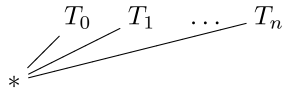
Fast track guide to cardinals for use with Lurie's Higher Topos Theory

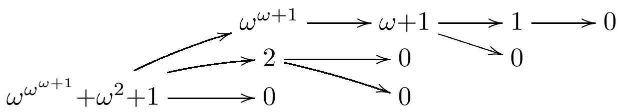
1. Ordinals

(1) An *ordinal* is a well-ordered set. That is, a totally ordered set such that every subset has a smallest element. Finite ordinals look like $n = \{0 < 1 < \cdots < n{-}1\}$. Infinite ordinals can be more complicated, e.g., $\omega \cdot 2 + 1 = \{0_0{<}1_0{<}2_0{<} \cdots < 0_1{<}1_1{<}2_1{<} \cdots < 0_2\}$.

(2) There is a bijection between finite rooted trees and countable ordinals. The bijection *Ord* is defined inductively by sending the tree with one node to $0 = \{\}$, and the tree



to $\omega^{Ord(T_1)} + \cdots + \omega^{Ord(T_n)}$.[1] So for example, $\omega^{\omega^{\omega+1}} + \omega^2 + 1$ corresponds to the tree



(3) An ordinal of the form $\alpha+1$ is called a *successor ordinal.* An ordinal which is not a successor ordinal is called a *limit ordinal.*

2. Cardinals

(1) A cardinal is an isomorphism class of sets.

(2) Cardinals have a relation: $\lambda \leq \kappa$ if there exists a monomorphism $L \to K$ (for representatives, $L \in \lambda$, $K \in \kappa$).

(3) One can consider the category $Set_{<\kappa}$ of all sets of cardinality $< \kappa$. Such sets are called *κ-small.*

(4) A cardinal κ is *regular* if for every $I \in Set_{<\kappa}$ and $K_i \in Set_{<\kappa}$ we have

$$\sqcup_I K_i \in Set_{<\kappa}.$$

Informally, κ is regular if the "universe" $Set_{<\kappa}$ has colimits. Formally, κ is regular if the category of κ-small sets has κ-small coproducts.

(5) The smallest regular cardinal is the cardinality of the set $\mathbb{N}$ of integers. It is denoted by $\aleph_0$ or ω_0 or ω.

$$\omega = |\mathbb{N}|.$$

The next smallest regular cardinal is the cardinality of the set of countable ordinals, or equivalently, the set $\mathbb{T}_{fin.root}$ of finite rooted trees. It is denoted by $\aleph_1$ or ω_1.

$$\omega_1 = |\mathbb{T}_{fin.root}|.$$

In particular:

A set is $\aleph_0$-small if and only if it is finite.
A set is $\aleph_1$-small if and only if it is finite or countable.

(6) A cardinal is *strongly inaccessible* if it is regular and when $K \in Set_{<\kappa}$ we have $\text{Subsets}(K) \in Set_{<\kappa}$. Informally, κ is strongly inaccessible if the "universe" $Set_{<\kappa}$ admits colimits and limits. Formally, κ is strongly inaccessible if the category of κ-small sets has κ-small coproducts and κ-small limits. It's hard to write down a strongly inaccessible cardinal.

3. κ-filtered

A category is *filtered* if every finite diagram admits a cocone. It is *κ-filtered* if every κ-small diagram admits a cocone. So a diagram is $\aleph_0$-filtered if every finite diagram admits a cocone, and $\aleph_1$-filtered if every countable diagram admits a cocone.

[1]For ordinals α, β, the ordinal $\alpha + \beta$ is the set $\alpha \sqcup \beta$ with $a \leq b$ for all $a \in \alpha, b \in \beta$. The ordinal $\alpha\beta$ is the set $\alpha \times \beta$ with the lexicographical ordering. The ordinal α^β is the set of all functions $\beta \to \alpha$ (of sets) with finite support, i.e., all but finitely many elements of β are sent to the minimal element of α. This latter is also given the lexicographical ordering. Of course, $\omega = \mathbb{N}$ with the usual ordering.

Remark. The set of real numbers $\mathbb{R}$ may or may not have regular cardinality, depending on which axioms your set theory satisfies. The statement $|\mathbb{R}| = \aleph_1$, known as the continuum hypothesis (CH), is independent of ZFC in the sense that, there are collections of sets which satisfy ZFC for which CH is true, and there are collections of sets which satisfy ZFC for which CH is false. So there is no harm in setting $|\mathbb{R}| = \aleph_1$, but it is just that: an extra axiom.

