Cost and Gaboriau's theorem

"Treeings realize the cost".

(based on the lecture

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For a countable group G

and its p.m.p. action $G \curvearrowright X$

on a standard probability space X

(where "p.m.p." stands for

"probability-measure-preserving"),

$$R(G \curvearrowright X) := \{(x, g^{-1}x) \mid x \in X, g \in G\}$$

the orbit equivalence relation

of the action $G \curvearrowright X$.

$$R(G \curvearrowright X) \simeq R(H \curvearrowright Y)$$

$$\overset{\Longleftrightarrow}{\scriptstyle df} \quad \exists f : X \xrightarrow{\sim} Y$$

$$\text{s.t. } f(Gx) = Hf(x) \quad \text{a.e. } x \in X$$

<u>Goal</u> F_n : the free group of rank n

$$\left(F_n = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n} = \pi_1 \left(\right. \right.$$  $$\left. \left. \right) \right)$$

Then $\mathcal{R}(F_n \curvearrowright X) \not\cong \mathcal{R}(F_m \curvearrowright Y)$ if $n \neq m$

ergodic free p.m.p.

We prove this by introducing cost,

an OE-invariant.

✡ The minimal number of generators of F_n is n.

Ref: D. Gaboriau, Coût des relations d'equivalence et des groupes, Invent. Math. 139 (2000), 41–98.

Standard probability spaces.

✓ Very "standard" behavior.

✓ include sufficiently many examples.

Def

- A Polish space
  := a separable top. space
  which is metrizable by a complete metric

- A standard Borel space
  := a measure space isom. to
  $(X, \mathcal{B})$ with X Polish space
  and $\mathcal{B}$ $\underbrace{\text{the } \sigma\text{-alg. of Borel sets}}_{\text{generated by open sets}}$

- A standard prob. sp.
  $:= (X, \mathcal{B}, \mu)$ s.t. $(X, \mathcal{B})$ standard Borel
  μ : prob. measure.

Facts (not proved)

- $(X, \mathcal{B})$ standard Borel. X not countable

$$\Rightarrow \quad (X, \mathcal{B}) \simeq ([0,1], \mathcal{B}_{[0,1]})$$

- $(X, \mathcal{B}, \mu)$ standard prob. sp.

  μ has no atom

$$\Rightarrow \quad (X, \mathcal{B}, \mu) \simeq ([0,1], \mathcal{B}_{[0,1]}, \overset{\text{Lebesgue}}{\lambda})$$

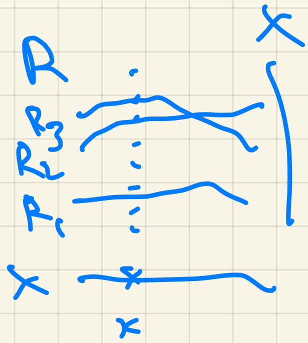
(In general, we define

$$(X, \mathcal{B}, \mu) \simeq (Y, \mathcal{C}, \nu)$$

as $\exists X' \subset X$, $Y' \subset Y$ conull
$\quad \in \mathcal{B} \qquad \in \mathcal{C}$

$$\exists f : (X', \mathcal{B}) \xrightarrow{\sim} (Y', \mathcal{C})$$

s.t. $f_* \mu = \nu$.)



Countable p.m.p. equivalence relations

(X, μ) standard prob. sp.

We mean by a countable p.m.p. eq. rel. R on (X, μ)

an eq. rel. $R \subset X \times X$ s.t.

- every eq. class $[x]_R$ is countable.
- R is a measurable subset of $X \times X$.
- $\exists\, R_n \subset R$ meas. s.t. $R = \bigcup_n R_n$ and

  $r : R \to X$ is inj. on each R_n.

  $(x,y) \mapsto x$

  called the range map

- R preserves μ, i.e., if $\varphi : A \xrightarrow{\sim} B$ $(A, B \subset X)$

  and graph $\varphi \subset R$, then $\varphi_* \mu|_A = \mu|_B$.

(introduced by Feldman - Moore (1977)
in connection with vN. alg.)

<u>E.g.</u> G countable group $\curvearrowright (X,\mu)$ p.m.p.

$\to$ $\mathcal{R} := \mathcal{R}(G \curvearrowright X)$ is a countable p.m.p. eq. rel. on (X,μ)

$\ast$ $\mathcal{R} = \bigcup_{g \in G} \operatorname{graph} g$.

$\ast$ $\forall g \in G$ $\operatorname{graph} g \subset X \times X$ meas.

(Consider $X \times X \to X \times X$ meas.
$(x,y) \longmapsto (gx, y)$
and the inverse image of Δ.) $\lrcorner$

<u>Thm</u> (not proved, a basic fact)
X, Y standard Borel sp.
$f : X \to Y$ inj. meas. map
Then $f(X) \subset Y$ meas. and $f : X \xrightarrow[\sim]{\text{meas.}} f(X)$

Def For $\mathcal{R}$ on (X,μ),

- $[[\mathcal{R}]] := \{\varphi : \underset{X}{A \overset{\sim}{\to} B} \mid \operatorname{graph}\varphi \subset \mathcal{R}\}$
  Φ

  the full pseudo-group of $\mathcal{R}$

- For $\Phi \subset [[\mathcal{R}]]$ and $x \in X$,

  $\Phi[x] :=$ the graph s.t.

  vertices $= [x]_{\mathcal{R}}$

  edges $= \{y, z\}$ s.t. $z = \varphi(y)$ for some $\varphi \in \Phi^{\pm 1}$

- Φ is a graphing of $\mathcal{R}$

  if for a.e. $x \in X$, $\Phi[x]$ is connected.

- $C(\Phi) := \sum_{\varphi \in \Phi} \mu(\operatorname{dom}\varphi)$: cost of Φ

  (for a countable Φ)

- $C(\mathcal{R}) := \inf_{\Phi : \text{graphing of } \mathcal{R}} C(\Phi)$: cost of $\mathcal{R}$.

e.g. $\mathcal{R} = \mathcal{R}(G \curvearrowright X)$

If G is generated by $S \subset G$.

then S defines a graphing of $\mathcal{R}$.

(For every $g \in G$ and $x \in X$,

$g = s_1 \cdots s_n$, $s_i \in S^{\pm 1}$,

$$x \underset{\substack{\Uparrow \\ \text{adjacent in } S[x]}}{\sim} s_n x \sim s_{n-1} s_n x \sim \cdots \sim \underset{\substack{\| \\ gx}}{s_1 \cdots s_n x}$$)

Hence $C(\mathcal{R}) \leq C(S) = |S|$.

Def For $\mathcal{R}$ on (X, μ) and $\Phi \subset [[\mathcal{R}]]$,

A graphing Φ of $\mathcal{R}$ is called a treeing of $\mathcal{R}$

if for a.e. $x \in X$, $\quad (\varphi_i \in \Phi^{\pm 1})$

$\forall y \in [x]_{\mathcal{R}}$ $\exists!$ reduced Φ-word $\varphi_1 \cdots \varphi_n$

i.e. $\forall i$ $\varphi_{i+1} \neq \varphi_i^{-1}$ s.t. $\varphi_1 \circ \cdots \circ \varphi_n(x) = y$.

(where a Φ-word means a word over $\Phi^{\pm 1}$!)

<u>e.g.</u> $\mathcal{R} = \mathcal{R}(F_n \curvearrowright X)$
free p.m.p.

$F_n = \langle a_1, \cdots, a_n \rangle$.

Then $\Phi := \{a_1, \cdots, a_n\}$ is a treeing of $\mathcal{R}$.

(because every $\gamma \in F_n$ is written as a unique reduced word over $\{a_1^{\pm 1}, \cdots, a_n^{\pm 1}\}$)

<u>Thm</u> (Gaboriau)

If Φ is a treeing of $\mathcal{R}$, then $C(\mathcal{R}) = C(\Phi)$.

<u>Cor</u> $C(\mathcal{R}(F_n \curvearrowright X)) = n$.
free p.m.p

Thus $\mathcal{R}(F_n \curvearrowright X) \not\cong \mathcal{R}(F_m \curvearrowright Y)$

if $n \neq m$.

e.g. R on (X,μ) is called finite
if every $[x]_R$ is finite.

Suppose $|[x]_R| = N \in \mathbb{N}$ $\forall x \in X$.

☆ $\exists$ $X = \bigsqcup_{i=1}^{N} E_i$ s.t.

each E_i meets every R-class at
exactly one point.

(i.e. E_i is a transversal of R)

Pf Regard $X = [0,1]$.

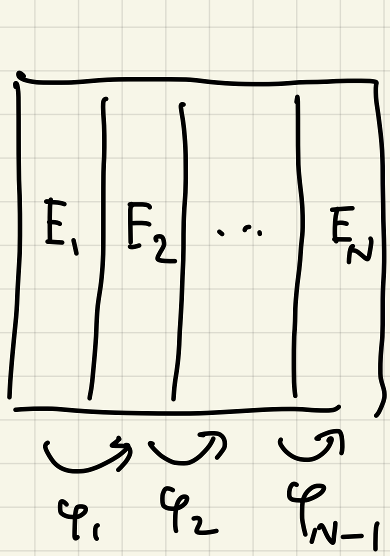
$$\{(x_1, \cdots, x_N) \in X^N \mid x_i \underset{R}{\sim} x_j,\ x_1 < x_2 < \cdots < x_N\}$$

meas. in X^N

$$\begin{array}{ccc} (x_1,\cdots,x_N) & & \\ \downarrow & \Big\downarrow p_i & \\ x_i & X & \end{array}$$

p_i is inj. $\Rightarrow$ $E_i := \mathrm{Im}\, p_i \subset X$ meas.

$\boxtimes$



$\forall x \in E_1 \ \exists!\, y \in E_2$

s.t. $(x, y) \in \mathcal{R}$.

So define $\varphi_1 : E_1 \xrightarrow{\sim} E_2$

by $x \mapsto y$

similarly define $\varphi_i : E_i \xrightarrow{\sim} E_{i+1}$

$i = 1, 2, \cdots, N-1$.

☆ $\Phi := \{\varphi_1, \cdots, \varphi_{N-1}\}$ is a treeing of $\mathcal{R}$.

$$x_1 \overset{\varphi_1}{\text{———}} x_2 \overset{\varphi_2}{\text{———}} x_3 \cdots\cdots x_{N-1} \overset{\varphi_{N-1}}{\text{———}} x_N \qquad \Phi[x_1]$$

$$C(\Phi) = 1 - \frac{1}{N}.$$

☆ If Ψ is a graphing of $\mathcal{R}$,

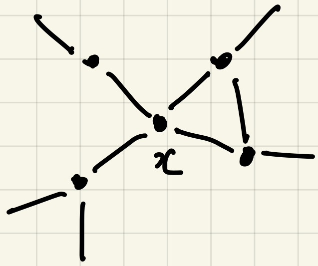
then $C(\Psi) \geq C(\Phi)$. Thus $C(\mathcal{R}) = 1 - \frac{1}{N}$.

(we prove this without Gaboriau's Thm)

Pf First, note that

$$C(\overline{\Phi}) \geq \frac{1}{2}\int_X \deg_{\overline{\Phi}[x]}(x)\, d\mu(x)$$

$\overline{\Phi}[x]$:



$\deg_{\overline{\Phi}[x]}(x) = 4.$

(since $\forall \varphi \in [[R]]$ s.t. $\varphi(x) \neq x$,

$$\mu(\mathrm{dom}\,\varphi) = \frac{1}{2}\int_X \deg_{\{\varphi\}[x]}(x)\, d\mu(x) \quad)$$

We need at least $N-1$ edges to connect N vertices.

