Universal homeomorphisms of and not of finite presentation

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July 4, 2018

Abstract

We show that on qcqs schemes any (completely decomposed) universal homeomorphism is a filtered limit of (completely decomposed) universal homeomorphisms of finite presentation, at least up to refinement.

Along the way we note that if $f : Y \to X$ is a finite universal homeomorphism of qcqs $\mathbb{F}_p$-schemes, then the nth iterated Frobenius factors as through it for n big enough, $Frob^n : Y \to X \to Y^{[p^n]}$.

Recall that a *universal homeomorphism* (uh) is a morphism of schemes $T \to S$ such that $S' \times_S T \to S$ induces a homeomorphism on underlying topological spaces for every S-scheme S'. A morphism $T \to S$ is *completely decomposed* (cd) if for every point $s \in S$ there is a factorisation $s \to T \to S$. In this note we prove the following theorem.

Theorem 1 (Theorem 27)**.** *Suppose that $T \to S_0$ is a uh (resp. cduh) of qcqs schemes. Then there exists a filtered system $(S_\lambda \to S_0)_{\lambda \in \Lambda}$ of uh (resp. cduh) of finite presentation, and a factorisation*

$$\varprojlim S_\lambda \to T \to S_0.$$

Remark 2. We point out that the equal characteristic case is not so hard: pulling back along the perfection of the seminormalisation of the reduction $(((S_0)^{\mathrm{red}})^{\mathrm{sn}})^{\mathrm{perf}}$ we can assume that S_0 is seminormal and $f : T \to S$ is completely decomposed. Since we can also assume T reduced, f is a cduh from a reduced scheme to a seminormal scheme, and therefore an isomorphism.

The trickier case is when S_0 is connected, not irreducible, and its irreducible components have different characteristics, cf. Prop.22.

On our way to proving Theorem 1 we prove the following which removes the excellent hypothesis from [Kol97, Prop.6.6]. See Cor.20, Cor.21 for versions without the "reduced" hypothesis.

Proposition 3 (cf. Cor.20)**.** *If S is a qc reduced $\mathbb{F}_p$-scheme and $f : T \to S$ a finite uh morphism, then there is some $q = p^n$ such that the nth iterated Frobenius factors as*

$$\begin{tikzcd} T \arrow[r, "Frob^n"] \arrow[d] & T \arrow[d] \\ S \arrow[r, "Frob^n"'] \arrow[ur] & S \end{tikzcd}$$

As a corollary of Theorem 1 we deduce the following. We let SCH_S denoted the category of separated qcqs S-schemes.

Theorem 4. *Suppose S is a qcqs scheme, $\mathcal{C}$ is a category admitting filtered limits, and $F : \mathsf{SCH}_S \to \mathcal{C}$ is a functor commuting with filtered limits. If F sends finite presentation uh's (resp. cduh's) to isomorphisms then F sends all uh's (resp. cduh's) to isomorphisms.*

Proof. Suppose $Y \to X$ is a (cd)uh in SCH_S. By Theorem 27, there is a filtered system $(X_\lambda \to X)_{\lambda \in \Lambda}$ of (cd)uh's of finite presentation and a factorisation $\varprojlim X_\lambda \to Y \to X$. The morphism $\varprojlim X_\lambda \to X$ is again a (cd)uh Thm.9, [EGAIV3, Thm.8.10.5(vi)(vii)(viii)], so $\varprojlim X_\lambda \to Y$ is also a (cd)uh, and we can apply the theorem again to find a filtered system $(Y_\rho \to Y)_{\rho \in R}$ of (cd)uh's of finite presentation and a factorisation $\varprojlim Y_\rho \to \varprojlim X_\lambda \to Y$. Now, As F commutes with filtered limits and sends finite presentation (cd)uh's to isomorphisms, we have the diagram

$$\begin{tikzcd} F(\lim Y_\rho) \arrow[r] \arrow[rr, bend left, equal] & F(\lim X_\lambda) \arrow[r] \arrow[rr, bend right, equal] & F(Y) \arrow[r] & F(X) \end{tikzcd}$$

and it follows that $F(Y) \to F(X)$ is an isomorphism. $\square$

This note was motivated by applications to K-theory, motivic cohomology, and topologies such as the cdh- and h-topologies. For example, we have the following version of [HK18, Cor.6.12] which does not assume the base to be Nagata. We let Sch_S denoted the category of separated finite type S-schemes.

Corollary 5. *Suppose that S is a separated noetherian scheme, and consider the Grothendieck topology* CDH *on* SCH_S *generated by the* cdh*-topology on* Sch_S*. Then the image of the functor*

$$h_{\mathrm{CDH}} : \mathsf{SCH}_S \to \mathsf{Shv}_{\mathrm{CDH}}(\mathsf{SCH}_S)$$

is the localisation of SCH_S *at the class of cduh's,*

$$\mathsf{SCH}_S[\mathrm{cduh}^{-1}] \xrightarrow{\sim} h_{\mathrm{CDH}}(\mathsf{SCH}_S).$$

In particular, for any $X, Y \in \mathsf{Sch}_S$*, we have*

$$h_{\mathrm{cdh}}(Y)(X) = \hom_{\mathsf{SCH}_S}(X^{\mathrm{sn}}, Y^{\mathrm{sn}}) = \hom_{\mathsf{SCH}_S}(X, Y^{\mathrm{sn}}).$$

Remark 6. Voevodsky proves a version of this statement for the h-topology on Sch_S when S is excellent [Voe96], and Rydh proves a version for the topology generated by open coverings and finite presentation universally subtrusive morphisms on SCH_S for S qcqs (this is the h-topology when restricted to noetherian schemes), [Ryd10, Thm.8.16].

Remark 7. For the CDH-topology we don't actually need the full power of Theorem 4. It is enough to use part 1 of Theorem 25.

Remark 8. We implicitly use that SCH_S is the category of schemes obtained as limits of filtered systems in Sch_S with affine transition morphisms, [Tem11, Thm.1.1.2].

