Rigidity of Margulis and Zimmer and measure equivalence rigidity of Furman

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Yoshikata Kida

Rigidity of Margulis and Zimmer

Thm (Margulis' superrigidity)

$$G = SL_3\mathbb{R} \supset H$$
$$\vee$$
$$\Gamma \text{ lattice} \qquad\qquad \{(z_{ij})_{i,j=1}^{3} \mid \det = 1\}$$

If $\pi : \Gamma \to H$ homo $\qquad \| \text{ alg. variety}$

s.t. $\pi(\Gamma) < H < SL_3\mathbb{C}$ is

Zariski dense,

then $\exists\, \tilde{\pi} : G \xrightarrow{\sim} H$ extending π. $\lrcorner$

✓ Zariski topology := closed sets are

zeros of a polynomial.

Rem (The Borel density)

Every lattice in $SL_3\mathbb{R}$ is

Zar. dense in $SL_3\mathbb{C}$.

<u>Cor</u> (The Mostow-type rigidity)

$$\begin{array}{ccc} G = SL_2\mathbb{R} & = & H \\ \vee & & \vee \\ \Gamma \text{ lattice} & & \Lambda \text{ lattice} \end{array}$$

If $\pi : \Gamma \xrightarrow{\sim} \Lambda$, then $\exists\, \tilde{\pi} : G \xrightarrow{\sim} H$ extending π.

<u>Rem</u> The same kind of rigidity holds for (P)$SL_n\mathbb{R}$ with $n \geq 3$, but does not hold for (P)$SL_2\mathbb{R}$.

Margulis' rigidity can extend to the following Zimmer's rigidity:

(Ref: R. J. Zimmer, Ergodic theory and semisimple groups. 1984)

Thm (Zimmer's cocycle superrigidity)

$G = SL_3\mathbb{R} = H$

$\curvearrowright (X, \mu)$ p.m.p. on a standard prob. sp.

$$\left(\begin{array}{ll} G \times X \to X & \text{measurable} \\ (g, x) \mapsto gx & \dots \text{ etc} \end{array} \right)$$

$\alpha : X \times G \to H$ cocycle

$$\left(\alpha(x, g_1 g_2) = \alpha(x, g_1)\, \alpha(g_1^{-1} x, g_2) \right)$$

with Zariski dense range.

Then $\exists\, \pi : G \xrightarrow{\sim} H$

$\exists\, \varphi : X \to H$ measurable

s.t. $\alpha(x, g) = \varphi(x)^{-1} \pi(g)\, \varphi(g^{-1} x)$. $\lrcorner$

Why does Zimmer generalize Margulis?

Mackey's virtual group viewpoint

G group lcsc

✓ $\{$subgroups of $G\}$ closed $/\text{conj.}$

L

$\downarrow$

$G \curvearrowright G/L$

$\updownarrow$ 1 : 1

$\{$transitive actions of $G\}/\text{isom}$

with a quasi-inv. Radon measure on a standard space

$\cap$

$\{$general actions of $G\}/\text{isom}$

$G \curvearrowright (X, \mu)$

call these actions a virtual subgroup of G.

✡ For a lcsc group G,

$\{\text{lattices in } G\}/\text{conj}$ $\qquad \Gamma$

$\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$

$\subset \{\text{p.m.p. actions of } G\}/\text{isom}$ $\qquad G \curvearrowright G/\Gamma$

<u>Q</u> What is a homomorphism from a virtual group $G \curvearrowright X$ into a group H?

The answer is a cocycle!

$$\alpha : X \times G \to H.$$

In fact, for $L < G$ closed, choose a section $s : G/L \to G$.

Then :

$$\{\text{homo } \pi : L \to H\}/_{H\text{-conj.}} \qquad \pi$$

$$\Big\updownarrow 1:1 \qquad\qquad \Big\downarrow$$

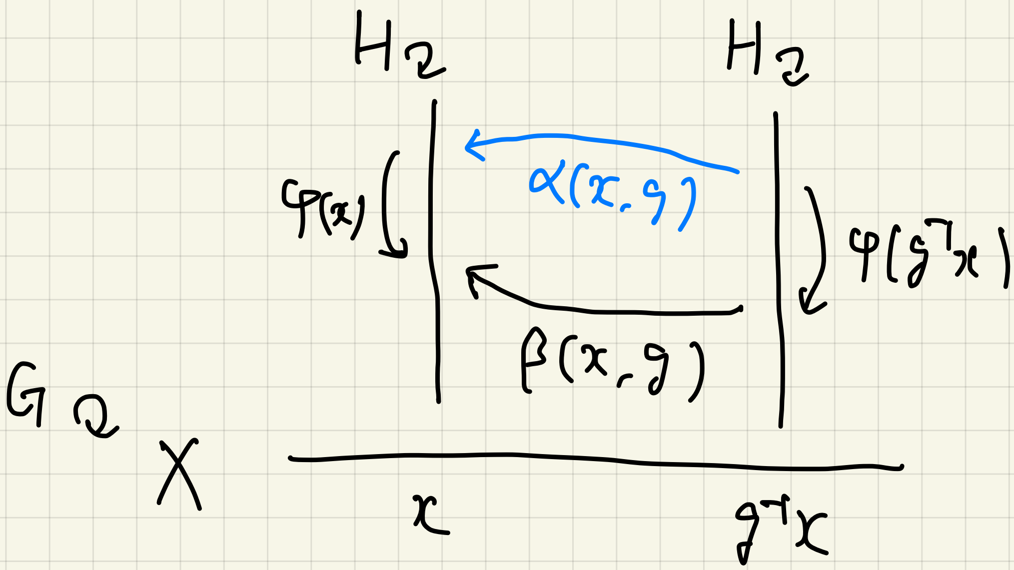
$$\{\text{cocycle } \alpha : G/L \times G \to H\}/_{\sim} \qquad \alpha_\pi$$