(seen by induction. Shrink an edge.)

Therefore :

$$C(\Psi) \geq \frac{1}{2} \int_X \deg_{\Psi[x]}(x)\, d\mu(x) \qquad (\varphi_0 := \mathrm{id}_{E_1})$$

$$= \frac{1}{2} \int_{E_1} \underbrace{\sum_{i=0}^{N-1} \deg_{\Psi[x]}(\varphi_i(x))}_{= 2\,\#(\text{edges of } \Psi[x])} d\mu(x)$$

$$\geq \int_{E_1} (N-1)\, d\mu(x) = 1 - \frac{1}{N}.$$

$$= C(\Phi) \qquad \boxed{/\!/}$$

<u>Prop</u> For a general R on (X, μ) and a graphing Φ of R,

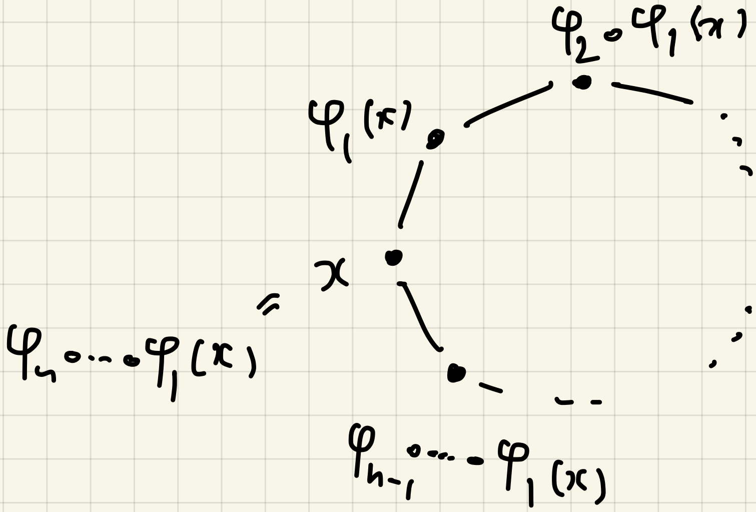
if $C(\Phi) = C(R)$, then Φ is a treeing.

<u>Pf</u> Otherwise $\exists\ \varphi_n \cdots \varphi_1$ reduced Φ-word $\quad \varphi_i \in \Phi^{\pm 1}$

$\exists\ A \subset X$ pos. meas.

s.t. $\mathrm{dom}(\varphi_n \cdots \varphi_1) \supset A$

$\forall x \in A \quad \varphi_n \circ \cdots \circ \varphi_1(x) = x.$



We may assume $\forall x \in A$

$$x,\ \varphi_1(x),\ \varphi_2 \circ \varphi_1(x),\ \cdots,\ \varphi_{n-1} \circ \cdots \circ \varphi_1(x)$$

are mutually distinct.

We want to delete $\varphi_1|_A$ from $\overline{\mathcal{L}}$,

keeping the graph connected.

☆ In general, $A \subset X$. $\mu(A) > 0$

$\varphi, \psi : A \to X$ inj. $\varphi(x) \neq \psi(x)$ $\forall x \in A$

Then $\exists B \subset A$ s.t. $\mu(B) > 0$,

$\varphi(B) \cap \psi(B) = \phi$.

If ✡ is true, then

$\exists B \subset A$ s.t. $B, \varphi_1(B), \varphi_2 \circ \varphi_1(B), \cdots,$

$\mu(B) > 0$ $\qquad \varphi_{n-1} \circ \cdots \circ \varphi_1(B)$

are mutually disjoint.

Delete $\varphi_1|_B$ from Φ.

Then we get a graphing of $\mathcal{R}$

with cost $< C(\Phi) = C(\mathcal{R})$.

contradiction $\qquad \blacksquare$

Pf of ✡

We may assume $\mu(\varphi(A) \triangle \psi(A)) = 0$

and $\varphi(A) = \psi(A)$.

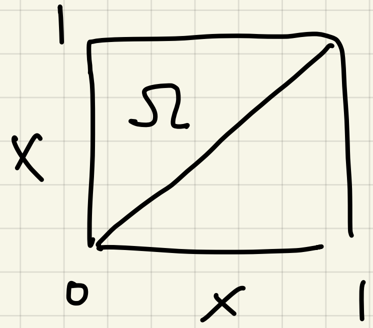
We may also assume $\psi = id_A$.

So $\varphi : A \to X$ inj. $\quad \varphi(x) \neq x \quad \forall x \in A$.

Regard $X = (0,1)$, $\mu = \text{Leb}$.

$$\Omega := \{ (x,y) \in X \times X \mid x < y \}$$

We may assume $\operatorname{graph} \varphi \subset \Omega$

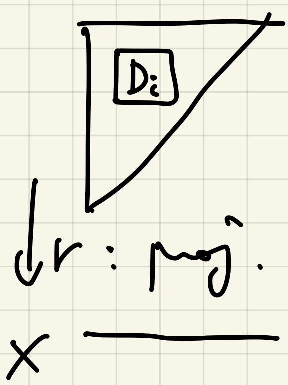


Take a countable family

$(D_i)_{i \in \mathbb{N}}$ of closed rectangles

s.t. $D_i \subset \Omega$, $\Omega = \bigcup_{i \in \mathbb{N}} D_i$.

Then $\exists i$



s.t. $\mu(r(D_i \cap \operatorname{graph} \varphi)) > 0$

(consider $r_1 : \operatorname{graph} \varphi \cong A$

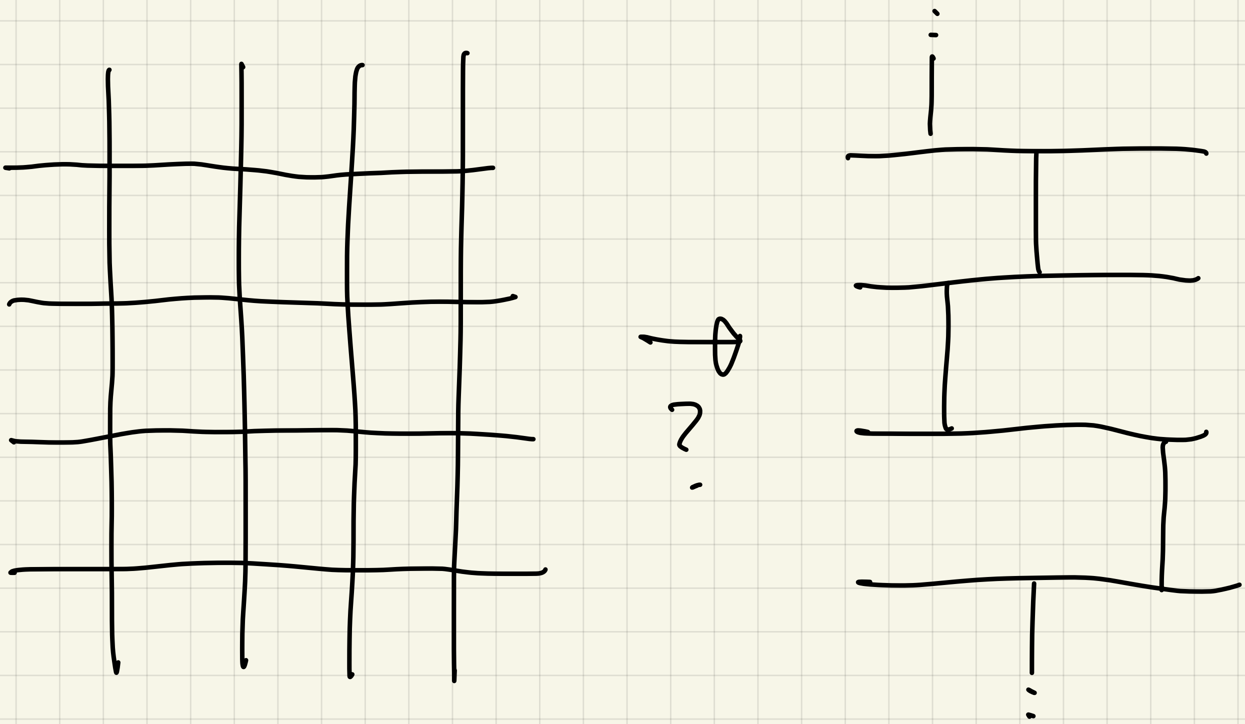
and $(r_1^{-1})_* \mu|_A$ on $\operatorname{graph} \varphi \subset \Omega$.)

$B := r(D_i \cap \operatorname{graph} \varphi)$ is a desired set. $\boxed{}$

Question 淀

Given a graphing Φ of $\mathcal{R}$,
can one get a treeing of $\mathcal{R}$
by repeating to remove edges
in circuits?

e.g. $\mathcal{R} = \mathcal{R}(\mathbb{Z}^2 \curvearrowright X)$ free, $\Phi = \{(1,0), (0,1)\}$



$\bar{\Phi}[x]$ $\qquad\qquad$ tree

<u>Prop</u> For every $\mathcal{R}$ on (X, μ),

$\exists G$ countable group $\curvearrowright X$

s.t. $\mathcal{R} = \mathcal{R}(G \curvearrowright X)$.

<u>Pf</u> $\mathcal{R} = \bigcup_n R_n$ s.t. r is inj. on each R_n.

$$R_n' := \{(y, x) \mid (x, y) \in R_n\}$$

$$r : \mathcal{R} \to X, \quad (x, y) \mapsto x$$

Then $\mathcal{R} = \bigcup_{n,m} \underbrace{R_n \cap R_m'}$

$$s : \mathcal{R} \to X, \quad (x, y) \mapsto y$$

on this, r and s are inj.

We may assume $X = [0, 1)$.

$$\Omega := \{(x, y) \in X \times X \mid x < y\}$$

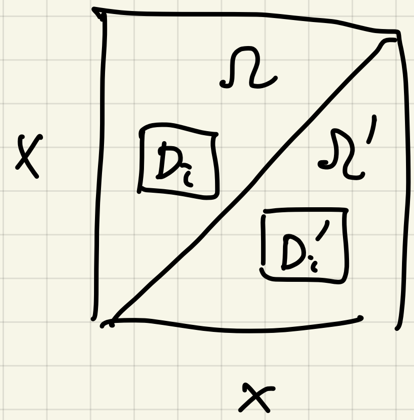
$$\mathcal{R} \setminus \Delta = \bigcup_{n,m,i} (R_n \cap R_m') \cap D_i$$

$$\cup \ (R_n \cap R_m') \cap D_i'$$

For $\lambda = (n, m, i)$, define φ_λ and φ'_λ by

$$\operatorname{graph} \varphi_\lambda = R_n \cap R'_m \cap D_i$$

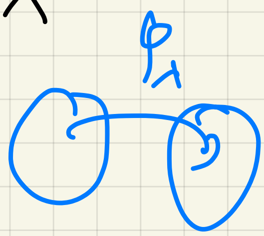
$$\operatorname{graph} \varphi'_\lambda = R_n \cap R'_m \cap D'_i .$$



Then $\operatorname{dom} \varphi_\lambda \cap \operatorname{ran} \varphi_\lambda$

$$\subset r(D_i) \cap s(D_i) = \phi .$$

Extend φ_λ to an auto. ψ_λ of X



s.t. $\psi_\lambda = \varphi_\lambda^{-1}$ on $\operatorname{ran} \varphi_\lambda$

$\psi_\lambda = \mathrm{id.}$ on $X \setminus (\operatorname{dom} \varphi_\lambda \cup \operatorname{ran} \varphi_\lambda)$.