Proof. First note that for any $T \in \mathsf{SCH}_S$, the system of cduh's to T has an initial element—the seminormalisation $T^{\mathrm{sn}} \to T$, cf. [HK18, Prop.2.8], [Swa80], and Lem.13. Consequently, the localisation $\mathsf{SCH}_S \to \mathsf{SCH}_S[\mathrm{cduh}^{-1}]$ not only exists, it is the retraction of SCH_S to the subcategory of seminormal S-schemes $(-)^{\mathrm{sn}} : \mathsf{SCH}_S \to \mathsf{SCH}_S^{\mathrm{sn}}$.

Now, certainly, finite presentation cduh's $Y \to X$ are sent to isomorphisms as they are covers and the diagonal induces an isomorphism $Y^{\mathrm{red}} \xrightarrow{\sim} (Y \times_X Y)^{\mathrm{red}}$. The functor $\mathsf{SCH}_S \to \mathsf{PreShv}(\mathsf{SCH}_S)$ preserves limits, and sheafification is exact, so we deduce from Theorem 4 that h_{CDH} sends all cduh's to isomorphisms, and obtain a canonical factorisation

$$\mathsf{SCH}_S \to \underbrace{\mathsf{SCH}_S[\mathrm{cduh}^{-1}]}_{\cong \mathsf{SCH}_S^{\mathrm{sn}}} \to h_{\mathrm{CDH}}(\mathsf{SCH}_S).$$

Now to finish it suffices to show that for $T, T' \in \mathsf{SCH}_S^{\mathrm{sn}}$ the canonical morphism $\hom_{\mathsf{SCH}_S}(T, T') \to h_{\mathrm{CDH}}(T')(T)$ is an isomorphism.

Injectivity: Suppose $f, g : T \rightrightarrows T'$ are two morphisms that become isomorphisms in $h_{\mathrm{CDH}}(T')(T)$. Then there is a CDH-cover $\mathcal{U} = \{\iota_j : U_j \to T\}$ such that $f \circ \iota_j = g \circ \iota_j$ for each j. But as the family $\mathcal{U}$ is surjective and completely decomposed, this implies $f|_\eta = g|_\eta$ for each generic point of T, so there is a dense open $U \subseteq T$ such that $f|_U = g|_U$, but then $f = g$ as T is reduced and T' separated.

Surjectivity: For every element a of $h_{\mathrm{CDH}}(T')(T)$ there exists a CDH-cover $\mathcal{U} = \{U_j \to T\}$ such that a can be represented as an element of $eq\left(\prod \hom(U_j, T') \rightrightarrows \prod \hom(U_i \times_T U_j, T')\right)$. We can assume that $\mathcal{U}$ is a composition of a finite presentation completely decomposed proper morphism $Y \to T$, and a completely decomposed finite family $\{U_j \to Y\}$ of finite presentation étale morphisms, [Kel18, Prop.34], [SV00, Prop.5.9]. For covers of this form, we have

$$\begin{aligned} eq&\left(\prod \hom(U_j, T') \rightrightarrows \prod \hom(U_i \times_T U_j, T')\right) \\ &= eq\Big(\hom(Y, T') \rightrightarrows \hom(Y \times_T Y, T')\Big) \end{aligned}$$

as the fppf topology is subcanonical. Now Y can be refined by a composition of a sequence of covers of the form $\{\iota : Z \to Y', \pi : B \to Y'\}$ where $Z \to Y'$ is closed and B is a proper morphism which is an isomorphism outside Z, [EGAIV3, Thm.8.8.2(ii),Thm.8.10.5], [SV00, proof of Prop.5.9]. So it suffices to show that

$$\hom(Y', T') \to \hom(Z, T') \times_{\hom(E,T')} \hom(B, T')$$

is surjective when Y' is seminormal and $E = Z \times_{Y'} B$.

Let $B' = \underline{\mathrm{Spec}}\, \pi_*\mathcal{O}_B$, cf. [Stacks, Tag 03GY]. As $B \to B'$ is proper, the target topological space has the quotient topology of the source. So by the cocycle condition, $B \to Z'$ factors through $B' \to Z'$ as a map of topological spaces. But then $\mathcal{O}_{B'} = \pi_*\mathcal{O}_B$ so it factors as a morphism of locally ringed spaces, or rather, as a morphism of schemes. Let $E' = Z \times_T B'$. As $E^{\mathrm{red}} \to (E')^{\mathrm{red}}$ is dominant with reduced source, we have produced an element of

$$\hom(Z^{\mathrm{red}}, T') \times_{\hom((E')^{\mathrm{red}},T')} \hom((B')^{\mathrm{red}}, T')$$

or equivalently, a morphism $Z^{\mathrm{red}} \sqcup_{(E')^{\mathrm{red}}} (B')^{\mathrm{red}} \to T'$, but since T' is seminormal $Z^{\mathrm{red}} \sqcup_{(E')^{\mathrm{red}}} (B')^{\mathrm{red}} \cong T$. See [HK18, Lem.2.10] for the existence of the pushout and the claimed isomorphism, noting that the statement of [HK18, Lem.2.10] is for Sch_S but the proof is valid for SCH_S, even replacing "finite" with "affine, quasi-finite", and certainly, $B' \to T$ (and therefore $E' \to Z$) is affine, quasi-finite, cf. [Stacks, Tag 0E0M]. □

1 Characterisations of universal homeomorphisms

Theorem 9. *Let $f : Y \to X$ be a morphism of schemes. The following are equivalent.*

1. *f is a* universal homeomorphism *(uh): For every X-scheme $T \to X$, the morphism $T \times_X Y \to T$ induces a homeomorphism on the underlying topological spaces.*

2. *f is integral, surjective, and universally injective.*

3. *f is affine, surjective, and universally injective.*

4. *f is a homeomorphism, and for every $y \in Y$, the field extension $k(y)/k(f(x))$ is purely inseparable.*

Proof. The equivalence of (1) and (2) is [EGAIV4, Cor.18.12.11] (equivalently [Stacks, Tag 04DF]). The equivalence of (2) and (3) is [Stacks, Tag 01WM], which actually says integral is equivalent to affine and universally closed. (1) implies (4) because a morphism is universally injective (on topological spaces) if and only if it is injective and all residue field extensions are purely inseparable, [Stacks, Tag 01S4]. Conversely, (4) implies (3) because homeomorphisms are affine, [Stacks, Tag 04DE]. $\square$

