Valuative tosets

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Abstract

In this short note we characterise those totally ordered sets which can appear as the set of prime ideals in a valuation ring.

These are the notes of a discussion we had in Cambridge after Marc asked which tosets can appear as Spec of a valuation ring. The result is not new, as Ko Aoki pointed out to me, the main result is Corollary 3.6 of the paper "The spectrum of a ring as a partially ordered set" written by Lewis.

By $<$ we will always mean $\lneq$. By *toset* we mean totally ordered set.

Proposition 1. *Consider the functor* Spec *from valuation rings to the category of tosets.*

$$\mathrm{Spec} : \{\ \textit{val.rings}\ \} \to \{\ \textit{tosets}\ \}$$

A toset T is in the image of this functor if and only if

(Inf) *T has infimums,*

(SS) *T is "successor separated" in the sense that for every $a < b \in T$ there exists $t_0, t_1 \in T$ such that T decomposes as $T = [-\infty, t_0] \sqcup [t_1, \infty]$ and $a \in [-\infty, t_0], b \in [t_1, \infty]$.*

Example 2. The easiest interesting example of such a toset is the subtoset $(\{0\}\times\mathbb{Q}) \cup (\{1\}\times\mathbb{R}) \subseteq \{0{<}1\}\times\mathbb{R}$ adjoin $\pm\infty$ (with the product given the lexicographical ordering). This is the spectrum of $\mathbb{Q}[[t^{\oplus_{\mathbb{Q}}\mathbb{Z}}]]$, cf. the proof of Prop. 7 and Prop. 10. Alternatively we could obtain this toset by considering rational functions on rationally many variables over the rationals $\mathbb{Q}(x_q : q \in \mathbb{Q})$ and defining a valuation $v : \mathbb{Q}(x_q : q \in \mathbb{Q})^* \to \oplus_{\mathbb{Q}}\mathbb{Z}$ sending x_q to the generator e_q where $\oplus_{\mathbb{Q}}\mathbb{Z}$ is given the lexicographical ordering. The valuation is uniquely determined by the properties $v(ab) = v(a) + v(b)$ and $v(a + b) = \min\{v(a), v(b)\}$ whenever $v(a) \neq v(b)$.

Remark 3. Recall that a topological space appears as the spectrum of a ring if and only if

1. it is compact, and T_0,
2. the compact open sets form a basis,
3. compact open sets are closed under intersection,
4. every nonempty irreducible closed subset has a unique generic point.

The condition (Inf) corresponds to existence of generic points, and (SS) corresponds to T_0 and compact opens forming a basis. Indeed, closed subsets of the spectrum correspond to subsets of the toset of the form $[-\infty, t]$, i.e., admitting a maximum, and the compact opens correspond to sets $[t_1, \infty]$ such that t_1 is a successor, cf. Prop.14, Prop.15.

Definition 4. *Recall that an* isolated subgroup *of a totally ordered abelian group G is a subgroup H such that: if $a \in H, b \in G$ satisfy $-a \leq b \leq a$ then $b \in H$.*

The following proposition is standard.

Proposition 5. *Let R be a valuation ring with fraction field K. The valuation*

$$v : K^* \to K^*/R^* =: G$$

induces an inclusion reversing bijection between primes of R and isolated subgroups of G. An isolated subgroup H corresponds to the prime $\mathfrak{p}_H = \{r \in R : H < v(r)\}$.

Remark 6. The group G is linearly ordered by $[a] \leq [b]$ if $b/a \in R$.

Proof. Follows directly from $v(ab) = v(a)+v(b)$ and $v(a+b) \geq \min\{v(a), v(b)\}$. □

Proposition 7. *Every totally ordered abelian group is the value group of some valuation ring.*

Proof. Let G be a totally ordered abelian group. The standard choice is *Hahn series* $\mathbb{Q}[[t^G]] \subseteq \hom_{Set}(G_{\geq 0}, \mathbb{Q})$. This is the set of functions whose support is well-ordered, with addition and multiplication induced in the way suggested by the notation $\sum_{g \in G_{\geq 0}} a_g t^g$ for a function $a_- : G_{\geq 0} \to \mathbb{Q}; g \mapsto a_g$. □

Due to the above two propositions we are now reduced to classifying tosets of isolated subgroups in totally ordered abelian groups.

Totally order abelian groups have analogues of Proposition 5 and Proposition 7.