where

- $\alpha_\pi(x, g) := \pi\left(s(x)^{-1}\, g\, s(g^{-1}x)\right).$

- For $G \curvearrowright X$ and $\alpha, \beta : X \times G \to H$ cocycles

$$\alpha \sim \beta \overset{\text{def}}{\iff} \exists\, \varphi : X \to H \quad \alpha(x, g) = \varphi(x)^{-1} \beta(x, g)\, \varphi(g^{-1}x)$$



✪ Apply Zimmer to $G \curvearrowright G/\Gamma$.
p.m.p.

$G = SL_2\mathbb{R} = H$
$\cup$
Γ lattice.

Then for every cocycle $\alpha : G/\Gamma \times G \to H$
with Zariski dense range,

$\exists\, \pi : G \xrightarrow{\sim} H \quad \exists\, \varphi : G/\Gamma \to H$

s.t. $\alpha(x, g) = \varphi(x)^{-1} \pi(g)\, \varphi(g^{-1}x)$.

i.e. $\alpha \sim \pi$

$$\left(\begin{array}{l} \pi \text{ defines the cocycle } G/\Gamma \times G \to H \\ \qquad\qquad\qquad\qquad\qquad (x, g) \mapsto \pi(g) \end{array} \right)$$

Since $\alpha \leftrightarrow$ homo : $\Gamma \to H$,

"Zimmer for $G \curvearrowright G/\Gamma$" is exactly "Margulis".

Cor (of Zimmer)

$G = SL_3\mathbb{R} = H$

$G \curvearrowright (X,\mu)$, $H \curvearrowright (Y,\nu)$ free p.m.p.

If these are OE, then they are isom

w.r.t. some $\pi : G \xrightarrow{\sim} H$.

Pf $\exists f : X \xrightarrow{\sim} Y$ $f(Gx) = Hf(x)$.

Define $\alpha : X \times G \to H$ by

$$f(x) = \alpha(x, g) f(g^{-1}x).$$

Then α is a cocycle (with Z. dense range).

Apply Zimmer. $\exists \pi : G \xrightarrow{\sim} H$

$\exists \varphi : X \to H$

s.t. $\alpha(x,g) = \varphi(x)^{-1} \pi(g) \varphi(g^{-1}x)$.

Define $\tilde{f} : X \to Y$ by

$$\tilde{f}(x) = \varphi(x) f(x).$$

Then

$$\tilde{f}(g^{-1}x) = \varphi(g^{-1}x) f(g^{-1}x)$$

$$= \pi(g)^{-1} \varphi(x) \alpha(x, g) f(g^{-1}x)$$

$$= \pi(g)^{-1} \varphi(x) f(x) = \pi(g)^{-1} \tilde{f}(x).$$

Hence $\tilde{f}$ is G-equiv. w.r.t. π.

f : inj., $\tilde{f}$: G-equiv. $\Rightarrow$ $\tilde{f}$: inj.

Since (almost) every orbit of $H \curvearrowright Y$ lies in $f(X)$, it also lies in $\tilde{f}(X)$.

Thus $\tilde{f} : (X, \mu) \xrightarrow{\sim} (Y, \nu)$.

$\boxtimes$

$\circledast$ (for actions of lattices)

$$\begin{array}{ccc} G = SL_2\mathbb{R} & = & H \\ \cup & & \cup \\ \Gamma \text{ lattice} & & \Lambda \text{ lattice.} \end{array}$$

$\Gamma \curvearrowright (X, \mu)$, $\Lambda \curvearrowright (Y, \nu)$ free p.m.p.

Assume these are OE. so

$\exists f : X \xrightarrow{\sim} Y \quad f(\Gamma x) = \Lambda f(x).$

and define the cocycle $\alpha : X \times \Gamma \to \Lambda$

by $f(x) = \alpha(x, \gamma) f(\gamma^{-1} x)$.

We define $\tilde{\alpha} : G/\Gamma \times X \times G \to \Lambda$:

$\Gamma \curvearrowright X \xrightarrow{\text{induced}} G \curvearrowright G/\Gamma \times X$

Let $s : G/\Gamma \to G$ be a section s.t. $s(e\Gamma) = e$.