Remark 10. Note there are no finiteness conditions. There are universal homeomorphisms that are not of finite type, [Bou64, Chap.6,Sec.8,Exercise 3b], and finite universal homeomorphisms that are not finitely presented. For example countably many affine lines glued perpendicularly at the origin $A = \varinjlim_{n \to \infty} \mathbb{F}_p[x_1, \dots, x_n]/\langle x_i x_j : i \neq j \rangle$ and the map $\phi : A \to A; x_1 \mapsto x_1^p$ which is the Frobenius on the first one. Note the topology on $\mathrm{Spec}(A)$ is a kind of "profinite-type" topology in the sense that any open U containing the origin there is an N such that U contains all the nth line for all $n > N$. To generate the kernel of $A[y] \to A; \sum a_m y^m \mapsto \sum \phi(a_m) x_1^m$, in addition to $y^p - x_1$ one needs all the $y x_n$ for $n > 1$.

Lemma 11. *Suppose that $Y \to X$ is a (cd)uh, and $\mathcal{A} \subseteq \mathcal{O}_Y$ is a sub-$\mathcal{O}_X$-algebra. Then both of $Y \to \underline{\mathrm{Spec}}(\mathcal{A}) \to X$ are (cd)uh's.*

Proof. By Theorem 9 it suffices to prove that $Y \to \underline{\mathrm{Spec}}(\mathcal{A})$ is surjective. But this follows from integrality, [Stacks, Tag 00GQ]. Cd-ness is clear. $\square$

Remark 12. Note that if A is reduced, then any uh $\phi : A \to B$ is injective. Indeed, if $\phi(a) = 0$, then $\phi(a) = 0$ in every residue field of B. But as ϕ is uh, this implies that $a = 0$ in every residue field of A, so a is contained in every prime of A, so a is nilpotent. But A is reduced so $a = 0$ in A.

2 Seminormalisation and perfection

Recall that a reduced ring A is *seminormal* if for every $b, c \in A$ such that $b^2 = c^3$, there exists $a \in A$ with $a^3 = b, a^2 = c$, [Swa80]. The inclusion $SemNor \to Red$ of seminormal reduced rings $SemNor$ into all reduced rings Red admits a left adjoint $(-)^{\mathrm{sn}} : Red \to SemNor$, [Swa80, Thm.4.1]. This induces a right adjoint $(-)^{\mathrm{sn}}$ to the inclusion functor

$$\left\{ \begin{array}{c} \text{Schemes s.t. } \mathcal{O}_X \text{ is a sheaf} \\ \text{of seminormal rings} \end{array} \right\} \rightleftarrows \{\text{All schemes}\} : (-)^{\mathrm{sn}}$$

[HK18, Prop.2.8]. On affine schemes we have $\mathrm{Spec}(A)^{\mathrm{sn}} = \mathrm{Spec}(A^{\mathrm{sn}})$, and on a general scheme X, the underlying topological space of X^{sn} is that of X, and structure sheaf is the sheaf associated to the presheaf $\mathcal{O}^{\mathrm{sn}}_{X^{\mathrm{red}}}$.

The following lemma is in [Swa80] but not stated as we want it.

Lemma 13. *It A is seminormal and B reduced, all cduh's* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *are isomorphisms.*

Note, in [Swa80] which only deals with reduced rings, cduh's are called subintegral extensions.

Proof. By Rem.12 $A \to B$ is injective, so it suffices to show surjectivity. Proof by contradiction. If $A \to B$ is not surjective, by [Swa80, Lem.2.6] there is $b \in B$ such that $b^2, b^3 \in A$, $b \notin A$. As A is seminormal there is also $a \in A$ such that $a^3 = b^3, a^2 = b^2$. By [Swa80, Lem.3.1] this implies $(a - b)^3 = 0$. But A and B are reduced, so $a = b$ contradicting $b \notin A$. $\square$

Recall that an $\mathbb{F}_p$-algebra A is *perfect* if $Frob : a \mapsto a^p$ is an isomorphism. The inclusion of perfect $\mathbb{F}_p$-algebras into all $\mathbb{F}_p$-algebras admits a left adjoint $(-)^{\mathrm{perf}}$ which can be explicitly calculated as:

$$A^{\mathrm{perf}} = \varinjlim \left(A \overset{Frob}{\to} A \overset{Frob}{\to} A \overset{Frob}{\to} \dots \right)$$

As we have $A[S^{-1}]^{\mathrm{perf}} = A^{\mathrm{perf}}[S^{-1}]$ and equalisers of perfect $\mathbb{F}_p$-algebras are perfect, $(-)^{\mathrm{perf}}$ on $\mathbb{F}_p$-algebras induces a right adjoint $(-)^{\mathrm{perf}}$ to the inclusion

$$\left\{ \begin{array}{c} \text{Schemes s.t. } \mathcal{O}_X \text{ is a sheaf} \\ \text{of perfect } \mathbb{F}_p\text{-algebras} \end{array} \right\} \rightleftarrows \{\text{All } \mathbb{F}_p\text{-schemes}\} : (-)^{\mathrm{perf}}$$

[HK18, proof of Prop.2.8]. On affine $\mathbb{F}_p$-schemes we have $\mathrm{Spec}(A)^{\mathrm{perf}} = \mathrm{Spec}(A^{\mathrm{perf}})$, and on a general $\mathbb{F}_p$-scheme X, the underlying topological space of X^{perf} is that of X, and structure sheaf is the sheaf associated to the presheaf $\mathcal{O}_X^{\mathrm{perf}}$.

Remark 14. The map $Frob : A^{\mathrm{perf}} \to A^{\mathrm{perf}}$ is surjective, so for any ideal $I \subseteq A^{\mathrm{perf}}$ the map $Frob : A^{\mathrm{perf}}/I \to A^{\mathrm{perf}}/I$ is surjective, so for any prime $\mathfrak{p} \subset A^{\mathrm{perf}}$ the map $Frob : k(\mathfrak{p}) \to k(\mathfrak{p})$ is surjective. That is, all residue fields of A^{perf} are perfect.

