What is the cdh topology?

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Abstract

In this short expository note we discuss how the cdh topology can be defined as the coarsest topology satisfying étale excision, and having a well-defined theory of cohomology with compact support.

Mayer-Vietoris $\geq$ Zariski topology

Recall that if we have a topological space X, and two open subspaces $U, V \subseteq X$ such that $U \cup V = X$, there is an associated long exact sequence of (singular, for example) cohomology with F-coefficients (for F a field, or more generally, a sheaf of abelian groups on X)

$$\begin{aligned} \cdots \to H^n(X,F) \to & H^n(U,F) \oplus H^n(V,F) \\ & \to H^n(U \cap V, F) \to H^{n+1}(X,F) \to \dots . \end{aligned} \tag{1}$$

Recall also that the Zariski cohomology of any sheaf F on an algebraic variety or a scheme X also has this property. In fact, it is possible to define the Zariski (Grothendieck) topology via this property.

Definition 1 (Cf. [BG73] for example)**.** *The Zariski topology on the category of noetherian schemes is the coarsest topology such that every sheaf $F : Sch^{op} \to Ab$ has*[1] *$F(\varnothing) = 0$, and for every cartesian square*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow j \\ U & \underset{i}{\longrightarrow} & X \end{array}$$

such that i and j are open immersions, and $j^{-1}(X \backslash U) \to X \backslash U$ is an isomorphism, the associated morphisms fit into a long exact sequence[2] *as in Equation* (1).

Equivalently, (cf. Appendix) for any open subspace $V \subseteq X$, and closed subspace $Z \subseteq X$ contained in V, (e.g., $Z = X \setminus U$ for the U from above), there are isomorphisms

$$H^n_Z(X,F) \xrightarrow{\sim} H^n_Z(V,F) \tag{2}$$

where $H^n_Z(X,F)$ are defined as the groups which fit into long exact sequences[3]

$$\begin{aligned} \cdots \to H^n_Z(X,F) \to & H^n(X,F) \\ & \to H^n(U,F) \to H^{n+1}_Z(X,F) \to \dots . \end{aligned} \tag{3}$$

Excision $\geq$ Nisnevich topology

If we are working with smooth manifolds then the isomorphism (2), sometimes called *excision*, means that, when calculating $H^n_Z(X,F)$, we can assume that X is a tubular neighbourhood of Z, or even that X is a vector bundle over Z, and Z is the zero section. In light of the long exact sequence (3), this makes calculating the cohomology with supports $H^n_Z(X,F)$ relatively accessible. When working with varieties though, even smooth varieties, Zariski locally we cannot make this assumption.

On the other hand, suppose we don't insist on open immersions, but allow étale morphisms. Étale locally on X, at least when $Z \to X$ is a closed immersion between smooth varieties, we *can* assume X is a vector bundle over Z, and Z is the zero section. Let us explain. Zariski locally on X, there exists an étale morphism $q : X \to \mathbb{A}^d$ such that $Z = q^{-1}(\mathbb{A}^{d-c} \times \{0, \dots, 0\})$ for some c, d. Using q we can build two étale morphisms $X \overset{\pi_1}{\leftarrow} X' \overset{\pi_2}{\to} Z \times \mathbb{A}^c$ such that $Z \cong \pi_1^{-1}(Z) = \pi_2^{-1}(Z \times \{0, \dots, 0\}))$, cf. [MV99, Proof of Lemma 3.2.28].

Example 2. Consider any rational point $x \in E$ in an elliptic curve E over some k. Choose a local parameter t in the dvr $\mathcal{O}_{E,x}$ and lift it to some $f \in \mathcal{O}_E(V)$ on some open affine neighbourhood $V \ni x$. This defines a map $f : V \to \mathbb{A}^1_k$ sending x to the origin $0 \in \mathbb{A}^1_k$ which is étale at x. As E is a curve, X is obtained by removing some points from E, that is, we have $V = E - \{x_1, \dots, x_n\}$. Removing some more points if necessary, we may assume that $V \to \mathbb{A}^1_k$ is étale everywhere, and is an isomorphism over $0 \in \mathbb{A}^1_k$. □

Now if we allow j to be an étale morphism in the square of Definition 1, then up to shrinking X a little, we automatically obtain isomorphisms $H^n_Z(X) \cong H^n_{Z \times (0, \dots, 0)}(Z \times \mathbb{A}^c)$, where $Z = X - U$. Indeed, asking for the excision isomorphisms (2) is equivalent to demanding that squares as in Definition 1 induce long exact sequences generalising Equation (1), where j is now allowed to be an étale morphism, cf. Appendix.