Define ψ'_λ similarly.

Then $R \setminus \Delta \subset \bigcup_\lambda \operatorname{graph} \psi_\lambda \cup \operatorname{graph} \psi'_\lambda$

Let G be the group of auto. of X generated by ψ_λ and ψ'_λ. $\boxtimes$

The induction formula

$\mathcal{R}$ on (X,μ). For $Y \subset X$,

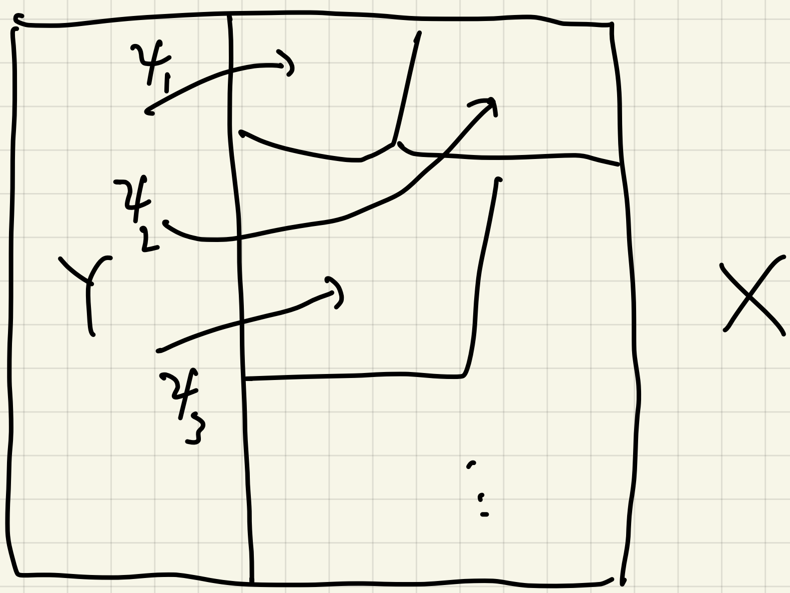
$$[Y]_{\mathcal{R}} := \{ x \in X \mid \exists y \in Y \ (x,y) \in \mathcal{R} \}$$

☆ This is measurable.

(If $\mathcal{R} = \mathcal{R}(G \curvearrowright X)$, then $[Y]_{\mathcal{R}} = GY$)

☆ If $[Y]_{\mathcal{R}} = X$, then $\exists \varphi_n \in [[\mathcal{R}]]$

$$\operatorname{dom} \varphi_n \subset Y, \quad X = \bigsqcup_n \operatorname{ran} \varphi_n$$



Thm (The induction formula)

R on (X,μ), $Y \subset X$ $[Y]_R = X$.

Then $C_\mu(R) - \mu(X) = C_\mu(R|_Y) - \mu(Y)$.

$$\left(\begin{array}{l} \text{where } R|_Y := R \cap (Y \times Y). \\ C_\mu(\Phi) = \sum_{\varphi \in \Phi} \mu(\mathrm{dom}\,\varphi) \\ C_\mu(R) := \inf_{\Phi} C_\mu(\Phi) \end{array} \right)$$

The formula is eq. to:

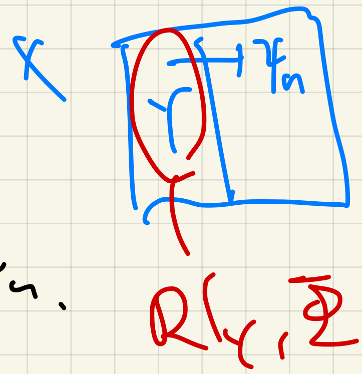
$$C_\mu(R) = C_\mu(R|_Y) + \mu(X \setminus Y)$$

<u>Pf of $\leq$</u>

Pick $\varepsilon > 0$ and a graphing $\Phi \subset [[R|_Y]]$ of $R|_Y$ s.t. $C_\mu(\Phi) < C_\mu(R|_Y) + \varepsilon$.

Take $\psi_n \in [[R]]$ s.t.

$\mathrm{dom}\,\psi_n \subset Y$, $\quad X = \bigsqcup_n \mathrm{ran}\,\psi_n$.



$$\tilde{\Phi} := \Phi \cup \{\psi_n\}_n \subset [[R]].$$

$$C(\tilde{\Phi}) = C(\Phi) + \underbrace{\sum_n \mu(\mathrm{dom}\,\psi_n)}_{\substack{\| \\ \sum_n \mu(\mathrm{ran}\,\psi_n) = \mu(X \setminus Y)}}$$

$$< C(R|_Y) + \varepsilon + \mu(X \setminus Y)$$

☆ $\tilde{\Phi}$ is a graphing of R.

$\left(\Rightarrow C(R) \leq C(R|_Y) + \mu(X \setminus Y) \right)$

<u>Pf of ☆</u> Set $Y_n := \mathrm{ran}\,\psi_n$, $Y_0 := Y$.

$\psi_0 := \mathrm{id}_Y$

For $(x, y) \in R$,

$\exists n, m$ s.t. $x \in Y_n$, $y \in Y_m$

$$x \underset{\psi_n}{\sim} \underset{\substack{\cap \\ Y}}{\psi_n^{-1}(x)} \underset{\Phi}{\sim} \underset{\substack{\cap \\ Y}}{\psi_m^{-1}(y)} \underset{\psi_m}{\sim} y$$

◻

Before proving "$C(\mathcal{R}) \geq C(\mathcal{R}|_Y) + \mu(X \setminus Y)$".

Suppose we are given

- a connected graph $\mathcal{G} = (V, E)$
- $V_0 \subset V$
- procedure to "bring down" each vertex in $V \setminus V_0$ to a vertex in V_0

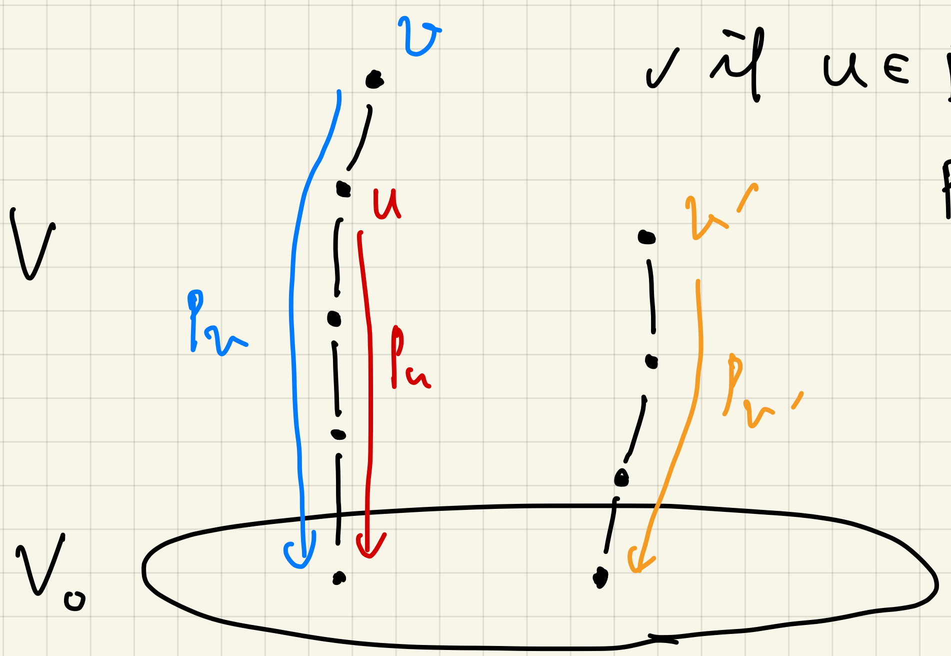
i.e. for each $v \in V \setminus V_0$,

a path p_v in $\mathcal{G}$ is given s.t.

✓ the terminating pt is the first vertex in p_v that belongs to V_0

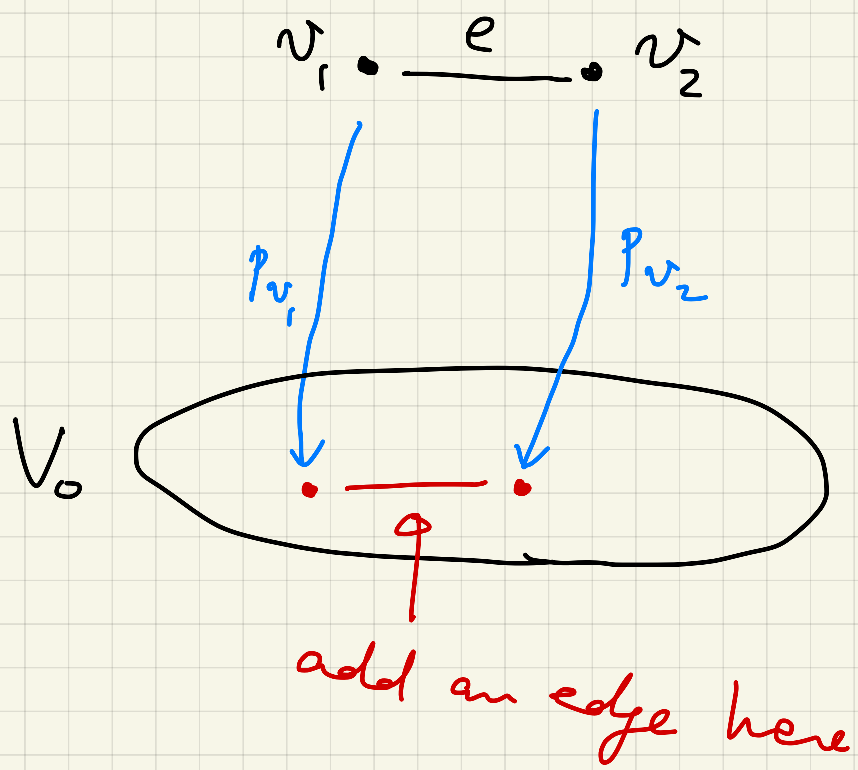
✓ if $u \in p_v$, then

$$p_u = p_v|_{\text{from } u}.$$



☆ Then we get a graph connecting all vertices in V_0 naturally.