Lemma 15. *If A is a reduced resp. seminormal ring, then so is A^{perf}.*

Proof. Let $a \in A^{\mathrm{perf}}$ with $a^n = 0$. For some $q = p^m$, $a^q \in A$. But then $(a^q)^n = a^{qn} = (a^n)^q = 0$, so $a^q = 0$, so since A^{perf} is perfect, $a = 0$. Let $b, c \in A^{\mathrm{perf}}$ with $b^2 = c^3$. Choose q such that $b^q, c^q \in A$. Then $0 = (b^2 - c^3)^q = (b^q)^2 - (c^q)^3$ so since A is seminormal there is (a unique) $a \in A$ with $a^3 = b^q, a^2 = c^q$. But $(-)^q$ is surjective on A^{perf}, so there is $a' \in A^{\mathrm{perf}}$ with $(a')^q = a$, and therefore $((a')^2 - c)^q = 0, ((a')^3 - b)^q = 0$, but $(-)^q$ is injective on A^{perf} so $(a')^2 = c, (a')^3 = b$. □

3 The affine case

Lemma 16. *Suppose that $A \subseteq B$ is cduh of reduced rings. Then B is the filtered colimit $B = \varinjlim B_\lambda$ of the cduh A-algebras of finite presentation.*

Proof. Certainly, every algebra is the filtered colimit $B = \varinjlim B_\lambda$ of the factorisations $A \to B_\lambda \to B$ through finite presentation algebras $A \to B_\lambda$, so it suffices to show that the cduh ones are cofinal. Given some $A \to B_\lambda \xrightarrow{\phi} B$, the image $\phi(B_\lambda)$ is also cduh, Lem.11. As such, $\phi(B_\lambda)$ is the filtered union of its subalgebras which are obtained by a finite number of elementary cduh, [Swa80, Thm.2.8], (recall that an *elementary* cduh is a morphism of reduced rings of the form $C \to C[b]$ such that $b^2, b^3 \in C$). As B_λ is finite presentation, $\phi(B_\lambda)$ is finite, so $\phi(B_\lambda)$ *itself* is obtained by a finite number of elementary cduh extensions. Say $\phi(B_\lambda) = A[b_1, \dots, b_n]$ with $b_i^2, b_i^3 \in A[b_1, \dots, b_{i-1}]$. For any elementary cduh $C \to C[b]$, there is clearly a surjection $C' = C[x]/\langle x^2 - b^2, x^3 - b^3\rangle \to C[b]; x \mapsto b$, and in fact $(C')^{\mathrm{red}} = C[b]$, [Swa80, Lem.4.4]. So inductively, we build an epimorphism

$$A' = A[x_1, \dots, x_n]/\langle x_i^2 - b_{i,2}, x_i^3 - b_{i,3} \mid i = 1, \dots, n\rangle \to A[b_1, \dots, b_n]; x_i \mapsto b_i$$

such that $(A')^{\mathrm{red}} = A[b_1, \dots, b_n] = \phi(B_\lambda)$ where

$$b_{j,2}, b_{j,3} \in A[x_1, \dots, x_{i-1}]/\langle x_i^2 - b_{i,2}, x_i^3 - b_{i,3} \;\mid\; i = 1, \dots, j-1\rangle$$

are lifts of $b_i^2, b_i^3 \in A[b_1, \dots, b_{i-1}]$. Now $(A')^{\mathrm{red}} = \phi(B_\lambda)$ is the filtered colimit $\varinjlim A'/I$ over finitely generated subideals $I \subseteq nil(A')$ so the morphism with finite presentation source $B_\lambda \to \phi(B_\lambda)$ factors through some $B_\lambda \to A'/I \to \phi(B_\lambda)$. As $A \to \phi(B_\lambda) = (A'/I)^{\mathrm{red}}$ is a cduh, so is $A \to A'/I$. $\square$

Lemma 17. *Suppose that A is an $\mathbb{F}_p$-algebra, and A' a sub-A-algebra of A^{perf}. Then A' is a filtered colimit $A' = \varinjlim B_\lambda$ of finite presentation uh A-algebras.*

Proof. As every A-algebra A' is the filtered colimit of the factorisations $A \to B_\lambda \to A'$ through A-algebras of finite presentation, it suffices to show that when $A' \subseteq A^{\mathrm{perf}}$, the uh's are cofinal. Given some $A \to B \xrightarrow{\phi} A'$, consider its image $\phi(B)$. For each generator $b_1, \dots, b_n$ of B, choose $a_i \in A$ and some p-power q such that $\phi(b_i)^q = \iota(a_i)$ in $A' \subseteq A^{\mathrm{perf}}$ where $A \xrightarrow{\iota} A'$ is the structural morphism. Replacing B with $B/\langle b_i^q - a_i \rangle$, we can assume there exists an epimorphism of the form $B' = A[x_1, \dots, x_n]/\langle x_i^q - a_i : i = 1, \dots, n \rangle \to B$. Now $A \to \phi(B)$ and $A \to B'$ induce uh's, so the closed immersions $\mathrm{Spec}(\phi(B)) \to \mathrm{Spec}(B) \to \mathrm{Spec}(B')$ are surjective, and therefore also uh's. Consequently, $A \to B$ is a uh. $\square$

Lemma 18. *If $\phi : A \to B$ is a uh of $\mathbb{F}_p$-algebras, then for every $b \in B$ there is some $q = p^n$ such that $b^q \in \phi(A)$.*

Proof. If its true for A, B reduced, its true in general: Suppose there is $a \in A$ and a p-power q such that $b^q = \phi^{\mathrm{red}}(a)$ in B^{red}. Then $b^q - \phi(a)$ is nilpotent in B. So for some large enough p power q', we have $(b^q - \phi(a))^{q'} = 0$ in B. Replacing q with qq' and a with $a^{q'}$, we can assume $b^q = \phi(a)$ in B.

