What is the relationship between Grothendieck pretopologies and Grothendieck topologies?

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Given a category C, there are three related notions one can consider.

1. Pretopologies on C, cf.[SGA4, Exp.II Def.1.2].
   A pretopology assigns to each object X a set of *covering families* of the form

   $$\{U_i \to X\}_{i \in I}.$$

   These are required to satisfy four axioms: (PT0) pullbacks of covering families exist, (PT1) pullbacks preserve covering families, (PT2) compositions of covering families are covering families, and (PT3) the identity singleton is a covering.

2. Topologies on C, cf. [SGA4, Exp.II Def.1.1].
   A topology assigns to each object X a set of *covering seives*, namely a set of subpresheaves

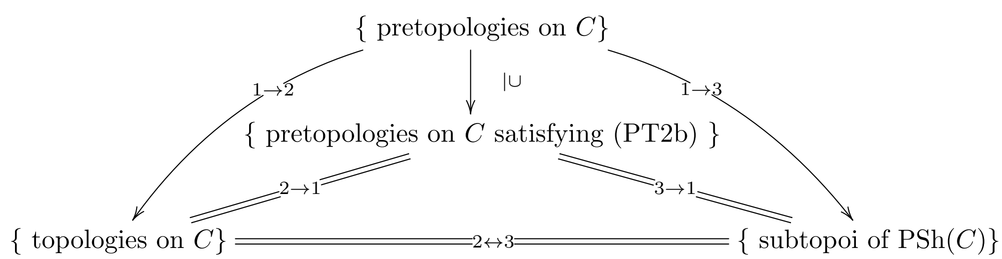
   $$R \subseteq h_X := \hom_C(-, X)$$

   of the presheaf represented by X. These are required to satisfy three axioms: (T1) pullbacks preserve covering sieves, (T2) pullbacks along covering sieves detect covering sieves, (T3) the maximal sieve $h_X \subseteq h_X$ is a covering sieve.[1]

3. Subtopoi of PSh(C), cf. [SGA4, Exp.IV Def.9.1.1].
   Namely, subcategories whose inclusion admits a left exact[2] left adjoint,

   $$\mathrm{PSh}(C) \underset{\supseteq}{\overset{a}{\rightleftarrows}} E.$$

Each of 1, 2, 3 can be used to define any of the others, and under these assignments we have a retraction and three bijections



[1] Note that SGA4 (and Suslin, Voevodsky, ...) also talk about *covering families* of a topology. These are the families $\{U_i \to X\}_{i \in I}$ whose image $im(\amalg_{i \in I} h_{U_i}) \subseteq h_X$ is a covering sieve, cf cf.[SGA4, Exp.II Def.1.2]. So seeing the term "covering family" does not remove the ambiguity about whether an author means 1 or 2, or rather, given the equivalences below, it does not remove the ambiguity about whether the author is assuming (PT2b) (see below) or not. Since this has no bearing on which presheaves are sheaves, this is usually harmless.

[2] Concretely, the image under a of any limit diagram of PSh(C) is a limit diagram of E.

where the extra axiom[3] is:

(PT2b) If $\mathcal{Y}$=$\{Y_i \to X\}_{i\in I}$ is a family, and there exists a covering family $\{U_j \to X\}_{j\in J}$ such that for each j there exists a factorisation

$$\begin{array}{ccc} U_j & \dashrightarrow & Y_{i_j} \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

for some i_j, then $\mathcal{Y}$ is also a covering family.

The operations $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 1$

Here are the procedures for getting between 1. pretopologies, 2. topologies, and 3. subtopoi. Modulo the retraction,[4] the below procedures are all mutually inverse to each other.[5]

(1 → 2) Covering sieves are those sieves admitting a factorisation

$$\amalg_{i\in I} h_{U_i} \dashrightarrow R \hookrightarrow h_X$$

for some covering family $\{U_i \to X\}_{i\in I}$. In other words, the sieves containing an image $im(\amalg U_i) \subseteq R \subseteq h_X$ of some covering family. Cf. [SGA4, Exp.II Prop.1.4].

(1 → 3) E is the category of presheaves F such that

$$F(X) \to \prod_{i\in U} F(U_i) \rightrightarrows \prod_{i,j\in U} F(U_i \times_X U_j)$$

is exact (i.e., an equaliser diagram) for every covering family $\{U_i \to X\}_{i\in I}$. Cf. [SGA4, Exp.II Cor.2.4, Thm.3.4].

(2 → 3) E is the category of presheaves F such that

$$\hom(h_X, F) \to \hom(R, F)$$

is a bijection for every covering sieve $R \subseteq h_X$. Cf. [SGA4, Exp.II Def.2.1, Thm.3.4].

(3 → 2) Covering sieves are those sieves $R \subseteq h_X$ such that

$$a(R) \to a(h_X)$$

is an isomorphism in E. Cf. [SGA4, Exp.II (Proof of) Thm.4.4].

(2 → 1) Covering families are the families $\{U_i \to X\}_{i\in I}$ such that $im(\amalg_{i\in I} U_i) \subseteq h_X$ is a covering sieve. Cf. [SGA4, Exp.II Para.1.3.1].

(3 → 1) Apply 3 → 2 then 2 → 1. Equivalently, covering families are those $\{U_i \to X\}_{i\in I}$ such that

$$a(\amalg_{i\in I} h_U) \to a(h_X)$$

is a categorical epimorphism in E. Cf. [SGA4, Exp.II Thm.4.4].

[3] There is a clear relationship between (PT1) and (T1) and between (PT3) and (T3). Furthermore, it is clear that (T2) contains a version of (PT2b). In fact, (T2) corresponds to the *combination* of (PT2) and (PT2b).