✓ For every edge $e = (v_1, v_2)$, $v_i \in V \setminus V_0$ which belongs to no path p_v,

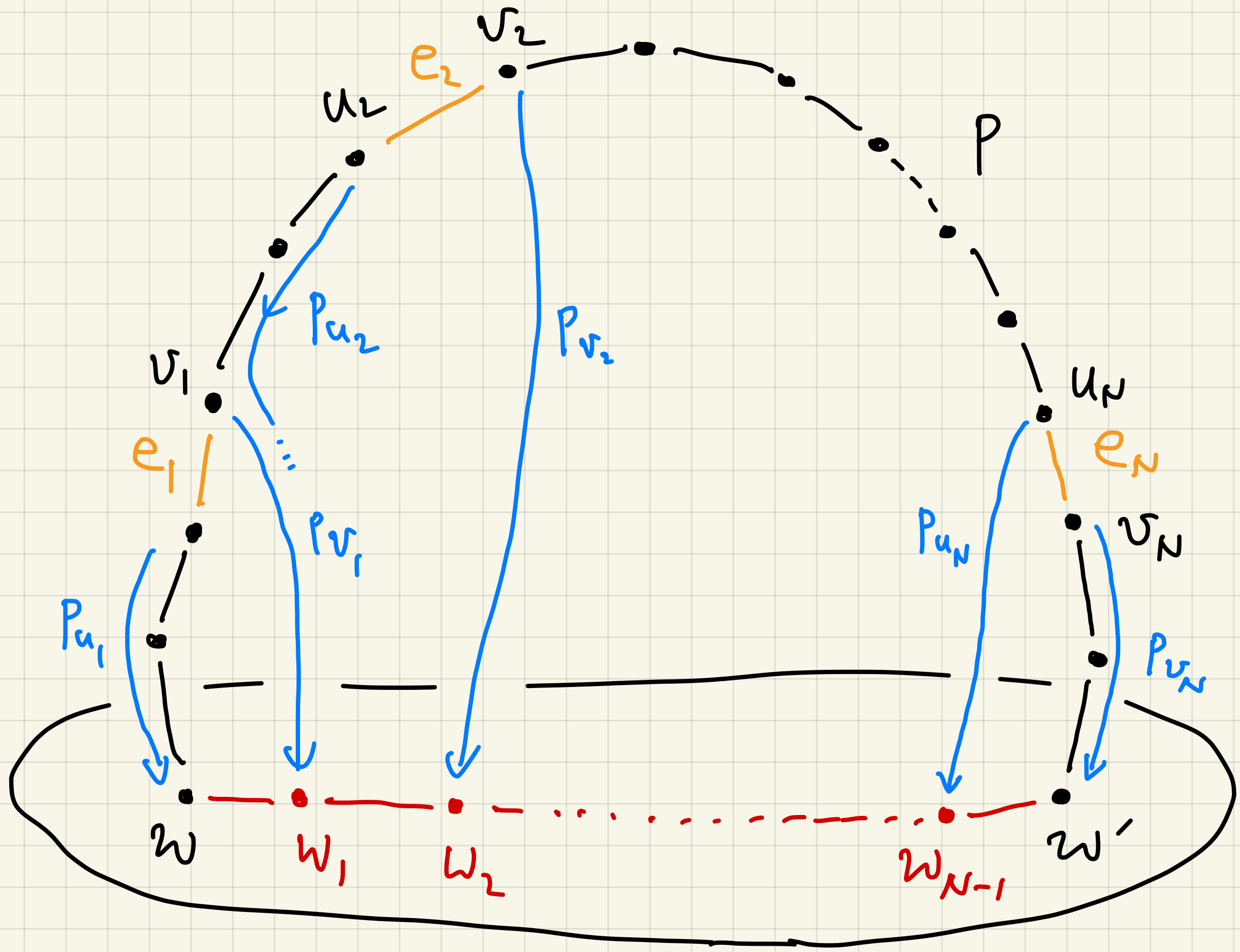


✓ $E|_{V_0 \times V_0}$ and these new edges connect all vertices in V_0.

For $w, w' \in V_0$, $\exists$ path p in $\mathcal{G} = (V, E)$ from w to w'.

We may assume $p \setminus \{w, w'\} \subset V \setminus V_0$.

$e_1, \cdots, e_N$: edges in the path which belong to no p_v.



Pf of "$C(\mathcal{R}) \geq C(\mathcal{R}|_Y) + \mu(X \setminus Y)$"

$\mathcal{R}$ on (X, μ), $Y \subset X$ s.t. $[Y]_{\mathcal{R}} = X$.

Pick $\varepsilon > 0$ and let Φ be a graphing of $\mathcal{R}$

s.t. $C(\Phi) < C(\mathcal{R}) + \varepsilon$.

Define $d : X \to \mathbb{N} \cup \{0\}$ by

$d(x) :=$ the minimal distance of x

from vertices of Y in the graph $\Phi[x]$.

$Y_n := \{d = n\}$.

Then $Y_0 = Y$, $\quad X = \bigsqcup_{n=0}^{\infty} Y_n$.

Write $\Phi = \{\varphi_j \mid j = 1, 2, \cdots\}$

For each $x \in Y_n$, choose the minimal j
$n \geq 1$

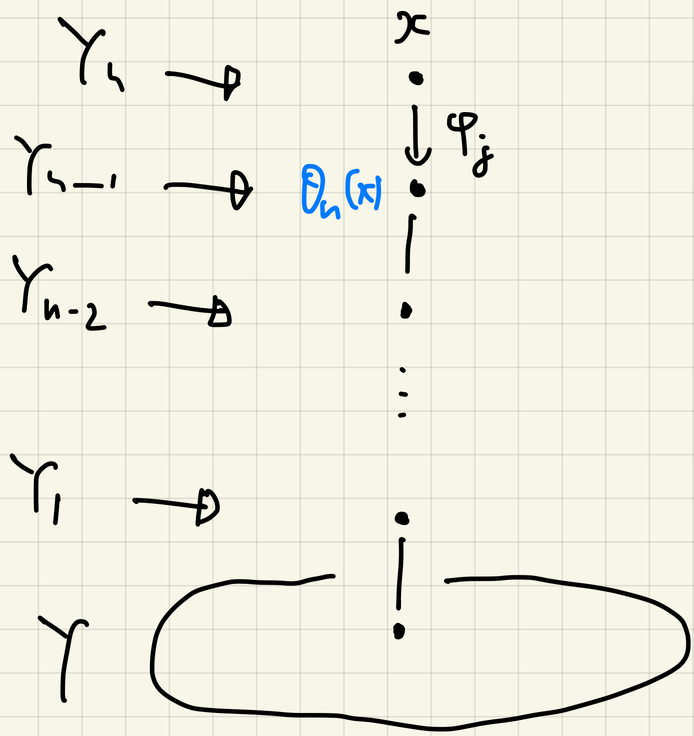
s.t. $\varphi_j(x) \in Y_{n-1}$ or $\varphi_j^{-1}(x) \in Y_{n-1}$.

Let $\theta_n(x)$ be $\varphi_j(x)$ if the former holds,

and be $\varphi_j^{-1}(x)$ if the latter holds.

(If both hold, let $\theta_n(x)$ be $\varphi_j(x)$.)

$\rightarrow$ $\theta_n : Y_n \to Y_{n-1}$



✓ $\{\theta_n \mid n = 1, 2, \cdots\}$ gives a procedure to bring down every vertex in $X \setminus Y$ into a vertex in Y.

$\to$ a path p_v in $\bar{\Phi}[x]$

for each $v \in [x]_R \setminus Y$.

From $\bar{\Phi}$ and $\{\theta_n \mid n = 1, 2, \cdots\}$,

we get a graphing $\bar{\bar{\Phi}}$ of $R|_Y$

by bringing down every edge

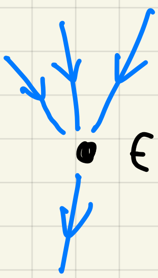
which appears in no path p_v.

$$C(\bar{\Phi}) \geqslant \frac{1}{2} \int_X \deg_{\bar{\Phi}[x]}(x)\, d\mu(x)$$

$$= \frac{1}{2} \int_X \Big[\#(\text{edges } e \text{ s.t. } x \in \partial e,\ e \in Y \times Y)$$

$$+ \#(e \text{ s.t. } x \in \partial e,\ e \text{ in some } p_v)$$

$$+ \#\left(\begin{array}{l} e \text{ s.t. } x \in \partial e,\ e \text{ in no } p_v \\ \qquad e \notin Y \times Y \end{array} \right) \Big] \, d\mu(x)$$



$\in X \setminus Y$

$\exists!$ outgoing edge

$$= \frac{1}{2}\int_X \Big[\#(e \text{ s.t. } x \in \partial e,\ e \in Y \times Y)$$

$$+ \#(\underline{e \text{ in } \Psi} \text{ added in the procedure})\Big]$$

$$d\mu(x)$$

$$+ \underline{\mu}(X \setminus Y)$$

$$= C(\Psi) + \mu(X \setminus Y)$$

Thus $C(\Psi) \leq C(\Phi) - \mu(X \setminus Y)$

$$< C(R) + \varepsilon - \mu(X \setminus Y)$$

and $C(R|_Y) \leq C(R) - \mu(X \setminus Y)$

$\boxtimes$

<u>Cor</u> $\mathcal{R}$ on (X, μ)

If every $\mathcal{R}$-class is ∞, then $C(\mathcal{R}) \geq 1$.

<u>Pf</u> One can find $Y_n \subset X$

s.t. $[Y_n]_{\mathcal{R}} = X$ and $\mu(Y_n) \to 0$.

(This is obvious if $\mathcal{R}$ is ergodic.

Otherwise it is not so obvious.

We prove this later.)

$$\text{Then } C(\mathcal{R}) = C(\mathcal{R}|_{Y_n}) + \mu(X \setminus Y_n)$$

$$\geq \mu(X \setminus Y_n) \to 1. \quad \blacksquare$$

<u>Cor</u> $C(\mathcal{R}(\mathbb{Z} \curvearrowright X)) = 1$

free p.m.p.

<u>Lem</u> R on (X, μ)

Assume every R-class is ∞. $\varepsilon > 0$

Then $\exists\, Y \subset X$

$$\mu(Y) < \varepsilon \quad \text{and} \quad [Y]_R = X.$$

<u>Pf</u> Pick $q \in \mathbb{N}$ s.t. $\frac{1}{q} < \varepsilon$.

Pick $G \curvearrowright X$ s.t. $R = R(G \curvearrowright X)$

Enumerate all $(q-1)$-tuples

of distinct elements of G

$\to \lambda_1, \lambda_2, \cdots$

Write $\lambda_i = (g_1^i, \cdots, g_{q-1}^i)$.

To define $A_1 \subset X$, consider $A \subset X$

s.t. $A, g_1^1 A, \cdots, g_{q-1}^1 A$ are

mutually disjoint.

Let A_1 be such an A which is maximal (modulo null set).

This exists by Zorn's lemma. 

For each $i \geq 2$, define $A_i \subset X$ to be maximal among sets

$$A \subset X \setminus [A_1 \cup \cdots \cup A_{i-1}]_R \text{ s.t.}$$

$A, g_1^i A, \ldots, g_{q-1}^i A$ are mutually disjoint.

Let $Y := \bigcup_{i=1}^{\infty} A_i$.

Note $[A_i]_R$ are mutually disjoint.

$$\begin{array}{c} [A_i]_R \\ \cup \\ A_i \sqcup g_1^i A_i \sqcup \cdots \sqcup g_{q-1}^i A_i \end{array}$$

So $\mu(Y) < \frac{1}{q} < \varepsilon$.

We prove $[Y]_R = X$.

Otherwise $X \setminus [Y]_R$ has pos. measure.

Since every R-class is ∞,

$\exists\, i$ s.t. $x, g_1^i x, \cdots, g_{g-1}^i x$ are mutually

distinct for all x in a subset

of $X \setminus [Y]_R$ of pos. meas.

$\exists\, B \subset X \setminus [Y]_R$ pos. meas.

$B, g_1^i B, \cdots, g_{g-1}^i B$ are mutually

disjoint.

This contradicts the max. of A.

$\boxtimes$

Treeings realize the cost.

$\mathcal{R}$ on (X, μ).

Thm (Gaboriau)

If Ψ is a treeing of $\mathcal{R}$, then $C(\mathcal{R}) = C(\Psi)$.