If its true for A seminormal, B reduced, its true for A, B reduced: Consider the seminormalisation A^{sn} of A, [Swa80, Thm.4.1]. There is a p-power and $a \in A^{\mathrm{sn}}$ such that $b^q = a$ in $(A^{\mathrm{sn}} \otimes_A B)^{\mathrm{red}}$. So $b^q - a$ is nilpotent in $A^{\mathrm{sn}} \otimes_A B$, and as before, we can assume $b^q = a$ in $A^{\mathrm{sn}} \otimes_A B$. As $A \subseteq A^{\mathrm{sn}}$ is subintegral, A^{sn} is the subintegral closure of A in A^{sn}, so by [Swa80, Thm.2.8] there is some finite number of elementary subextensions $A \subseteq A[c_1] \subseteq \cdots \subseteq A[c_1, \dots, c_n] \subseteq A^{\mathrm{sn}}$ such that $a \in A[c_1, \dots, c_n]$. By definition, $c_i^2, c_i^3 \in A[c_1, \dots, c_{i-1}]$ for all i, so in particular, $A[c_1, \dots, c_i]^p \subseteq A[c_1, \dots, c_{i-1}]$, and by induction, $A[c_1, \dots, c_n]^{p^n} \subseteq A$. It follows that $a^{p^n} \in A$, so replacing a with a^{p^n} and q with qp^n, we have found $a \in A$ and q such that $b^q - a = 0$ in $A^{\mathrm{sn}} \otimes_A B$. But $B \to A^{\mathrm{sn}} \otimes_A B$ is a cduh from a reduced ring, so it is injective, Rem.12, and so $b^q = a$ in B.

If its true for A seminormal perfect, B reduced, its true for A seminormal, B reduced: There is $a \in A^{\mathrm{perf}} = \varinjlim(A \xrightarrow{Frob} A \xrightarrow{Frob} \dots)$ such that $b^q = a$ in

$(A^{\mathrm{perf}} \otimes_A B)^{\mathrm{red}}$. As before, making q bigger and replacing a, we can assume $b^q = a$ in $A^{\mathrm{perf}} \otimes_A B$. By construction of A^{perf}, there exists some q' such that $a^{q'} \in A \subseteq A^{\mathrm{perf}}$, so replacing a with $a^{q'}$ and q with qq', we have $b^q = a$ in $A^{\mathrm{perf}} \otimes_A B$ for some $a \in A$. But $B \to A^{\mathrm{perf}} \otimes_A B$ is a uh from a reduced ring, so it is injective, Rem.12, so $b^q = a$ in B.

Its true for A seminormal perfect, B reduced: If A is seminormal and perfect, then ϕ is a cduh morphism from a seminormal ring to a reduced ring, so it is an isomorphism, Lem.13. $\square$

Corollary 19. *If S is a qc $\mathbb{F}_p$-scheme and $f : T \to S$ a finite uh morphism, then there is some $q = p^n$ such that $\mathcal{O}_T^q \subseteq \mathrm{im}(\mathcal{O}_S)$.*

Proof. Since S is qc, there exists a finite open affine cover $\{U_i\}_{i=1}^n$, and since $T \to S$ is finite, for each i there exists finitely many generators $f_{ij} \in \mathcal{O}_T(U_i)$. Applying the previous part to each f_{ij} and choosing the largest q we obtain a p power q such that $\mathcal{O}_T^q \subseteq \mathrm{im}(\mathcal{O}_S)$. As S is reduced, $\mathcal{O}_S = \mathrm{im}(\mathcal{O}_S)$ so we obtain the desired factorisation. $\square$

Corollary 20. *Suppose S is a qc $\mathbb{F}_p$-scheme and $f : T \to S$ a finite uh morphism. The nth iterated Frobenius factors as*

$$\begin{tikzcd}
T \arrow[r, "Frob^n"] \arrow[d] & T \arrow[d] \\
S \arrow[r, "Frob^n"'] \arrow[ur] & S
\end{tikzcd}$$

for some n, if and only if $Frob^{n'} \ker(\mathcal{O}_S \to \mathcal{O}_T) = 0$ for some n', (e.g., if S is reduced, Rem.12).

Proof. Suppose $Frob^{n'} \ker(\mathcal{O}_S \to \mathcal{O}_T) = 0$. Then $Frob^{n'}$ factors as $\mathcal{O}_S \to \mathrm{im}(\mathcal{O}_S \to \mathcal{O}_T) \to \mathcal{O}_S$. Next, Cor.19 gives an inclusion $\mathcal{O}_T^{p^m} \subseteq \mathrm{im}(\mathcal{O}_S \to \mathcal{O}_T)$ for some m. So we obtain the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}_T \arrow[r] \arrow[rr, bend left, "Frob^m"] & \mathcal{O}_T^{p^m} \arrow[r] \arrow[dr] & \mathcal{O}_T \arrow[r, "Frob^{n'}"'] & \mathcal{O}_T \\
 & & \mathrm{im}(\mathcal{O}_S \to \mathcal{O}_T) \arrow[u] \arrow[dr] & \\
\mathcal{O}_S \arrow[uu] \arrow[rr, "Frob^m"'] & & \mathcal{O}_S \arrow[u] \arrow[r, "Frob^{n'}"'] & \mathcal{O}_S \arrow[uu]
\end{tikzcd}$$

Applying $\underline{\mathrm{Spec}}$ produces the result. Conversely, if we have such a factorisation, then $\ker(\mathcal{O}_S \to \mathcal{O}_T) \subseteq \ker Frob^n$. $\square$

Corollary 21. *Suppose S is a qc $\mathbb{F}_p$-scheme and $f : T \to S$ a finite uh morphism. Then there is a factorisation*

$$\begin{tikzcd}
T^{\mathrm{perf}} \arrow[r] \arrow[d] & T \arrow[d] \\
S^{\mathrm{perf}} \arrow[r] \arrow[ur] & S
\end{tikzcd}$$

Proof. The kernel of $\mathcal{O}_S \to \mathcal{O}_T$ consists of nilpotents, Rem.12, and the kernel of $\mathcal{O}_S \to \mathcal{O}_S^{\mathrm{perf}}$ is the ideal of nilpotents so the argument of Cor.20 produces the dashed morphism, with $Frob^{n'}$ replaced with the canonical $\iota : \mathcal{O} \to \mathcal{O}^{\mathrm{perf}}$. Here ι are the canonical morphisms.