$\beta : G/\Gamma \times G \to \Gamma$

$\beta(b, g) := s(b)^{-1} g\, s(g^{-1} b)$.

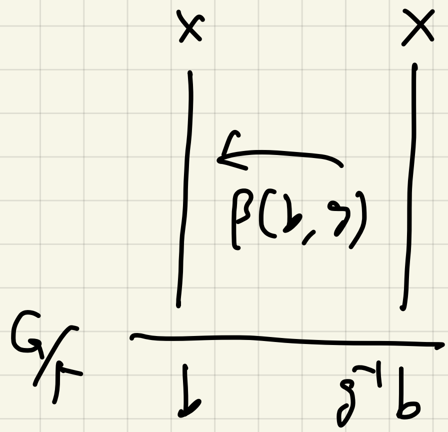
Now define $G \curvearrowright G/\Gamma \times X$ by

$$g(b,x) := (gb, \beta(gb, g)x)$$

Define $\tilde{\alpha} : G/\Gamma \times X \times G \to \Lambda$ by

$$\tilde{\alpha}(b,x,g) = \alpha(x, \beta(b,g)).$$

Apply Zimmer to $\tilde{\alpha}$.



$\exists\, \pi : G \twoheadrightarrow H$

$\exists\, \varphi : G/\Gamma \times X \to H$

s.t. $\tilde{\alpha}(b,x,g) = \varphi(b,x)^{-1}\pi(g)\varphi(g^{-1}(b,x))$.

We suppose that a this holds

for $b = e\Gamma$ and $\forall g \in \Gamma$.

Then :

$\forall \gamma \in \Gamma$

$$\tilde{\alpha}(e\Gamma, x, \gamma) = \varphi(e\Gamma, x)^{-1} \pi(\gamma)\, \varphi(\underbrace{\gamma^{-1}(e\Gamma, x)})$$

$$\tilde{\alpha}(e\Gamma, x, \gamma) = \alpha(x, \gamma)$$

$$\gamma^{-1}(e\Gamma, x) = (e\Gamma, \beta(e\Gamma, \gamma^{-1})x) = (e\Gamma, \gamma^{-1}x)$$

$$\varphi_0(x) := \varphi(e\Gamma, x)$$

$$\varphi_0 : X \to H$$

$$= \varphi_0(x)^{-1} \pi(\gamma)\, \varphi_0(\gamma^{-1}x)$$

Thus $\alpha : X \times \Gamma \to \Lambda < H$

is equiv. $(\sim)$ to $\pi : \Gamma \to H$

(as H-valued cocycles).

However these are not necessarily equiv.

as Λ-valued cocycles.

Rem Margulis & Zimmer are generalized into : $G = SL_k\mathbb{R}$, $H = SL_\ell\mathbb{R}$ $k, \ell \geq 3$

$\curvearrowright (X, \mu)$ p.m.p.

$\alpha : X \times G \to H$ cocycle with Z. dense range

Then $\exists \pi : G \xrightarrow{\simeq} H$ (thus $k = \ell$)

$\exists \varphi : X \to H$

s.t. $\alpha(x, g) = \varphi(x)^{-1} \pi(g) \varphi(g^{-1}x)$.

Hence if $\Gamma < G$, $\Lambda < H$ lattices

and $\Gamma \curvearrowright (X, \mu) \underset{OE}{\sim} \Lambda \curvearrowright (Y, \nu)$,

free p.m.p.

then the argument of ⊛ proves

$G \cong H$ (but not necessarily $\Gamma \simeq \Lambda$)

$\lrcorner$

Measure equivalence couplings

A motivating example:

G: lcsc group $\quad \Gamma, \Lambda < G$ lattices.

$\to \quad \Gamma \times \Lambda \curvearrowright (G, \mathrm{Haar})$

$$(\gamma, \lambda) g := \gamma g \lambda^{-1}$$

Def (Gromov) $\quad \Gamma, \Lambda$ countable groups

A (Γ, Λ)-coupling is a standard measure space (Σ, m)

σ-finite, non-zero

endowed with measure-preserving actions $\Gamma \curvearrowright (\Sigma, m)$

$\Lambda \curvearrowright$

s.t.

- the two actions commute i.e.

$$\gamma(\lambda x) = \lambda(\gamma x) \quad \forall \gamma \in \Gamma,\ \lambda \in \Lambda,\ x \in \Sigma.$$

(so the Λ-action is denoted by $(\lambda, x) \mapsto x\lambda^{-1}$.)

- Each of the Γ-action and the Λ-action admits a fundamental domain of finite measure i.e.

$\exists\, X, Y \subset \Sigma$ measurable

s.t. $m(X) < \infty$, $m(Y) < \infty$

and $\Sigma = \bigsqcup_{\gamma \in \Gamma} \gamma Y = \bigsqcup_{\lambda \in \Lambda} X \lambda^{-1}$. ∎

In this case, Γ and Λ are called **measure equivalent**.