Idea Let Φ be a graphing of $\mathcal{R}$

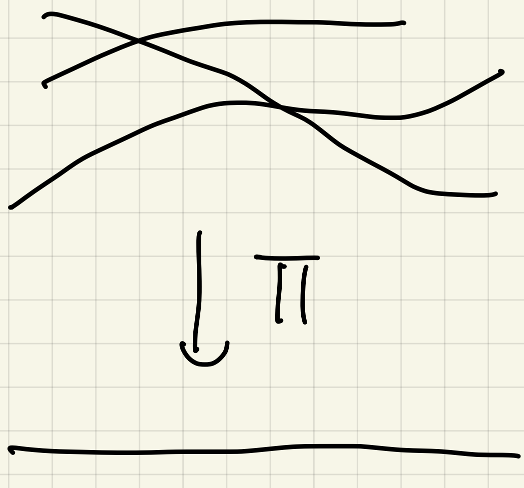
s.t. $C(\Phi) < C(\mathcal{R}) + \varepsilon$.

Using Φ, we construct a "cover" $\overline{\Theta}$

of Ψ.

Then we make "foldings" of $\overline{\Theta}$.

$\overline{\Theta}$



Ψ

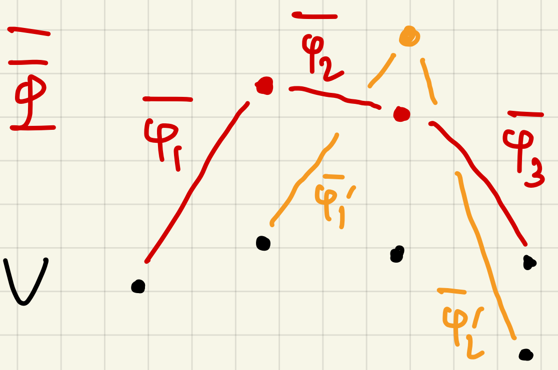
The single-graph case

T : tree V : the set of vertices

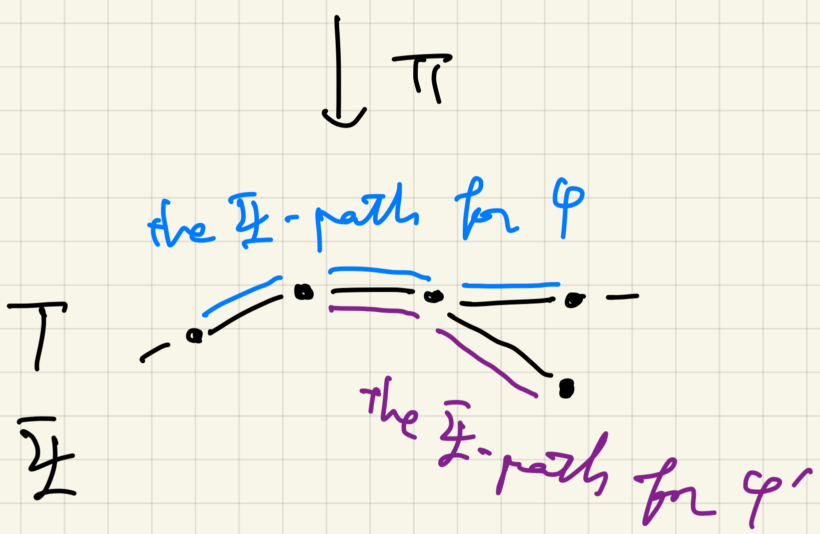
$\overline{\Psi}$: the set of edges of T

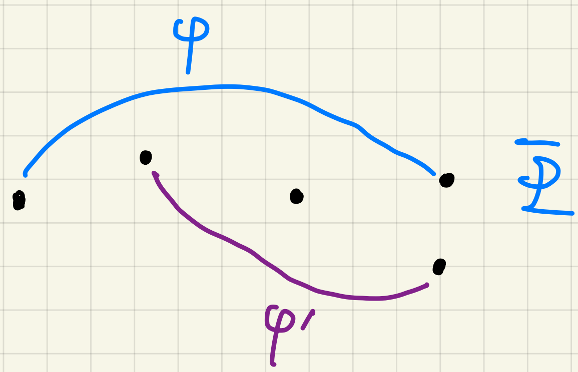
We take a connected graph whose set of vertices is V, and let Φ be its set of edges.

$\to$ $\overline{\Phi}$: cover of T.



New vertices and edges are added for each $\varphi \in \Phi$.

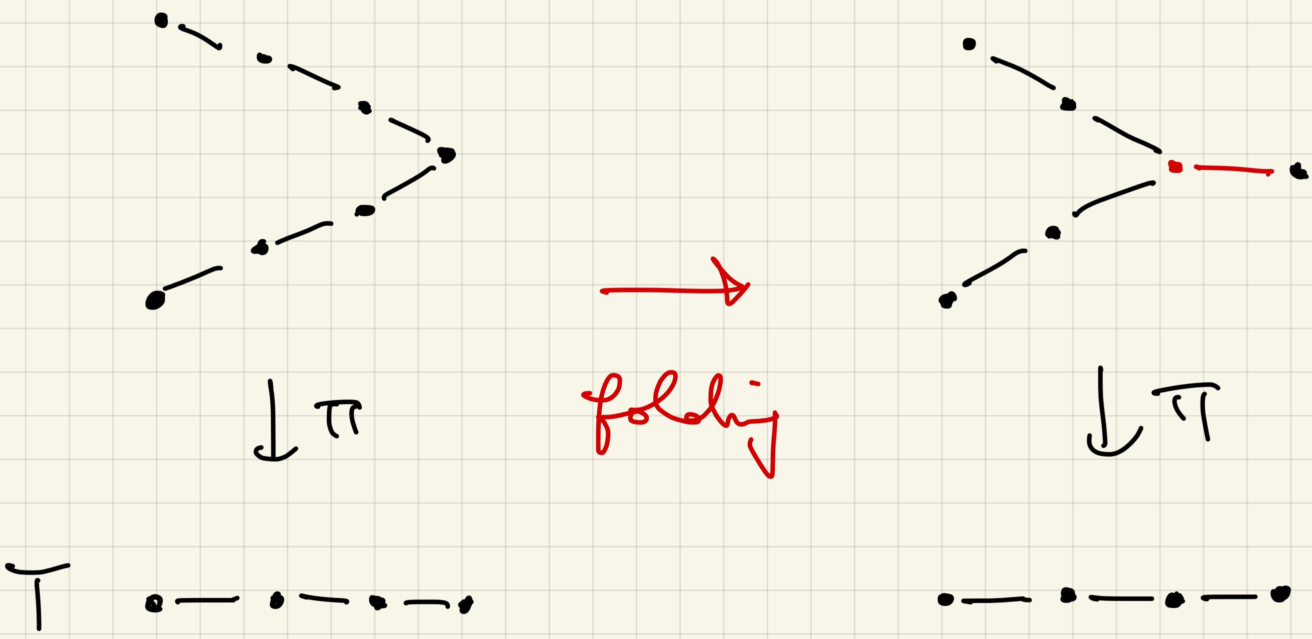




" $\#(\text{edges of } \bar{\Phi}) - \#(\text{vertices of } \bar{\Phi})$

$= \#(\text{edges of } \overline{\Phi}) - \#(\text{vertices of } \overline{\Phi})$ "

We make foldings of $\overline{\Phi}$.



$$\overline{\Phi} \longrightarrow \overline{\Phi}_1 \longrightarrow \overline{\Phi}_2 \longrightarrow \cdots \longrightarrow \overline{\Phi}_N$$

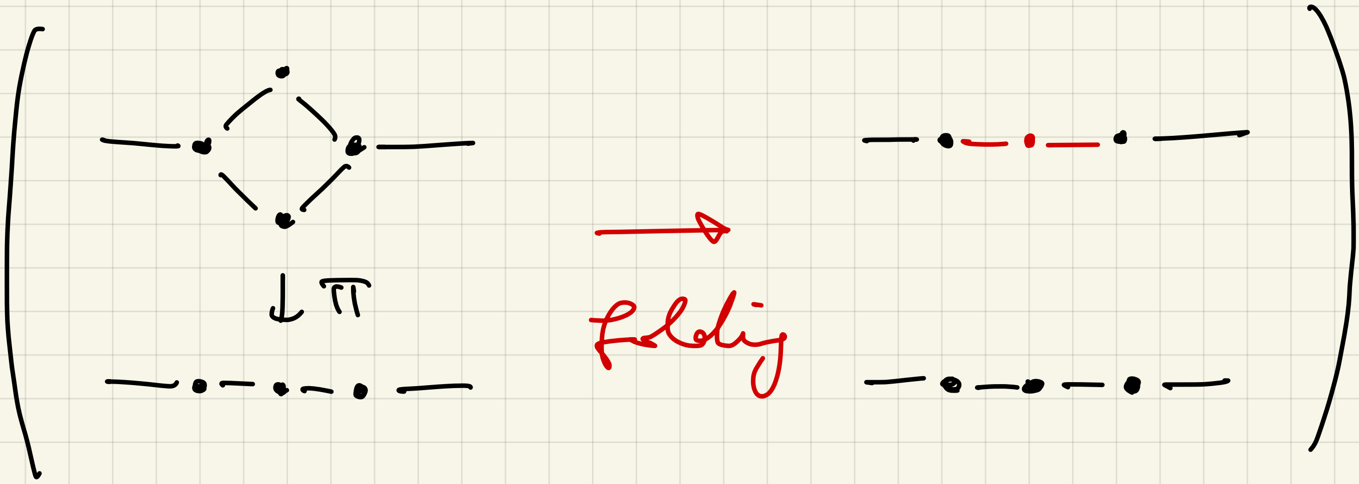
foldings

$\exists N \quad \overline{\Phi}_N \xrightarrow[\pi]{\sim} \overline{\Phi}$ (if we are lucky)

The value

"$\#(\text{edges of } \overline{\Phi}) - \#(\text{vertices of } \overline{\Phi})$"

does not increase under foldings.



Thus

$$\#(\text{edges of } \overline{\Phi}) - \#(\text{vertices of } \overline{\Phi})$$

$$= \#(\text{edges of } \overline{\Phi}) - \#(\text{vertices of } \overline{\Phi})$$

$$\geq \#(\text{edges of } \overline{\Phi}_N) - \#(\text{vertices of } \overline{\Phi}_N)$$

$$= \#(\text{edges of } \overline{\Psi}) - \#(\text{vertices of } \overline{\Psi})$$

This is like $C(\overline{\Phi}) - \mu(X) \geq C(\overline{\Psi}) - \mu(X)$.

To make the foldings terminate in finite steps

We assume the following:

$\exists\, L \in \mathbb{N}$ s.t.

$$\begin{cases} \text{① every } \varphi \in \overline{\Phi} \text{ has } \overline{\Psi}\text{-length} \leq L. \\ \text{② every } \psi \in \overline{\Psi} \text{ has } \overline{\Phi}\text{-length} \leq L. \end{cases}$$

$\overline{V}$: the set of vertices of $\overline{\Phi}$.