$$\begin{tikzcd}
T^{\mathrm{perf}} \arrow[d] \arrow[rrr, bend left, "\iota"] & T^{\mathrm{perf}} \arrow[l, "Frob^m"', "\cong"] \arrow[r, "\iota"] \arrow[d] & T \arrow[r, "Frob^m"] & T \arrow[d] \\
S^{\mathrm{perf}} \arrow[rrr, bend right, "\iota"'] & S^{\mathrm{perf}} \arrow[l, "\cong"', "Frob^m"] \arrow[r, "\iota"'] \arrow[urr] & S \arrow[r, "Frob^m"'] & S
\end{tikzcd}$$

$\square$

Proposition 22. *Suppose* $\mathrm{Spec}(\phi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is a uh morphism from a reduced affine scheme to a seminormal affine scheme. Then* $\mathrm{Spec}(B)$ *is a filtered limit of uh morphisms of finite presentation.*

Proof. It suffices to show that for every $b \in B$, there is a uh inducing A-algebra of finite presentation B' and a morphism $B' \to B$ of A-algebras such that b is in the image of B'. This is because finite presentation uh A-algebras are closed under finite colimits[1] and for any such $\phi : B' \to B$, and $c \in \ker \phi$, B'/c is again a uh A-algebra of finite presentation: $\mathrm{Spec}(B'/c) \to \mathrm{Spec}(B')$ is a closed immersion which is surjective because the homeomorphism $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ factors through it.

[1] To have all finite colimits it suffices to have finite coproducts and coequalisers. Clearly, if $A \to B, B'$ are two finite presentation uh A-algebras, then $B \otimes_A B'$ is again a finite presentation uh A-algebra, so we have finite coproducts exist. The coequaliser of two morphisms $\phi, \psi : B \rightrightarrows B'$ between finite presentation uh A-algebras is $B \otimes_{B \otimes_A B} (B' \otimes_A B')$. Since B and $B \otimes_A B$ are finite presentation uh A-algebras, $B \otimes_A B \to B$, and therefore also $B' \otimes_A B' \to B \otimes_{B \otimes_A B} (B' \otimes_A B')$, is a finite presentation uh morphism. Since $B' \otimes_A B'$ is a finite presentation uh A-algebra, it follows that the coequaliser is as well.

It is convenient to replace B with $A[b] \subseteq B$, in other words, we can and do assume that $B = A[b]$. Firstly, as $A \to B$ is integral, there is some monic $f(x) \in A[x]$ such that

$$f(b) = 0 \text{ in } B.$$

Next, $A_{\mathbb{Q}} \to B_{\mathbb{Q}}$ is a uh morphism from a seminormal $\mathbb{Q}$-algebra to a reduced ring, hence, an isomorphism, so there is some $n > 0$ and $a \in A$ such that

$$n(b - a) = 0 \text{ in } B.$$

Note $b = a$ in $B[\frac{1}{n}]$ Let $p_1, \dots, p_m$ be the primes dividing n. Since $\phi/p_i : A/p_i \to B/p_i$ is a uh morphism of $\mathbb{F}_p$-algebras, there is some p_i power q_i and $a_i \in A$ such that $b^{q_i} = a$ in B/p_i, Lem.18. Now since $b^{q_i} - a_i = 0$ in B/p_i, there is some $b'_i \in p_i B$ such that $b^{q_i} - a_i - b'_i = 0$ in B. Now we use the assumption from the beginning that $B = A[b]$. Since $B = A[b]$, we have $p_i B = p_i A[b]$, so there is some $f_i(x) \in A[x]$, such that $p_i f_i(b) = b'_i$. Rewriting, we get

$$b^{qi} - a_i - p_i f_i(b) = 0 \text{ in } B.$$

Now consider

$$B' = A[x] \Big/ \left\langle \begin{array}{c} f(x) \\ n(x-a) \\ (p_1 \dots \widehat{p_i} \dots p_m)(x^{q_i} - a_i - p_i f_i(x)) : i = 1, \dots, m \end{array} \right\rangle$$

with its canonical morphism $x \mapsto b$ to B. This is certainly finitely presented. We claim it is a uh morphism. It suffices to show that it is integral, and the morphism on Spec's is surjective and injective with all field extensions radicial. As $f(x)$ is monic, it is certainly integral. The other conditions can be checked after tensoring with each of $\mathbb{Z}[\frac{1}{n}], \mathbb{F}_{p_1}, \dots, \mathbb{F}_{p_m}$. Over $\mathbb{Z}[\frac{1}{n}]$, we have the retraction $B'[\frac{1}{n}] \to A[\frac{1}{n}]; x \mapsto a$ (note that a satisfies the other relations, as $a = b$ over $\mathbb{Z}[\frac{1}{n}]$), and plainly, $A[\frac{1}{n}] \to B'[\frac{1}{n}]$ is surjective (because $x = a$). So $A[\frac{1}{n}] \cong B'[\frac{1}{n}]$. Over $\mathbb{F}_{p_i}$, the relations $n(x-a)$ and $(p_1 \dots \widehat{p_j} \dots p_m)(x^{q_j} - a_j - p_j f_j(x))$ for $j \neq i$ vanish, $(p_1 \dots \widehat{p_i} \dots p_m)$ is a unit, and we are left with $x^{q_i} - a_i$. Pulling back $A/p_i \to A/p_i[x]/x^{q_i} - a_i$ to any residue field of A/p_i gives either a purely inseparable extension, or a nilpotent thickening, according to whether $a_i = 0$ or not in that field. So we have indeed produced a finitely presented uh A-algebra equipped with a surjection $B' \to B = A[b]$. □

Corollary 23. *Suppose that $A \to B$ is finite type, and $I \subseteq B$ is an ideal such that* $\mathrm{Spec}(B/I) \to \mathrm{Spec}(A)$ *is a cduh of reduced affine schemes (resp. a uh from a reduced scheme to a seminormal scheme). Then there is a finitely generated ideal $J \subseteq I$ such that* $\mathrm{Spec}(B/J) \to \mathrm{Spec}(A)$ *is a cduh (resp. uh).*