(In fact, this is an equivalence relation.)

✡ G lcsc. group $\Gamma, \Lambda < G$ lattices

Then $\Gamma \times \Lambda \curvearrowright (G, \mathrm{Haar})$ is

$$(\gamma, \lambda) g := \gamma g \lambda^{-1}$$

a (Γ, Λ)-coupling.

<u>Thm</u> (a reformulation of ✡)

$G = SL_3\mathbb{R}$, $\Gamma, \Lambda < G$ lattices

$\forall (\Sigma, m)$: (Γ, Λ)-coupling

$\exists! \, \Phi : \Sigma \to \mathrm{Aut}(G)$ $\Gamma \times \Lambda$-equiv.

where $\Gamma \times \Lambda \curvearrowright \mathrm{Aut}(G)$

from left and right.

(<u>Rem</u> $G \underset{\text{finite index.}}{<} \mathrm{Aut}(G)$ as inner auto.)

<u>Pf</u> $\exists X, Y \subset \Sigma$

$$\Sigma = \bigsqcup_{\gamma \in \Gamma} \gamma Y = \bigsqcup_{\lambda \in \Lambda} X \lambda^{-1}$$

$$\Gamma \curvearrowright X \simeq \Sigma / \Lambda \ , \quad \Lambda \curvearrowright Y \simeq \Sigma / \Gamma$$

$$(\gamma, x) \mapsto \underset{\text{dot}}{\gamma \cdot x} \qquad (\lambda, y) \mapsto \underset{\text{dot}}{\lambda \cdot y}$$

Define $\alpha : \Gamma \times X \to \Lambda$ so that

$$(\gamma \cdot x =)\ \gamma x \alpha(\gamma, x)^{-1} \in X .$$

Then α is a cocycle i.e.

$$\alpha(\gamma_1 \gamma_2, x) = \alpha(\gamma_1, \gamma_2 \cdot x)\, \alpha(\gamma_2, x) .$$

Apply the argument in $\circledast$ to α

(we can!).

$$\exists \pi : G \xrightarrow{\sim} G \quad \exists \varphi : X \to G$$

s.t. $\alpha(\gamma, x) = \varphi(\gamma \cdot x)\, \pi(\gamma)\, \varphi(x)^{-1}$.

Let $\Phi : \Sigma \to \mathrm{Aut}(G)$ by

$$\Phi(x\lambda^{-1}) = (\varphi(x)\pi)^{-1}\lambda^{-1}$$

for $x \in X$ and $\lambda \in \Lambda$.

$G < \mathrm{Aut}(G)$
$\pi \cdot g \cdot \pi^{-1}$
$= \pi(g)$
for $g \in G$.

Then Φ is Λ-equiv and

$\gamma \in \Gamma$, $x \in X$

$$\begin{aligned}\Phi(\gamma x) &= \Phi((\gamma \cdot x)\alpha(\gamma, x)) \\ &= (\varphi(\gamma \cdot x)\pi)^{-1}\alpha(\gamma, x) \\ &= \pi^{-1}\ \pi(\gamma)\ \varphi(x)^{-1} \\ &= \gamma\,\pi^{-1}\varphi(x)^{-1} = \gamma\,\Phi(x).\end{aligned}$$

Thus Φ is $\Gamma \times \Lambda$-equiv.

For uniqueness of Φ, we use:

<u>Fact</u> δ_e is the only prob. measure on $\mathrm{Aut}(G)$ that is Γ-conj. inv.

Let $\Phi_1, \Phi_2 : \Sigma \to \mathrm{Aut}(G)$ be

$\Gamma \times \Lambda$-equiv.

Define $\Theta : \Sigma \to \mathrm{Aut}(G)$ by

$$\Theta(x) := \Phi_1(x)\, \Phi_2(x)^{-1}.$$

Then $\Theta(\gamma x \lambda^{-1}) = \gamma\, \Theta(x)\, \gamma^{-1}$

$\Theta \rightsquigarrow \Sigma/\Lambda \longrightarrow \mathrm{Aut}(G)$

$m \mapsto \Gamma$-conj. inv. prob. meas.

By Fact, $\Theta(\Sigma) = \{e\}$.

Therefore $\Phi_1(x) = \Phi_2(x)$ a.e. $\blacksquare$

We further get the following:

Thm (Furman)

$G = SL_3\mathbb{R}$, $\Gamma, \Lambda < G$ lattices

Let (Σ, m) be a (Γ, Λ)-coupling.

$\to \exists! \ \Phi : \Sigma \to \mathrm{Aut}(G)$ $\Gamma \times \Lambda$-equiv.

Then $\Phi_* m$ is a linear combination of the Haar measure and atomic measures.

That is, every (Γ, Λ)-coupling arises from:

- $\Gamma \times \Lambda \curvearrowright (G, \text{Haar})$ and
- countable couplings.

(proved by using Ratner's measure classification theorem.)