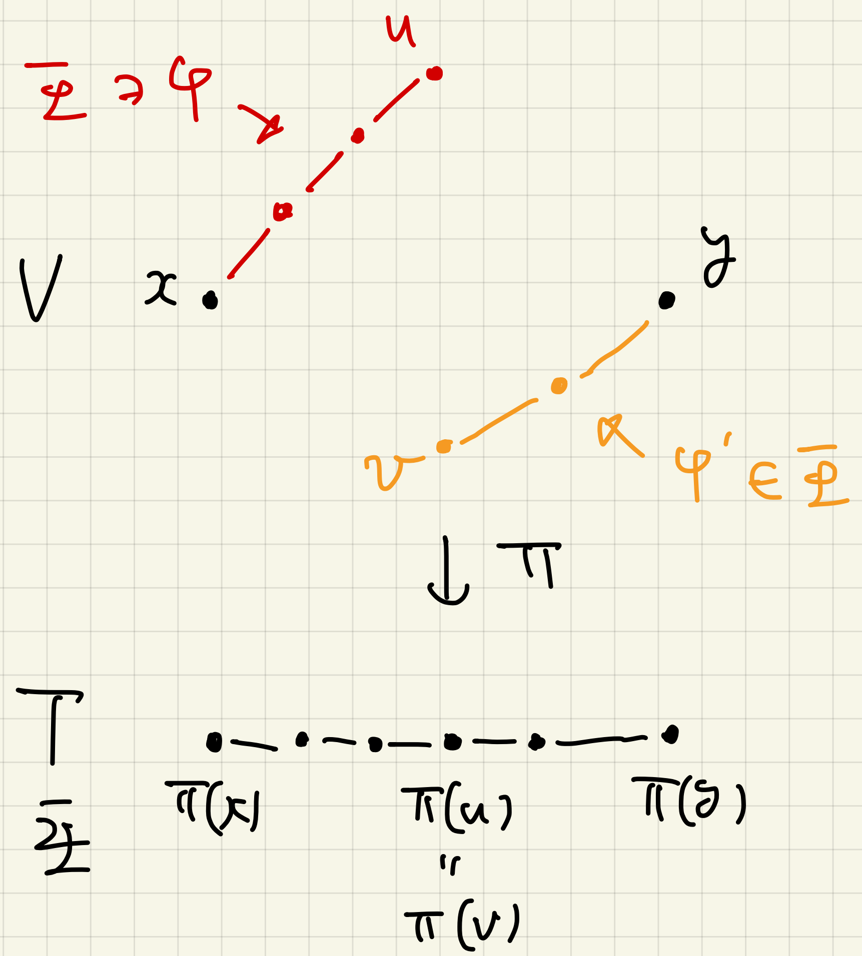
$\cup$

V

✩ If $u, v \in \overline{V}$ and $\pi(u) = \pi(v)$, then u and v are joined by a $\overline{\Phi}$-path of length $\leq 2L + 2L^3$.

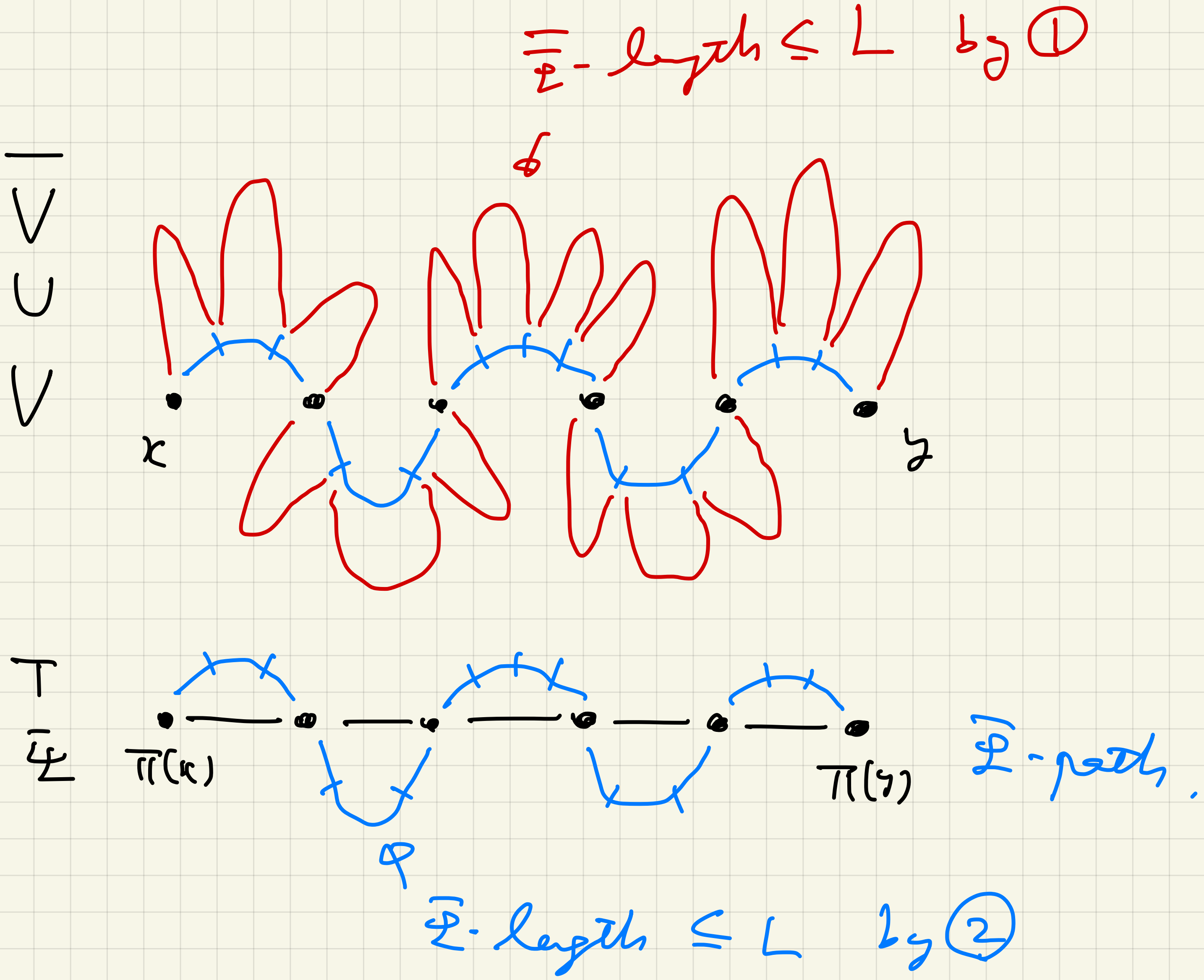
$$\Rightarrow \quad \overline{\Phi}_{L+L^3} \xrightarrow[\pi]{\simeq} \overline{\Psi}.$$

Pf of ✡ $\exists x, y \in V \subset \bar{V}$



$$\left.\begin{array}{l} u \to x \;:\; \bar{\Phi}\text{-length} \le L \\ v \to y \;:\; \bar{\Phi}\text{-length} \le L \end{array}\right\} \text{by ①}$$

$$\pi(x) \xrightarrow[\bar{\Phi}\text{-length} \le L]{} \pi(u) = \pi(v) \xrightarrow[\bar{\Phi}\text{-length} \le L]{} \pi(y) \qquad \text{by ①}$$



Therefore $x \to y$

: $\bar{\Phi}$ - length $\leq L \cdot L \, (L + L)$

$$= 2L^3$$

and $u \to v$

: $\bar{\Phi}$ - length $\leq 2L + 2L^3$. $\boxed{/\!/}$

<u>Thm</u> (Gaboriau) R on (X, μ).

If Ψ is a treeing of R, then $C(R) = C(\Psi)$.

<u>Pf of Thm</u> First assume Ψ is finite.

Take $\varepsilon > 0$ and a graphing Φ of R

s.t. $C(\Phi) < C(R) + \varepsilon$.

Write $\Phi = \{\varphi_1, \varphi_2, \cdots\}$.

<u>Lem</u> $\exists\, \Theta$: finite graphing of R $\exists\, L > 0$

s.t. (i) $C(\Theta) < C(R) + 2\varepsilon$.

(ii) Every element of Θ can be written as a Ψ-word of length $\leq L$

(iii) Every element of Ψ can be written as a Θ-word of length $\leq L$ after subdividing its domain.

Pf We may assume every element of $\overline{\Psi}$ is written as a single $\overline{\Psi}$-word.

$\overline{\Phi}_N := \{\varphi_1, \cdots, \varphi_N\}$.

$\exists N$: large $\exists L_1 > 0$ s.t.

every $\psi \in \overline{\Psi}$ can be written as a $\overline{\Phi}_N$-word of length $\leq L_1$

on some $D_\psi \subset \mathrm{dom}\,\psi$ (after subdivision)

with $\mu(\mathrm{dom}\,\psi \setminus D_\psi)$ so small that

$$\sum_{\psi \in \overline{\Psi}} \mu(\mathrm{dom}\,\psi \setminus D_\psi) < \varepsilon.$$

$\Theta := \overline{\Phi}_N \cup \{ \psi|_{\operatorname{dom}\psi \setminus D_\psi} \}_{\psi \in \overline{\Psi}}$

✓ Θ is a finite graphing of $\mathcal{R}$.

✓ $C(\Theta) = C(\overline{\Phi}_N) + \sum_{\psi \in \overline{\Psi}} \mu(\operatorname{dom}\psi \setminus D_\psi)$

$$\leq C(\overline{\Psi}) + \varepsilon < C(\mathcal{R}) + 2\varepsilon.$$

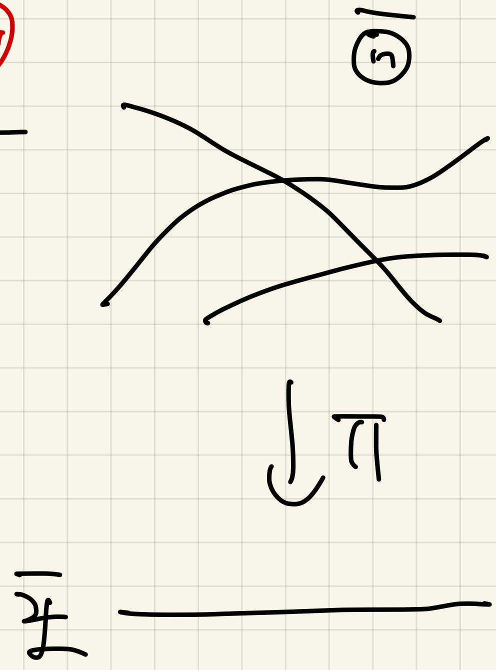
✓ Every $\psi \in \overline{\Psi}$ is written as a Θ-word of length $\leq L_1$ (after subdivision)

✓ Every $\theta \in \Theta$ is written as a $\overline{\Psi}$-word of length $\leq {}^{\exists}L_2$

(since each of $\varphi_1, \cdots, \varphi_N$ is written as a single $\overline{\Psi}$-word.)

$\boxed{/\!/}$

Construction of a cover $\overline{17}$



Given $\Theta \in$ (4).

write $\Theta = \psi_\ell^{\varepsilon_\ell} \cdots \psi_2^{\varepsilon_2} \psi_1^{\varepsilon_1}$

as a reduced

$\overline{\mathcal{Z}}$-word with $\varepsilon_i \in \{\pm 1\}$.