This may look odd at first since one usually says "the" set of real numbers, so how can we choose its cardinality? But if we look closer it's not so strange. A real number is essentially a choice of subsets of $\mathbb{N}$.[2] So the cardinality of $\mathbb{R}$ is intimately connected which the notion of "subset", which is precisely what the axioms describe.

In fact, it's possible for $|\mathbb{R}|$ to be a successor cardinal or a limit cardinal, and either a regular cardinal or a singular cardinal.

4. Accessible categories

Just as topoi can be defined as categories of the form $\mathrm{Shv}(C)$ for some small category C equipped with a Grothendieck topology, *κ-accessible categories* can be defined as categories of the form $\mathrm{Ind}_\kappa(C)$ for some small category C where κ is a regular cardinal κ. Examples abound.

1. *Rings.* The category of rings is $\mathrm{Ind}_\omega(\mathrm{Ring}^{fp})$ where Ring^{fp} is the category of rings of finite presentation, i.e., of the form $\mathbb{Z}[x_1, \dots, x_d]/\langle f_1, \dots, f_c \rangle$.
2. *Fields.* The category of fields is $\mathrm{Ind}_\omega(\mathrm{Field}^{fg})$ where Field^{fg} is the category of finitely generated fields, i.e., of the form $\mathbb{Q}(x_1, \dots, x_n)[\alpha_1, \dots, \alpha_m]$ or $\mathbb{F}_p(x_1, \dots, x_n)[\alpha_1, \dots, \alpha_m]$ where the α_i are algebraic.
3. *Presheaves.* The category $\mathrm{PSh}(C)$ of presheaves on a small category C is $\mathrm{Ind}_\omega(\mathrm{PSh}(C)^\omega)$ where $\mathrm{PSh}(C)^\omega$ is the subcategory of those presheaves which are retracts of finite colimits of representables.
4. *Sets.* For any regular cardinal κ we have
$$\mathrm{Set} \cong \mathrm{Ind}_\kappa(\mathrm{Set}_{\leq\kappa}).$$

Probably you notice a theme here. A category C is κ-accessible if and only if[3] $C \cong \mathrm{Ind}_\kappa(C^\kappa)$ where C^κ is the full subcategory of *κ-compact* objects, namely, those objects X such that $\hom(X, -)$ commutes with κ-filtered colimits.

A functor
$$F : \mathrm{Ind}_\kappa(C) \to \mathrm{Ind}_\kappa(D)$$
between κ-accessible categories is *κ-accessible* if it is the left Kan extension of its restriction to C. That is, if and only if $F(\text{“}\mathrm{colim}\text{”} X_\lambda) = \mathrm{colim}\, F(X_\lambda)$.

5. Presentable categories

In the above three examples, you might notice that (1), (3), and (4) have all small colimits, while (2) does not. Equivalently, in (1), (3) and (4) the subcategory of κ-compact objects admits κ-small colimits while (2) does not.

A category C is *presentable* if:

1. It has all small colimits.
2. $C \cong \mathrm{Ind}_\kappa(C^\kappa)$ for some regular cardinal κ.

Since $\mathrm{Ind}_\kappa(C^\kappa)$ admits all small colimits, the canonical inclusion $\mathrm{Ind}_\kappa(C^\kappa) \subseteq \mathrm{PSh}(C^\kappa)$ admits a left adjoint
$$L : \mathrm{PSh}(C^\kappa) \to \mathrm{Ind}_\kappa(C^\kappa) \cong C.$$
Such a localisation can be considered as a *presentation* for C, where C^κ is the generating category.

Reference: [Adamek, Rosicky, Locally Presentable and Accessible Categories].

[2]Real numbers in $\mathbb{R}$ are in bijection with real numbers in the open unit interval $(0, 1)$, Writing $x \in (0, 1)$ in base 2, we get an $\mathbb{N}$-indexed sequence of 0s and 1s. Thinking of this as a characteristic function, we get a bijection between real numbers $0 < x < 1$ and proper non-empty subsets of $\mathbb{N}$.

[3]We implicitly demand that this equivalence restricts to the identity on $C \supseteq C^\kappa \subseteq \mathrm{Ind}_\kappa(C^\kappa)$.