Definition 3. *The Nisnevich topology on the category of noetherian schemes is the coarsest topology such that any sheaf F has $F(\varnothing) = 0$, and for every cartesian square*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow j \\ U & \underset{i}{\longrightarrow} & X \end{array}$$

such that i is an open immersion, j is an étale morphism, and $j^{-1}(X \backslash U) \to X \backslash U$ is an isomorphism, the associated morphisms fit into a long exact sequence[4] *as in Equation* (1).

[1] The condition $F(\varnothing) = 0$ is equivalent to asking that the empty family $\{\}$ is a covering family of $\varnothing$.

[2] Actually, the condition we really need is that the canonical morphism from $C^\bullet(X,F)$ to the deshifted cone of the canonical morphism $C^\bullet(U,F) \oplus C^\bullet(V,F) \to C^\bullet(U \cap V, F)$ is a quasi-isomorphism, where $C^\bullet(X,F), C^\bullet(U,F), C^\bullet(V,F), C^\bullet(U \cap V, F)$ are the complexes calculating $H^n(X,F), H^n(U,F), H^n(V,F), H^n(U \cap V, F)$. For the sake of the reader uncomfortable with triangulated categories, we will pretend that the long exact sequences condition is the same as the distinguished triangle condition, but a priori the long exact sequence condition is weaker.

[3] By which we really mean, that $H^n_Z(X,F)$ is the $(n-1)$th cohomology of $Cone(C^\bullet(X,F) \to C^\bullet(U,F))$., cf. Footnote 2.

[4] Cf. Footnote 2.

Equivalently, it is the coarsest topology such that for all sheaves F, we have $F(\varnothing) = 0$, and:

(Exc) *For any closed immersion $Z \to X$, and étale morphism $j : V \to X$ such that $j^{-1}(Z) = Z$ we have isomorphisms $H^n_Z(X, F) \cong H^n_Z(V, F)$.*

Example 4. Continuing with Example 2, we find that for any topology at least as fine as the Nisnevich topology, we have $H^n_{\{O\}}(E, F) \cong H^n_{\{0\}}(\mathbb{A}^1_{\mathbb{C}}, F)$. This is true for the étale topology, for example.

Compact support ≥ cdh topology

Two of the defining characteristics of cohomology with compact support of topological spaces, are

(CS1) If X is compact then $H^n_c(X, F) = H^n(X, F)$.

(CS2) If $U \subseteq X$ is an open subspace and $Z = X \setminus U$ its closed complement, there is a long exact sequence

$$\dots \to H^n_c(U, F) \to H^n_c(X, F) \to H^n_c(Z, F) \to H^{n+1}_c(U, F) \to \dots.$$

For cohomology theories of algebraic varieties, its not always clear what "compact support" should mean, however, we can force the above two properties with the following "definition".

"Definition" 5. *If F is a sheaf on the category of varieties equipped with some topology τ, define $H^n_{\tau,c}(X, F)$ to be the groups which fit into a long exact sequence*[5]

$$\dots \to H^{n-1}_\tau(\partial X, F) \to H^n_{\tau,c}(X, F) \to H^n_\tau(\overline{X}, F) \to H^n_\tau(\partial X, F) \to \dots$$

where $X \to \overline{X}$ is an open immersion into a proper variety, and $\partial X = \overline{X} \setminus X$.

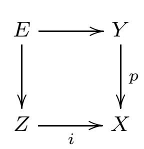
The obvious problem with "Definition" 5 is that it depends on the choice of compactification. The cdh topology adresses this.