Rem

Since $\mathrm{Aut}(G) = \bigsqcup_{\gamma \in \Gamma} \gamma Y$

for some $Y \subset \mathrm{Aut}(G)$

and $\Omega = \bigsqcup_{\gamma \in \Gamma} \gamma \Phi^{-1}(Y)$,

the measure $\Phi_* m$ is σ-finite.

Hence every atom of $\Phi_* m$ has finite measure.

<u>Rem</u> $G = SL_3\mathbb{R}$, $\Gamma = \Lambda < G$ lattices

$\Gamma \times \Lambda \curvearrowright (G, \text{Haar})$
from left and right

$$G = \bigsqcup_{\gamma \in \Gamma} \gamma Y = \bigsqcup_{\lambda \in \Lambda} X \lambda^{-1}.$$

$\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$

$\alpha : \Gamma \times X \to \Lambda$ s.t. $\underbrace{\gamma x \alpha(\gamma, x)^{-1}}_{= \gamma \cdot x} \in X$.

Then $\alpha(\gamma, x) = (\gamma \cdot x)^{-1} \gamma x$

so if $\varphi : X \to G$, $\varphi(x) := x$,

then $\alpha(\gamma, x) = \varphi(\gamma \cdot x)^{-1} \gamma \varphi(x)$.

Uniqueness of Φ implies that

α is not equiv. to $\mathrm{id} : \Gamma \to \Lambda$

as a Λ-valued cocycle.

Furman's homomorphisms

Q If a countable group Λ is ME to a lattice in $SL_3\mathbb{R}$, (meas. eq.)

what can we say about Λ?

Thm (Furman)

$\Gamma < G = SL_3\mathbb{R}$ lattice

Λ: arbitrary countable group

(Σ, m) : (Γ, Λ)-coupling

Then $\exists\, \rho : \Lambda \to \mathrm{Aut}(G)$ homo.

s.t. $\#\ker\rho < \infty$, $\rho(\Lambda) < \mathrm{Aut}(G)$ lattice

Moreover $\exists\, \Phi_0 : \Sigma \to \mathrm{Aut}(G)$ $\Gamma\times\Lambda$-equiv.

i.e. $\Phi_0(\gamma x \lambda^{-1}) = \gamma\, \Phi_0(x)\, \rho(\lambda)^{-1}$. $\lrcorner$

Pf Let $\Gamma \times \Gamma \times \Lambda \curvearrowright \Sigma \times \Sigma$ by

$$(\gamma_1, \gamma_2, \lambda)(x, y) = (\gamma_1 x \lambda^{-1}, \gamma_2 y \lambda^{-1}).$$

Define $\Omega := \Sigma \times \Sigma / \Lambda$, and

we have $\Gamma \times \Gamma \curvearrowright \Omega$.

Then Ω is a (Γ, Γ)-coupling.

By the last Thm,

$$\exists !\ \Phi : \Omega \to \mathrm{Aut}(G) =: \overline{G}$$

$\Gamma \times \Gamma$-equiv.

We aim to find

- a homo $\rho : \Lambda \to \overline{G}$ and
- a $\Gamma \times \Lambda$-equiv. map $\Psi : \Sigma \to \overline{G}$

s.t. $\Phi([x, y]) = \Psi(x) \Psi(y)^{-1}$.

$\Phi\ (x,y,z) \mapsto \Phi([x,z])\,\Phi([y,z])^{-1}$.

does not depend on z for a.e. $(x,y) \in \Sigma^2$.

$\underline{H}$ Define $\Theta : \Sigma^4 \to \overline{G}$ by

$$\Theta(x,y,z,w) = \Phi([x,z])\,\Phi([y,z])^{-1}$$
$$\Phi([y,w])\,\Phi([x,w])^{-1}.$$

Then $\forall \gamma \in \Gamma \quad \forall \lambda \in \Lambda$

- $\Theta(\gamma x, y, z, w) = \gamma\, \Theta(x,y,z,w)\,\gamma^{-1}$
- $\Theta(x, \gamma y, z, w) = \Theta(x,y,z,w)$

$$= \Theta(x,y,\gamma z,w) = \Theta(x,y,z,\gamma w).$$

- $\Theta(\lambda x, \lambda y, \lambda z, \lambda w) = \Theta(x,y,z,w)$.

$\Lambda \curvearrowright \Sigma$ is denoted by $(\lambda, x) \mapsto \lambda x$.

So Θ induces :

$$\underbrace{\left[\Sigma \times \Sigma/\Gamma \times \Sigma/\Gamma \times \Sigma/\Gamma\right]/\Lambda}_{\Gamma \curvearrowright} \to \bar{G} \qquad \Gamma \curvearrowright \text{conj.}$$

Γ-equiv. map.

This has a Γ-inv. prob. meas.

