ *is ME rigid.*

We note that there are many flags of $\mathbb{R}^n$ with $n \geq 3$ such that the amalgamated free products constructed as above satisfy Assumption (‡) and therefore they are ME rigid. It is an interesting problem to find subgroups of mapping class groups such that the associated amalgamated free products satisfy Assumption (‡).

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