Recall that given a second open immersion $X \to \overline{X}'$ into a proper variety with closed complement $\partial X'$, there exists a compactification dominating the two. That is, we can find proper morphisms $\overline{X}'' \to \overline{X}$ and $\overline{X}'' \to \overline{X}'$ which are isomorphisms over X. Now, again using Lemma 7, we see that "Definition" 5 is independent of the choice of compactification, *if and only if* for any proper morphism $p : \overline{X}' \to \overline{X}$ which is an isomorphism over X, we have a long exact sequence

$$\dots \to H^n_\tau(\overline{X}, F) \to H^n_\tau(\overline{X}', F) \oplus H^n_\tau(\partial X, F) \to H^n_\tau(\partial X', F) \to H^{n+1}_\tau(\overline{X}, F) \to \dots \quad (4)$$

where $\partial X' = \overline{X}' \times_{\overline{X}} \partial X$.

Definition 6. *The cdh topology on the category of noetherian schemes is the coarsest topology which is finer than the Nisnevich topology, and such for every sheaf F and every cartesian square*



such that i is a closed immersion, p is proper, and $Y \setminus E \to X \setminus Z$ is an isomorphism, the associated morphisms fit into a long exact sequence

$$\dots \to H^n(X, F) \to H^n(Y, F) \oplus H^n(Z, F) \to H^n(E, F) \to H^{n+1}(X, F) \to \dots. \quad (5)$$

Equivalently, it is the coarsest topology finer than the Nisnevich topology, such that for any open immersion $U \to X$, and proper morphism $p : Y \to X$ such that $p^{-1}(U) = U$ we have isomorphisms[6]

$$H^n(X, X-U; F) \xrightarrow{\sim} H^n(Y, Y-U; F) \quad (6)$$

and $F(\varnothing) = 0$, for all sheaves F.

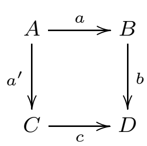
More heuristically:

Definition 6'. *The cdh topology is the coarsest topology for which*

1. *excision is satisfied, and*
2. *cohomology with compact support is well-defined.*

Appendex 1. Some homological algebra

Exercise 7. In the category of chain complexes of abelian groups, suppose that the square



is commutative. Then $Cone\left(A \overset{a+a'}{\to} B \oplus C\right) \overset{b-c}{\to} D$ is a quasi-isomorphism if and only if $Cone(a) \to Cone(c)$ is a quasi-isomorphism.

Hint: Show that $Cone(Cone(A \to B) \to Cone(C \to D))$ is *equal* to $Cone(Cone(A \to B \oplus C) \to D)$ as a chain complex and note that this complex is acyclic if and only if the two morphisms in question are quasi-isomorphisms.

Corollary 8. *Equation* (1) *(resp.Equation* (5)*) is a long exact sequence if and only if Equation* (2) *(resp.Equation* (6)*) is an isomorphism.*

Appendex 2. Covering families

Often the Nisnevich, and cdh topologies are defined as topologies generated by certain covering families $\{U \to X, V \to X\}$, $\{Z \to X, E \to X\}$ associated to squares as in Definitions 3 and 6. For the equivalence of these definitions see [Voe10].

References

[BG73] Kenneth S Brown and Stephen M Gersten. Algebraic K-theory as generalized sheaf cohomology. In *Higher K-theories*, pages 266–292. Springer, 1973.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Publications mathématiques de l'IHÉS*, 90(1):45–143, 1999.

[Voe10] Vladimir Voevodsky. Homotopy theory of simplicial sheaves in completely decomposable topologies. *Journal of Pure and Applied Algebra*, 214(8):1384–1398, 2010.

[5] More precisely, we define $H^n_c(X)$ to be the $(n-1)$th cohomology of $Cone(C^\bullet(\overline{X}, F) \to C^\bullet(\partial X, F))$ where $C^\bullet(\overline{X}, F), C^\bullet(\partial X, F)$ are complexes calculating the groups $H^n(\overline{X}, F), H^n(\partial X, F)$.

[6] Here, by $H^n(X, X-U; F)$ we mean the groups fitting into long exact sequences $\dots \to H^n(X, F) \to H^n(X-U, F) \to H^n(X, X-U; F) \to H^{n+1}(X, F) \to \dots$, or more precisely, the $(n-1)$th cohomology of $Cone(C^\bullet(X, F) \to C^\bullet(X-U, F))$.