Since δ_e is the only prob. meas. on $\bar{G}$ that is Γ-conj. inv,

we get $\Theta(\Sigma^4) = \{e\}$. $\blacksquare$

For each $x \in X$, define a cocycle

$$\alpha_x : \Lambda \times \Sigma \to \bar{G}$$

$$\alpha_x(\lambda, y) = \Phi([x, \lambda y])\, \Phi([x, y])^{-1}.$$

Claim For a.e. $x \in \Sigma$, $\forall \lambda \in \Lambda$,

$\alpha_x(\lambda, y)$ does not depend on y.

Pf By ✡, for a.e. $(x, y, z, w) \in \Sigma^4$

$$\Phi([x,z])\Phi([y,z])^{-1} = \Phi([x,w])\Phi([y,w])^{-1}$$

So

$$\Phi([x,w])^{-1}\Phi([x,z]) = \Phi([y,w])^{-1}\Phi([y,z]).$$

Hence for a.e. $(z,w) \in \Sigma^2$,

the map $x \mapsto \Phi([x,w])^{-1}\Phi([x,z])$

is const a.e.

For a.e. $(z,w) \in \Sigma^2$, a.e. $x \in \Sigma$, $\forall \lambda \in \Lambda$

$$\Phi([x,w])^{-1}\Phi([x,z])$$

$$= \Phi([\lambda x, w])^{-1}\Phi([\lambda x, z]).$$

So

$$\Phi([x,z])\,\Phi([\lambda x,z])^{-1}$$
$$= \Phi([x,w])\,\Phi([\lambda x,w])^{-1}.$$

$z \to \lambda z$, $w \to \lambda w$

Thus for a.e. $(z,w) \in \Sigma^2$,

for a.e. $x \in \Sigma$, $\forall \lambda \in \Lambda$

$$\Phi([x,\lambda z])\,\Phi([x,z])^{-1}$$
$$= \Phi([x,\lambda w])\,\Phi([x,w])^{-1}$$

$\boxed{}$

Claim and the cocycle id

$$\alpha_x(\lambda_1\lambda_2, y) = \alpha_x(\lambda_1, \lambda_2 y)\,\alpha_x(\lambda_2, y)$$

imply that for a.e. $x \in \Sigma$,

$$\rho_x : \Lambda \to \overline{G}, \quad \rho_x(\lambda) = \alpha_x(\lambda, y)$$

is a homo.

Fix a certain $x = x_0 \in \Sigma$.

Set $\overline{\Phi}_0(y) := \overline{\Phi}([x_0, y])^{-1}$.

Then
$$\begin{array}{ccc} \alpha_{x_0}(\lambda, y) & = & \overline{\Phi}([x_0, \lambda y])\, \overline{\Phi}([x_0, y])^{-1} \\ \| & & \| \\ \rho_{x_0}(\lambda) & & \overline{\Phi}_0(\lambda y)^{-1}\, \overline{\Phi}_0(y). \end{array}$$

Put $\rho = \rho_{x_0}$, and we get

$$\overline{\Phi}_0(\lambda y) = \overline{\Phi}_0(y)\, \rho(\lambda)^{-1}.$$

Thus $\overline{\Phi}_0 : \Sigma \to \overline{G}$ is $\Gamma \times \Lambda$-equiv.

It remains to show that

① $\# \ker \rho < \infty$.

② $\rho(\Lambda)$ is a lattice in $\overline{G}$.

Pf of ①

Pick $Y \subset \overline{G}$ s.t. $\overline{G} = \bigsqcup_{\gamma \in \Gamma} \gamma Y$.

Then $\Sigma = \bigsqcup_{\gamma \in \Gamma} \gamma \, \Phi_0^{-1}(Y)$

and hence $m(\Phi_0^{-1}(Y)) < \infty$.

$\forall \lambda \in \ker \rho \quad Y \rho(\lambda)^{-1} = Y$

and $\Phi_0^{-1}(Y) \lambda^{-1} = \Phi_0^{-1}(Y)$.

Thus $\# \ker \rho < \infty$. ▨

Pf of ②

Recall $\Omega = \Sigma \underset{\wedge}{\times} \Sigma$.

Let η be the measure on Ω.

We have $\Phi : \Omega \to \overline{G}$

$\Gamma \times \Gamma$-equiv. map

By the last Thm, $\Phi_* \eta$ is a linear comb. of the Haar measure and atomic measures.

We assume $\Phi_* \eta$ is atomic.

(otherwise we need a further fact, the p-adic ver. of Zimmer's CSR.)

Recall we defined

$$\overline{\Phi}_0(\gamma) := \overline{\Phi}([x_0, \gamma])^{-1} \in [\operatorname{Supp} \overline{\Phi}_* \eta]^{-1}$$

So we may assume $\overline{\Phi}_0(\Sigma)$ is countable.

Let $g \in \overline{\Phi}_0(\Sigma)$ be an atom of $\overline{\Phi}_{0*} m$.

Consider $\overline{\Phi}_0 : \Gamma \overline{\Phi}_0^{-1}(g) \Lambda \to \Gamma g \rho(\Lambda)$.

$$\rightarrow \underbrace{\Gamma \overline{\Phi}_0^{-1}(g) \Lambda / \Lambda}_{\text{This has a } \Gamma\text{-inv. prob. meas.}} \to \Gamma g \rho(\Lambda) / \rho(\Lambda) \quad \Gamma\text{-equiv.}$$

Hence $\# \Gamma g \rho(\Lambda) / \rho(\Lambda) < \infty$.