Add the following spaces

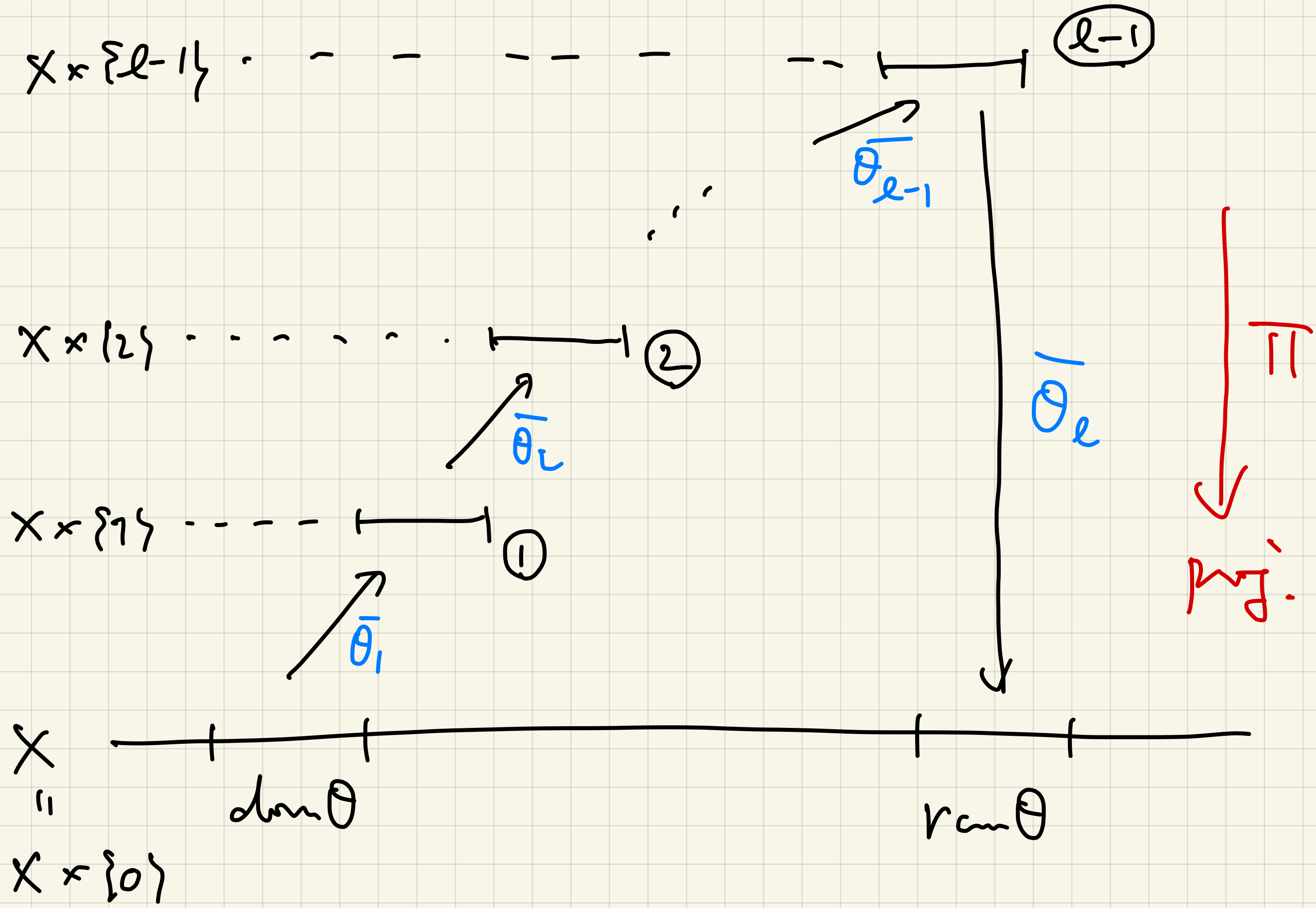
to X $(= X \times \{0\})$:

① $\psi_1^{\varepsilon_1}(\mathrm{dom}\,\Theta) \times \{1\}$

② $\psi_2^{\varepsilon_2}\psi_1^{\varepsilon_1}(\mathrm{dom}\,\Theta) \times \{2\}$

$\vdots$

($\ell-1$) $\psi_{\ell-1}^{\varepsilon_{\ell-1}} \cdots \psi_1^{\varepsilon_1}(\mathrm{dom}\,\Theta) \times \{\ell-1\}$.



Define $\overline{\theta}_i$ s.t. $\pi(\overline{\theta}_i) = \psi_i^{\varepsilon_i}$.

$\Uparrow$

$\mathrm{dom}\,\overline{\theta}_i$, $\mathrm{ran}\,\overline{\theta}_i$ are defined as above.

Carry out this for each $\theta \in \Theta$

So that the added spaces are disjoint if θ are distinct.

$\bar{X}$: the obtained space with

$\downarrow \pi$ proj. a measure extending μ

X (denoted by the same μ)

$\bar{\Theta}$:= the set of $\bar{O}_i$'s. ! Θ : finite

$\Downarrow$

$\mu(\bar{X}) < \infty$

☆ $\mu(\bar{X}) - \mu(X) = C(\bar{\Theta}) - C(\Theta)$.

Hence $C(\bar{\Theta}) - \mu(\bar{X}) = C(\Theta) - \mu(X)$.

$(= \#\text{edges} - \#\text{vertices})$.

☆ If $x, y \in \bar{X}$, $\pi(x) = \pi(y)$, then x and y can be connected by a $\bar{\Theta}$-path of length $\leq 2L + 2L^3$.

(This follows from Lem and the argument in the single-graph case.)

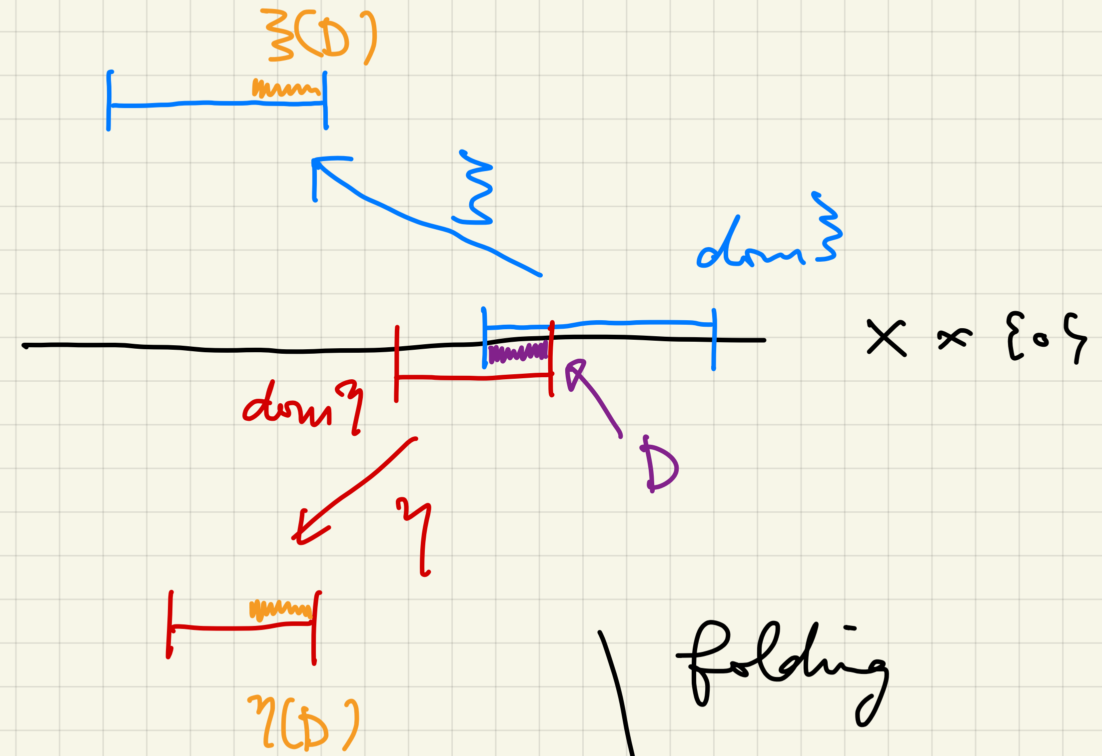
Foldings

Let $\xi, \eta \in \overline{\mathcal{G}}^{\pm 1}$ and $D \subset \operatorname{dom}\xi \cap \operatorname{dom}\eta$

s.t. $\mu(D) > 0$ and $\pi(\xi|_D) = \pi(\eta|_D)$.

Suppose further D is maximal.

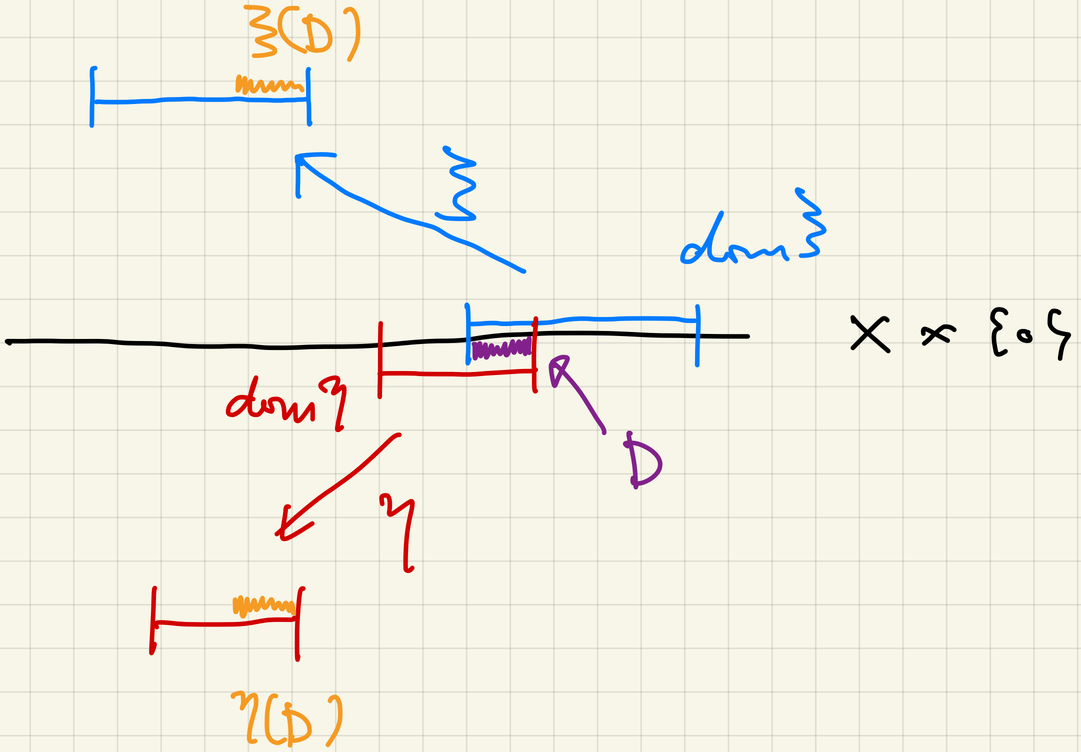
($\checkmark$ $D \subset X \times \{0\}$.)