Proof. By Lem.16 (resp. Prop.22) we can write B/I as a filtered colimit of cduh (resp. uh) A-algebras of finite presentation $\varinjlim B_\lambda = B/I$. As B is a finitely generated A-algebra, there is some λ such that $B_\lambda \to B/I$ is surjective. On the other hand, the A-algebra B/I is the filtered colimit $\varinjlim B/J_\rho = B/I$ over finitely generated subideals $J_\rho \subseteq I$. Lift the surjection $B_\lambda \to B/I$ to a surjection $B_\lambda \to B/J_\rho$ producing the sequence

$$\mathrm{Spec}(B/I) \xrightarrow{\text{cl.imm.}} \mathrm{Spec}(B/J_\rho) \xrightarrow{\text{cl.imm.}} \mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(A).$$

By hypothesis, the composition and the unnamed morphism are cduh's (resp. uh's), so the two closed immersions must be surjective. Consequently, $\mathrm{Spec}(B/J_\rho) \to \mathrm{Spec}(A)$ is a cduh (resp. uh). $\square$

4 The general case

Lemma 24. *Let S be a qcqs scheme. Then $S^{\mathrm{red}} \to S$ is a filtered limit of nilpotent thickenings of finite presentation.*

Proof. The sheaf of nilpotents $\mathcal{N} \subseteq \mathcal{O}_S$ is the filtered union of its finite type sub-quasicoherent sheaves, [EGAI, Cor.6.9.9]. $\square$

Proposition 25. *Let $f : T \to S$ be a morphism of qcqs schemes and τ a class of morphisms such that either*

1. *S is reduced, f is the morphism $S^{\mathrm{sn}} \to S$, and $\tau =$ cduh, or*
2. *S is seminormal, T is reduced, and $\tau =$ uh.*

Then $f : T \to S$ is the filtered limit over factorisations $T \to S' \to S$ such that $S' \to S$ is a τ of finite presentation.

Proof. As $T \to S$ is affine, T is the limit of a filtered system $(S_\lambda \to S)_{\lambda\in\Lambda}$ of affine S-schemes of finite presentation, $T = \varprojlim S_\lambda$, [EGAI, Prop.6.9.16(iii)]. This implies that, in fact, it is the limit of all factorisations $T \to S' \to S$ through an affine S-scheme of finite presentation, [EGAIV3, Cor.8.13.2]. We claim that for any such factorisation, there exists a finite type sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_{S'}$ such that $\underline{\mathrm{Spec}}(\mathcal{O}_{S'}/\mathcal{J}) \to S$ is a τ. This claims implies that the factorisations $T \to S' \to S$ through a τ of finite presentation are final in the system of all factorisations through finite presentation affine morphisms, and the proposition follows.

Now we prove the claim. For each member of an open affine cover $\{U_i \to S\}$ of S, using Cor.23 applied to $\mathcal{O}_S(U) \to \mathcal{O}_{S'}(U)$ and $I = \ker(\mathcal{O}_{S'}(U) \to \mathcal{O}_T(U))$, we find $J_i \subseteq \ker(\mathcal{O}_{S'}(U_i) \to \mathcal{O}_T(U_i))$ such that $\mathrm{Spec}(\mathcal{O}_{S'}(U_i)/J_i) \to U_i$ is τ. Now for each i there is a finite type subideal $\mathcal{J}_i \subseteq \ker(\mathcal{O}_{S'} \to \mathcal{O}_T)$ with $\mathcal{J}|_{U_i} = J_i^{\sim}$ [EGAI, Cor.6.9.3, Cor.6.9.9]. Let $\mathcal{J} \subseteq \ker(\mathcal{O}_{S'} \to \mathcal{O}_T)$ be the ideal generated by the $\mathcal{J}_i$. Then $\underline{\mathrm{Spec}}(\mathcal{O}_{S'}/\mathcal{J}) \to S$ is τ: Let $T' = \underline{\mathrm{Spec}}(\mathrm{im}(\mathcal{O}_{S'} \to \mathcal{O}_T))$. Since $T' \to S$ is τ, Lem 11, to show that $\underline{\mathrm{Spec}}(\mathcal{O}_{S'}/\mathcal{J}) \to S$ is τ, it suffices to show that the closed immersion $T' \to \underline{\mathrm{Spec}}(\mathcal{O}_{S'}/\mathcal{J})$ is surjective. For this it suffices to show it is surjective over each U_i. This follows from the factorisation

$$T' \times_S U_i \overset{\text{cl.imm.}}{\to} \mathrm{Spec}(\mathcal{O}_{S'}(U)/\mathcal{J}(U)) \overset{\text{cl.imm.}}{\to} \mathrm{Spec}(\mathcal{O}_{S'}(U_i)/J_i) \overset{\tau}{\to} U_i,$$

whose composition is a τ. □

Lemma 26. *If $(S_\lambda \to S)_{\lambda\in\Lambda}, (T_\mu \to \varprojlim S_\lambda)_{\mu \in M}$ are two filtered systems of uh (resp. cduh) of finite presentation, then there is a filtered system of uh (resp. cduh) of finite presentation $(U_\nu \to S)_{\nu \in N}$ with $\varprojlim T_\mu = \varprojlim U_\nu$.*

Proof. We can assume that the filtered categories Λ, M are filtered posets, [**?**, Expo.1, Prop.8.1.6].

For each $\mu \in M$, the uh (resp. cduh) there is a $\lambda_\mu \in \Lambda$ and a uh (resp. cduh) of finite presentation $U_\mu \to S_{\lambda_\mu}$ such that T_μ is the pullback of U_μ to $\varprojlim S_\lambda$, [EGAIV3, Thm.8.8.2(ii),Thm.8.10.5(vi, vii, viii)].[2]

Define $N_0 \subseteq \Lambda \times M$ to be the subposet of those pairs $(\lambda, \mu) \in \Lambda \times M$ such that $\lambda \leq \lambda_\mu$, define $U_{(\lambda,\mu)} = S_\lambda \times_{S_{\lambda_\mu}} U_\mu$, and define the objects of N to be the elements of N_0. Define $\hom_N((\lambda,\mu),(\lambda',\mu')) \subseteq \hom_S(U_{(\lambda,\mu)}, U_{(\lambda',\mu')})$ to be the preimage of the morphism $T_{\mu\to\mu'}$.