(since $\Gamma \curvearrowright \Gamma g \rho(\Lambda) / \rho(\Lambda)$ is transitive.)

$\exists\, \Gamma_1 < \Gamma$ finite index

s.t. $\forall \gamma \in \Gamma_1 \quad \gamma g \rho(\Lambda) = g \rho(\Lambda)$

So $\quad g^{-1} \Gamma_1 g < \rho(\Lambda)$. $\qquad (1)$

The argument for $_{\Gamma}\backslash \Gamma g \rho(\Lambda)$ implies

$\exists\, \Lambda_1 < \rho(\Lambda)$ finite index

s.t. $\forall \lambda \in \Lambda_1 \quad \Gamma g \lambda^{-1} = \Gamma g$.

So $\quad g \Lambda_1 g^{-1} < \Gamma$. $\qquad (2)$

Then
$$\begin{array}{ccc} g^{-1}\Gamma_1 g \cap \Lambda_1 & \underset{f.i.}{\overset{(1)}{<}} & g^{-1}\Gamma_1 g . \\ {\scriptstyle (2)}\ \wedge\ {\scriptstyle f.i.} & & \\ \Lambda_1 & & \end{array}$$

Therefore $g^{-1}\Gamma_1 g \cap \Lambda_1$ is of finite index in both $g^{-1}\Gamma g$ and $\rho(\Lambda)$.

Thus $\rho(\Lambda) < \bar{G}$ is a lattice. $\boxtimes$

Further examples of groups with rigidity

Some countable group Γ admits a
countable group C s.t.

(i) $\Gamma < C$ and δ_e is the only prob. meas. on C that is Γ-conj. inv.

(ii) every (Γ, Γ)-coupling has a $\Gamma \times \Gamma$-equiv. quotient onto C.

<u>e.g.</u> ① The mapping class group of a cpt ori. surface (with a few exceptions)

(Kida)

② $SL_3\mathbb{Z} \underset{P}{*} SL_3\mathbb{Z}$, $P = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$.

Thm Assume a countable group Γ admits a countable group C satisfying (i) and (ii).

(Then Γ is called taut w.r.t. C.)

Let Λ be an arbitrary countable group and (Σ, m) a (Γ, Λ)-coupling.

Then $\exists\, \rho : \Lambda \to C$ homo s.t. $\# \ker \rho < \infty$ and $\Gamma \cap \rho(\Lambda)$ is of finite index in both Γ and $\rho(\Lambda)$.

Moreover $\exists\, \Phi_0 : \Sigma \to C$ $\Gamma \times \Lambda$-equiv.

(In particular, if $\Gamma \underset{ME}{\sim} \Lambda$, then Γ and Λ are "almost" isom.)

From OE to a coupling.

Assume $\Gamma \curvearrowright (X,\mu) \underset{\substack{OE \\ f}}{\sim} \Lambda \curvearrowright (Y,\nu)$.

$\to \alpha : \Gamma \times X \to \Lambda$

$$f(\gamma x) = \alpha(\gamma, x) f(x).$$

Define $\Gamma \times \Lambda \curvearrowright X \times \Lambda$ by

$$(\gamma, \lambda)(x, \lambda') = (\gamma x, \alpha(\gamma, x)\lambda'\lambda^{-1}).$$

We also have the cocycle

$$\beta : \Lambda \times Y \to \Gamma$$

defined by $f^{-1}(\lambda y) = \beta(\lambda, y) f^{-1}(y)$.

Let $\Gamma \times \Lambda \curvearrowright \Gamma \times Y$ by

$$(\gamma, \lambda)(\gamma', y) = (\gamma\gamma'\beta(\lambda, y)^{-1}, \lambda y)$$

We define $F : X \times \Lambda \xrightarrow{\sim} \Gamma \times Y$.

First let $F(x, e) := (e, f(x))$.

Next extend F in a $\Gamma \times \Lambda$-equiv. way:

$$F((\gamma, \lambda)(x, e)) = (\gamma, \lambda) F(x, e).$$

(this is well-defined! )

We set $\Sigma := X \times \Lambda \simeq \Gamma \times Y$.

Then Σ is a (Γ, Λ)-coupling.

$\lrcorner$

Applying tautness of Γ in Thm,

we get the following OE rigidity:

Cor Γ : the group in Thm.

Λ : arbi. countable group.

$\Gamma \curvearrowright (X, \mu)$, $\Lambda \curvearrowright (Y, \nu)$ free pmp.

Assume these are OE. Then

① They are virtually conjugate,

i.e., there exist

- $1 \to \underset{\text{finite}}{N} \to \Lambda \to \Lambda/N \to 1$

$$\begin{array}{c} \Lambda/N \\ \vee \text{ finite index} \\ \Lambda_1 \end{array}$$

- $\Gamma_1 < \Gamma$ finite index.