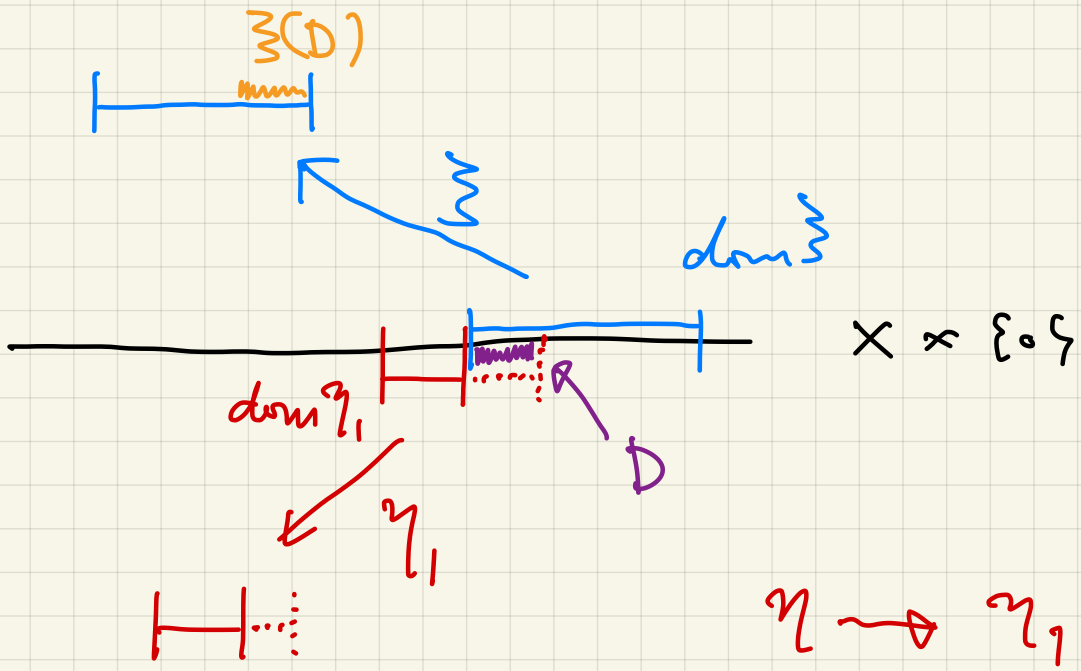
$\big\downarrow$ folding

Identify $\xi(D)$ and $\eta(D)$

and also identify $\xi|_D$ and $\eta|_D$.



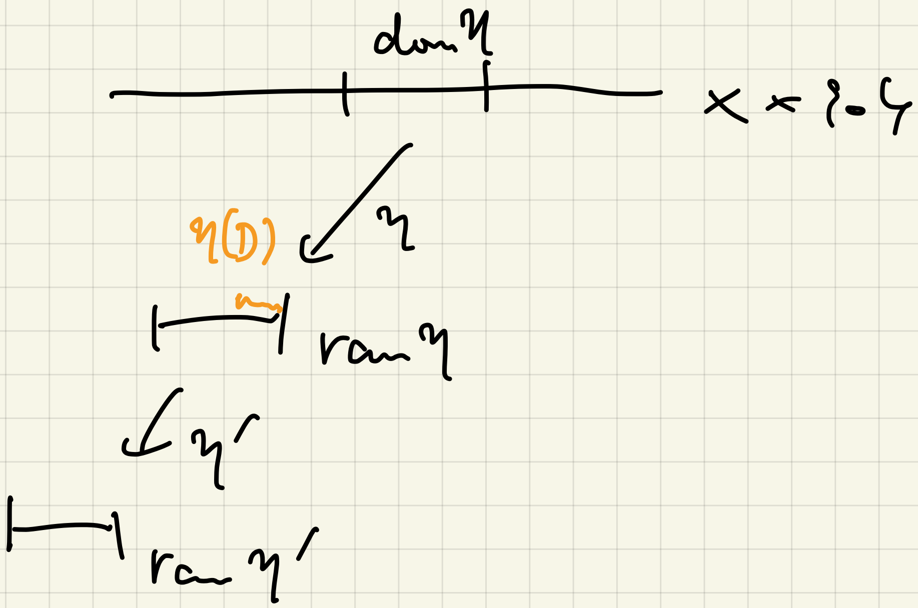
$\downarrow$ folding.



$\eta \rightsquigarrow \eta_1$

$\overline{\Theta} \to \overline{\Theta}_1$, $\overline{X} \to \overline{X}_1$

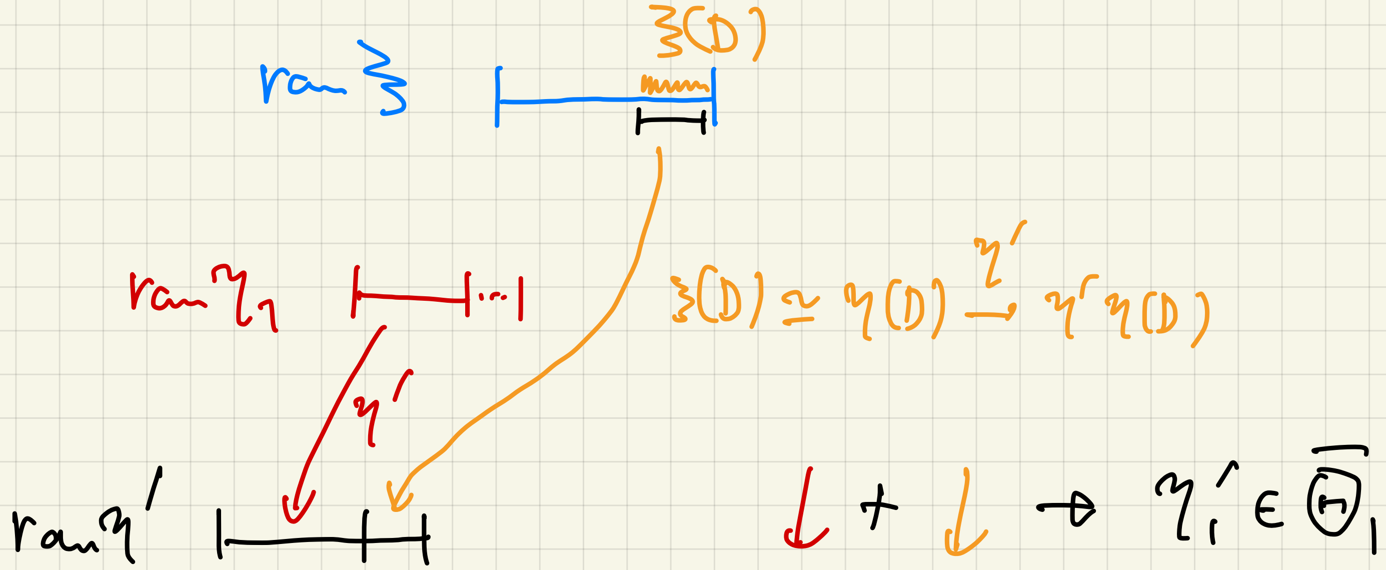
Rem If $\exists \eta'$ defined on $\operatorname{ran}\eta$,



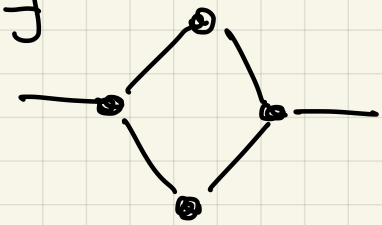
then after the folding,

$$\eta'|_{\eta(D)} \text{ is defined on } \xi(D).$$

(Do not delete $\eta'|_{\eta(D)}$!)



☆ $C(\bar{\Theta}) - \mu(\bar{X}) \geq C(\bar{\Theta}_1) - \mu(\bar{X}_1)$.

(If $\exists$  , then " $>$ " holds.)

☆ $M := L + L^3$.

$$\bar{\Theta} \to \bar{\Theta}_1 \to \bar{\Theta}_2 \to \cdots \to \bar{\Theta}_M$$

foldings.

Then $\Pi : \bar{X}_M \xrightarrow{\sim} X$.

(because all two points in the same Π-fiber are identified after M foldings, as seen in the single-graph case.)

☆ $C(\overline{\Theta}_M) \geq C(\overline{\mathcal{T}})$.

<u>Pf</u> Every $\theta \in \overline{\Theta}_M$ is identified with

some element of $\overline{\mathcal{T}}$ under $\pi : \overline{X}_M \xrightarrow{\simeq} X$.

I.e. $\pi(\theta)$ is a $\overline{\mathcal{T}}$-edge.

Since $\overline{\Theta}_M$ is a graphing of R and

$\overline{\mathcal{T}}$ is a treeing,

every $\overline{\mathcal{T}}$-edge must appear

as an $\pi(\overline{\Theta}_M)$-edge.

Therefore $C(\overline{\Theta}_M) \geq C(\overline{\mathcal{T}})$. ▨

$$
\begin{array}{ccc}
\overline{X}_M & \text{———} & \overline{\Theta}_M \\
\simeq \big\downarrow \pi & & \\
X & \text{———} & \overline{\mathcal{T}} : \text{treeing}
\end{array}
$$

Conclusion

$$
\begin{aligned}
C(\overline{\Theta}) - \mu(\overline{X}) &\geq C(\overline{\Theta}_1) - \mu(\overline{X}_1) \\
&\geq C(\overline{\Theta}_2) - \mu(\overline{X}_2) \\
&\vdots \quad \text{folding} \\
&\geq C(\overline{\Theta}_M) - \mu(\overline{X}_M) \\
&\geq C(\overline{\Psi}) - \mu(X)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
C(\overline{\Theta}) - \mu(\overline{X}) &= C(\Theta) - \mu(X) \\
&< C(R) + 2\varepsilon - \mu(X)
\end{aligned}
$$

Thus $\quad C(\overline{\Psi}) < C(R) + 2\varepsilon$.

This completes the Pf

in the case where Ψ is finite. $\lrcorner$

Finally assume $\bar{\Psi}$ is infinite.

(e.g. $C(\bar{\Psi}) = \infty$).

$\bar{\Psi} = \{\psi_1, \psi_2, \cdots\}$ a treeing of R.

Let $\bar{\Phi}$ be a graphing of R.

We show $C(\bar{\Phi}) \geq C(\bar{\Psi})$.

After subdivisions of each element of $\bar{\Phi}$, we may assume every $\varphi \in \bar{\Phi}$ is written as a single $\bar{\Psi}$-word.

$\bar{\Psi}_k := \{\psi_1, \cdots, \psi_k\}$.

$\bar{\Phi}_n := \{\varphi_1, \cdots, \varphi_n\}$.

Fix k. Choose a large $n_k \in \mathbb{N}$

s.t. $\text{graph } \psi_i|_{\text{dom}\,\psi_i \setminus \underset{\text{small}}{\underline{E_i}}} \subset R_{\bar{\Phi}_{n_k}}$, $\forall i \leq k$

where $E_i \subset \mathrm{dom}\,\psi_i$ $(i \leq k)$ is so small that $\mu(E_i) < \dfrac{1}{2^k}$.

By our assumption on $\overline{\Phi}$,

$\exists\, \ell > k$ large s.t. every element of $\overline{\Phi}_{n_k}$ is written as a single $\overline{\Psi}_\ell$-word.

$$\Theta := \overline{\Phi}_{n_k} \cup \{\psi_i|_{E_i}\}_{i=1}^{k} \cup \left(\overline{\Psi}_\ell \setminus \overline{\Psi}_k\right).$$

$$\rightarrow \quad R_\Theta = R_{\overline{\Psi}_\ell}.$$

$$C(\Theta) < C(\overline{\Phi}_{n_k}) + \frac{k}{2^k} + C(\overline{\Psi}_\ell \setminus \overline{\Psi}_k)$$

$$\geq$$

$$C(R_\Theta) \underset{\overline{\Psi}_\ell :\ \text{finite}}{=} C(\overline{\Psi}_\ell) = C(\overline{\Psi}_\ell \setminus \overline{\Psi}_k) + C(\overline{\Psi}_k)$$

Therefore

$$C(\overline{\Phi}_{n_k}) + \frac{k}{2^k} \geq C(\overline{\Psi}_k)$$

As $k \to \infty$, $C(\overline{\Phi}) \geq C(\overline{\Psi})$. $\blacksquare$

Ref: D. Gaborian, Around the orbit equivalence theory of the free groups.

Available at his webpage.