The poset $\Lambda \times M$ is filtered because Λ, M are, and N_0 is filtered because it is final. One checks that for every $(\lambda,\mu), (\lambda',\mu')$ in N there are morphisms $(\lambda'',\mu'') \to (\lambda,\mu), (\lambda'',\mu'') \to (\lambda',\mu')$ using [EGAIV3, Thm.8.8.2], which says that $\hom_{\varprojlim S_\lambda}(T_\mu, T_{\mu'}) = \varinjlim_{\lambda\leq\alpha} \hom(S_\lambda \times_{S_{\lambda_\mu}} U_{\lambda_\mu}, S_\lambda \times_{S_{\lambda'_{\mu'}}} U_{\lambda'_{\mu'}})$ where $\alpha \in \Lambda$ is any element with $\alpha \leq \lambda_\mu, \lambda'_{\mu'}$. One uses the same theorem to show that for any two parallel morphisms $(\lambda,\mu) \rightrightarrows (\lambda',\mu')$ in N, there is a morphism $(\lambda'',\mu'') \to (\lambda,\mu)$ such that the two compositions are equal.

The functor $N \to \mathsf{Sch}_S$ is clear from the definition. On the other hand, the pullback functors $(\varprojlim S_\lambda) \times_{S_\lambda} -$ induce a functor $N \to \mathsf{Sch}_{\varprojlim S_\lambda}$ which

[2]For cd-ness, it suffices to note that for any point $s \in \varprojlim S_\lambda$ with images $s_\lambda \in S_\lambda$, we have $k(s) = \varinjlim k(s_\lambda)$. Since each $S_\lambda \to S$ is a cduh, the transitions $S_\mu \to S_\lambda$ are cduh's, so $k(s) = k(s_\lambda)$ for all $\lambda \in \Lambda$. The same applies to $(S_\lambda \times_{S_{\lambda_\mu}} U_\mu)_{\lambda \leq \lambda_\mu}$ so it follows that $T_\mu \to S_{\lambda_\mu}$ is cd iff $U_\mu \to S_{\lambda_\mu}$ is cd.

factors through $T_- : M \to \mathsf{Sch}_{\varprojlim S_\lambda}$ via the morphism $N \to M; (\lambda, \mu) \mapsto \mu$. Using the universal property of the limit, one checks the left equality,

$$\varprojlim(N \to \mathsf{Sch}_S) = \varprojlim(U_- : N \to \mathsf{Sch}_{\varprojlim S_\lambda}) = \varprojlim(T_- : M \to \mathsf{Sch}_{\varprojlim S_\lambda}),$$

and the right equality follows from $N \to M$ being final.

As at the beginning, we can replace N with a poset if we desire, [**?**, Expo.1, Prop.8.1.6]. □

Theorem 27. *Suppose that $T \to S$ is a uh (resp. cduh) of qcqs schemes. Then there exists a filtered system $(S_\lambda \to S)_{\lambda \in \Lambda}$ of uh (resp. cduh) of finite presentation, and a factorisation*

$$\varprojlim S_\lambda \to T \to S.$$

Proof. By Lem.26, it suffices to prove the theorem for $S' \times_S T \to S'$ when S' is a filtered limit of uh's (resp. cduh's) of finite presentation. By Lem.24 $S^{\mathrm{red}} \to S$ is such a scheme, and by Prop.25 $(S^{\mathrm{red}})^{\mathrm{sn}} \to S^{\mathrm{red}}$ is also such a scheme. So we can assume that S is seminormal. Replacing T with T^{red}, we can also assume that T is reduced. But then $T \to S$ is a cduh (resp. uh) from a reduced scheme to a seminormal scheme, and is therefore an isomorphism, Lem.13, (resp. filtered limit of uh of finite presentation, Prop.25). □

References

[Bou64] N. Bourbaki. *Éléments de mathématique. Fasc. XXX. Algèbre commutative. Chapitre 5: Entiers. Chapitre 6: Valuations.* Actualités Scientifiques et Industrielles, No. 1308. Hermann, Paris, 1964.

[EGAI] A. Grothendieck. Éléments de géométrie algébrique. I. le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.*, (4):228, 1960.

[EGAIV3] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas, troisième partie. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966. Revised in collaboration with Jean Dieudonné. Freely available on the Numdam web site at http://www.numdam.org/numdam-bin/feuilleter?id=PMIHES_1966__28_.

[EGAIV4] Alexandre Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas

IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967. Revised in collaboration with Jean Dieudonné. Freely available on the Numdam web site at http://www.numdam.org/numdam-bin/feuilleter?id=PMIHES_1967__32_.

[HK18] Annette Huber and Shane Kelly. Differential forms in positive characteristic II: cdh-descent via functorial Riemann-Zariski spaces. *Algebra and Number Theory*, Forthcoming, 2018. Preprint arXiv:1706.05244.

[Kel18] Shane Kelly. A better comparison of cdh- and ldh-cohomologies, 2018.

[Kol97] János Kollár. Quotient spaces modulo algebraic groups. *Annals of mathematics*, 145(1):33–79, 1997.

[Ryd10] David Rydh. Submersions and effective descent of étale morphisms. *Bull. Soc. Math. France*, 138:181–230, 2010.

[SV00] Andrei Suslin and Vladimir Voevodsky. Bloch-Kato conjecture and motivic cohomology with finite coefficients. In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 548 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 117–189. Kluwer Acad. Publ., Dordrecht, 2000.

[Stacks] The Stacks Project Authors. *Stacks Project*. http://stacks.math.columbia.edu, 2014.

[Swa80] Richard G. Swan. On seminormality. *J. Algebra*, 67(1):210–229, 1980.

[Tem11] Michael Temkin. Relative Riemann-Zariski spaces. *Israel Journal of Mathematics*, 185(1):1–42, 2011.

[Voe96] Vladimir Voevodsky. Homology of schemes. *Selecta Math. (N.S.)*, 2(1):111–153, 1996.