- $\rho : \Lambda_1 \to \Gamma_1$ isom.

- $X_1 \subset X$ Γ_1-inv.

- $Y_1 \subset Y/N$ Λ_1-inv.

s.t. - $\Gamma_1 \curvearrowright X_1 \underset{\text{isom}}{\sim} \Lambda_1 \curvearrowright Y_1$

w.r.t. ρ.

- $\Gamma \curvearrowright X$ is induced from $\Gamma_1 \curvearrowright X_1$.

  $\left(\Leftrightarrow \; X = \bigsqcup_{\gamma\Gamma_1 \in \Gamma/\Gamma_1} \gamma X_1 \right)$

- $\Lambda/N \curvearrowright Y/N$ is induced from $\Lambda_1 \curvearrowright Y_1$

$\square$

② If every finite index of Γ acts on (X,μ) ergodically (e.g. mixing)

then $\Gamma \curvearrowright (X,\mu) \underset{\text{isom}}{\sim} \Lambda \curvearrowright (Y,\nu)$

w.r.t. some $\rho : \Lambda \xrightarrow{\sim} \Gamma$.

Pf of ② $\Gamma \curvearrowright (X,\mu) \underset{\substack{OE \\ f}}{\sim} \Lambda \curvearrowright (Y,\nu)$

$\to \Sigma := X \times \Lambda \simeq \Gamma \times Y.$

By tautness, $\exists\, \rho : \Lambda \to \underset{\Gamma}{\overset{\vee}{C}}$ homo

$\# \ker \rho < \infty$ and

$\Gamma \cap \rho(\Lambda)$ is of finite index

in both Γ and $\rho(\Lambda)$

Moreover $\exists\, \Phi_0 : \Sigma \to C$ $\Gamma \times \Lambda$-equiv.

$\overline{\Phi}_0 \to \Sigma/\Lambda \to \Phi_0(\Sigma)/\rho(\Lambda)$ Γ-equiv.

$$\begin{matrix} \Sigma/\Lambda \\ \shortparallel \\ \Gamma \curvearrowright X \end{matrix}$$

where $\Phi_0(\Sigma)$ is the support of $\Phi_{0*}m$.

Every pt of $\Phi_0(\Sigma)$ has finite measure.
g

(since $\Phi_0^{-1}(g)$ is contained in a fund. domain for $\Gamma \curvearrowright X$.)

$\Gamma \curvearrowright \Phi_0(\Sigma)/P(\Lambda)$ admits an inv. prob. meas.

Hence this is finite.

Since every f.i. subgroup of Γ acts on (X, μ) ergodically, $\Phi_0(\Sigma)/P(\Lambda)$ must be a single pt.

Pick $g \in \Phi_0(\Sigma)$.

Then $\Phi_0(\Sigma) = g P(\Lambda)$ and $\Gamma < g P(\Lambda) g^{-1}$.

✓ $\Phi_0^{-1}(g) \subset \Sigma$ is contained in a fund. domain for $\Gamma \curvearrowright \Sigma$.

✓ $\Phi_0^{-1}(g)$ is $\ker\rho$- inv.

A fund. domain for $\ker\rho \curvearrowright \Phi_0^{-1}(g)$ is

a fund. domain for $\Lambda \curvearrowright \Sigma$.

Since $m(X) = m(Y)$,

we must have $\ker\rho = \{e\}$.

and $\Gamma = g\rho(\Lambda)g^{-1}$.

Therefore $X_1 = Y_1 := \Phi_0^{-1}(g)$ is a

fund. domain for both $\Gamma \curvearrowright \Sigma$

and $\Lambda \curvearrowright \Sigma$.

Then $\Gamma \curvearrowright X_1 \underset{\text{isom}}{\sim} \Lambda \curvearrowright Y_1$

w.r.t. $\Lambda \xrightarrow{\cong} \Gamma$ $\lambda \mapsto g\rho(\lambda)g^{-1}$

In fact for $x \in X_1$ and $\lambda \in \Lambda$,

$$\Phi_0 (g \rho(\lambda) g^{-1} x \lambda^{-1}) = g$$

So
$$\underset{\substack{\phi \\ \Gamma \curvearrowright X_1}}{[g\rho(\lambda)g^{-1}] \cdot x} = \underset{\substack{\| \\ \lambda \cdot x \\ \phi \\ \Lambda \curvearrowright Y_1}}{g \rho(\lambda) g^{-1} x \lambda^{-1}} \ \underset{\in Y_1}{\in X_1}$$

Since $\Gamma \curvearrowright X \underset{\text{isom}}{\sim} \Gamma \curvearrowright X_1$

$$\Lambda \curvearrowright Y \underset{\text{isom}}{\sim} \Lambda \curvearrowright Y_1,$$

we have $\Gamma \curvearrowright X \underset{\text{isom}}{\sim} \Lambda \curvearrowright Y$.

$\boxtimes$