[4] Applying $1 \to 2 \to 1$ or $1 \to 3 \to 1$ to a pretopology τ will produce the pretopology σ such that σ-coverings are those families that are refinable (in the sense of (PT2b)) by a τ-covering.

[5] Assuming C has all fibre products. Otherwise $3 \to 1$ and $2 \to 1$ probably don't satisfy (PT0).

So what is the difference between pretopologies and topologies?

There are two differences.

1. Do you really want $\{Y \xrightarrow{f} X\}_{\substack{Y \in C, \\ f \in \hom_C(Y,X)}}$ to be a covering of X?

   The first difference is, basically, do you want to impose (PT2b) or not? Imposing (PT2b) gives a bijective correspondance between nice sets of families and categories of sheaves. Without it, we can have multiple pretopologies defining the same category of sheaves.

   It *is* also actually useful in practice sometimes: the proper cdh-topology on the category Var_k of varieties over a field k is defined as being generated (see below) by coverings of the form $\{Z \to X, X' \to X\}$ where $X' \to X$ is a proper morphism, isomorphic outside a closed $Z \subseteq X$. However, for regular X, any $\{X' \to X\}$ such that $X' \to X$ is birational is a covering [SV00, Lem.5.10], [HKK17, Prop.2.12]. Clearly, the latter statement is not true if we use the pretopology generated by the families $\{Z \to X, X' \to X\}$ without imposing (PT2b).

   On the other hand, if we consider the Zariski topology on the category of varieties Var_k, it clearly a bit strange for $\{\mathbb{A}^1_k \setminus \{0\}, \mathbb{A}^1_k \setminus \{1\}, \{0,1\}\}$ to be a covering of $\mathbb{A}^1_k$, since it contains the closed immersion $\{0,1\} \to \mathbb{A}^1_k$. Things are nicer if we only allow open immersions in our covering families.

2. Do you really want to have to picture $R \subseteq \hom(-, X)$?

   The second difference is one of language. Do we want to use covering families, or covering sieves? Families are usually more natural for humans to visualise, but sieves can be easier to work with in some proofs (try and prove Lemma 1 below using covering families).

Every left exact localisation is a sheafification? Really?

It's perhaps not straightforward why a given subtopos $E \subseteq \mathrm{PSh}(C)$ should agree with its associated category of sheaves, i.e., why $3 \to 2 \to 3$ is the identity, so we provide a brief discussion.

Lemma 1. *Let C be a category and $E \subseteq \mathrm{PSh}(C)$ a subcategory whose inclusion admits a left exact left adjoint $a : \mathrm{PSh}(C) \to E$, and equip C with the topology whose covering sieves are those sieves $R \subseteq \hom(-, X)$ such that $aR \cong a\hom(-, X)$. Then $\mathrm{Shv}(C) = E$.*

Proof. The inclusion $E \subseteq \mathrm{Shv}(C)$ is straightforward. If $F \in E$, or equivalently, $F = aF$, then by adjunction we have $\hom_{\mathrm{PSh}}(h_X, F) = \hom_{\mathrm{PSh}}(h_X, aF) = \hom_E(ah_X, aF) = \hom_E(aR, aF) = \hom_{\mathrm{PSh}}(R, aF) = \hom_{\mathrm{PSh}}(R, F)$ for any covering seive.

For the inclusion $\mathrm{Shv}(C) \subseteq E$, we first prove the claim: any monomorphism of sheaves $F \subseteq G$ which becomes an isomorphism under a, is an isomorphism of presheaves. Since $F \to G$ is a monomorphism, it suffices to show that every $s : h_X \to G$ factors through F. Since a sends $F \to G$ to an isomorphism and commutes with pullbacks, it sends each sieve $h_X \times_G F \subseteq h_X$ to an isomorphism, i.e., these are covering seives. Now F is a sheaf, so $h_X \times_G F \to F$ factors as $h_X \times_G F \to h_X \xrightarrow{\phi} F$, and G is a sheaf, so $h_X \xrightarrow{s} G$ agrees with $h_X \xrightarrow{\phi} F \to G$ (because they agree on the covering $h_X \times_G F$). So we have found the desired factorisation.

Now for a general sheaf, consider the diagonal map $F \to F \times_{aF} F$. This is a monomorphism of sheaves (either projection provides a retraction) which is sent to an isomorphism under a (since $a(F \times_{aF} F) = aF \times_{aF} aF = aF$) so our above claim implies that $F = F \times_{aF} F$ as presheaves. In particular, $F \to aF$ is a monomorphism of presheaves. Applying the above claim a second time, we find that $F = aF$, so our sheaf F is in E. □

What does "generated by" mean?

Cf.[SGA4, Exp.II Para.1.1.6.]. The collection of all subtopoi (resp. pretopologies, resp. topologies) are partially ordered in an obvious way (although the natural choice of orderings for 3 and 1 (resp. 2) are opposite: the trivial pretopology (where only the $\{X = X\}$ are coverings) is the smallest collection of families which form a pretopology, but this corresponds to the largest subtopos $\mathrm{PSh}(C) \subseteq \mathrm{PSh}(C)$).

Moreover, the poset of (pre)topologies clearly admits infimums, since an intersection of sets of families of morphisms (resp. sieves) satisfying the axioms, will again satisfy the axioms.

Generated by then means the intersection of all (pre)topologies containing a given set of sieves (resp. families).

For pretopologies one can give a "concrete" description. Given a class of families of morphisms of the form $\{U_i \to X\}_{i \in I}$ of a category C[6], one gets a pretopology by considering all compositions of pullbacks of generators.[7] If the generators are already preserve by pullback, it's enough to take compositions.

References

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[6] with fibre products

[7] Lets say identities $\{X = X\}$ are compositions of an empty collection of families to be less wordy.