Definition 8. *Say that two positive elements $x, y \in G_{>0}$ of a totally ordered abelian group are* commensurate *if there exist positive integers $n, m > 0$ such that $x \leq ny$ and $y \leq mx$. Evidently, this is an equivalence relation, and the order relation on G induces a total order on the set $T = G_{>0}/\sim$ of equivalence classes via the canonical projection $G_{>0} \times G_{>0} \to T \times T$. In other words, $G_{>0} \to T$ is a surjection of tosets.*

Proposition 9. *Let G be a totally ordered abelian group and $p : G_{>0} \to G_{>0}/\sim := T$ the toset of commensurate equivalence classes. Then*

$$Sub(T) \to Sub(G)$$

$$S \subseteq T \mapsto \{g \in G : p(|g|) < S\} := H_S$$

induces an inclusion reversing bijection between the isolated subgroups of G and the right closed subsets of T.

Here $|g| = g$ if $0 \leq g$ and $-g$ if $g < 0$, and we define $p(0) = -\infty$. By right-closed *we mean $t \leq t', t \in S \Rightarrow t' \in S$.*

Proof. Suppose $S \subseteq T$ is a subset. We will show that H_S is a subgroup. For this, it suffices to show that given $0 < x \leq y$ in H_S we have $x + y \in H_S$ and $y - x \in H_S$. For the first one we observe that $y \leq x + y \leq y + y = 2y$ so $p(y) = p(x + y)$. For the second one we observe $y - x \leq y$ so $p(y - x) \leq p(y) < S$. Next, H_S is isolated: given $x, y \in G$ with $-x \leq y \leq x$ and $x \in H_S$, we have $p(|y|) \leq p(|x|) < S$. So we conclude that $S \mapsto H_S$ sends subsets to isolated subgroups.

Injectivity. Suppose that S, S' are two right closed subsets. If $S \subsetneq S'$, then there is some $p(|x|) \in S' \setminus S$. Since S is right closed, and $p(|x|) \notin S$, we must have $p(|x|) < S$, so $x \in H_S$. But $p(x) \in S'$, so $x \notin H_{S'}$. Hence, $S \neq S' \Rightarrow H_S \neq H_{S'}$.

Surjectivity. We make the following sequence of observations.

1. The set of left closed subsets of T is bijective to the set of right closed subsets under $L \mapsto S_L = \{t : L < t\}$ with inverse the assignment $S \mapsto L_S = \{t : t < S\}$.

2. The map in the proposition is $S \mapsto (-p^{-1}(L_S)) \cup \{0\} \cup (p^{-1}(L_S))$ where $p : G_{>0} \to T$ is the canonical projection.

3. Every subgroup H of G is uniquely determined by its set of positive elements $H = (-H_{>0}) \cup \{0\} \cup (H_{>0})$.

4. If H is an isolated subgroup, then $p^{-1}p(H_{>0}) = H_{>0}$. Indeed, if $h \in H_{>0}, g \in G_{>0}$ are commensurate, then $-nh \leq g \leq nh$ for some positive integer n, so $g \in H_{>0}$.

It follows from the above observations that any isolated subgroup H is the image of the right closed subset $S_{p(H_{>0})} \subseteq T$. Indeed,

$$\begin{aligned} H &= (-H_{>0}) \cup \{0\} \cup (H_{>0}) \\ &= -p^{-1}p(H_{>0}) \cup \{0\} \cup p^{-1}p(H_{>0}) \\ &= -p^{-1}(L_{S_{p(H_{>0})}}) \cup \{0\} \cup p^{-1}(L_{S_{p(H_{>0})}}) \\ &= im(S_{p(H_{>0})}) \end{aligned}$$

□

Proposition 10. *Every toset T appears as the toset of commensurate equivalence classes of some totally ordered abelian group G.*

Proof. Take $G = \bigoplus_{t \in T} \mathbb{Z}$ with the lexicographical ordering. □

So now we have reduced the problem to classifying tosets in the image of the functor $RCSub : Toset \to Toset$ which sends a toset to its set of right closed subsets with the relation $A \leq B$ if $A \supseteq B$ (this convention matches the fact that $v(a) \leq v(b) \iff aR \supseteq bR$ for $a, b \in R$).

Proposition 11. *A toset is in the image of $RCSub$ if and only if it satisfies (Inf) and (SS) from Proposition 1.*

Proof. Consider some $RCSub(T)$ in the image of $RCSub$. Since right closed subsets are closed under union, the toset $RCSub(T)$ satisfies (Inf). Consider $A \supseteq B$ in $RCSub(T)$. If $A \neq B$, there is some $a \in A \setminus B$. Define $T_0 := T_{\geq a}$ and $T_1 := T_{>a}$. We then have $A \supseteq T_0 \supset T_1 \supseteq B$. Moreover, for every right closed subset $C \in RCSub(T)$ either, $a \in C$ or $a \notin C$, so either $C \supseteq T_0$ or $T_1 \supseteq C$. So $RCSub(T)$ satisfies (SS).

On the other hand, suppose that T satisfies (Inf) and (SS). Define

$$S = \{t \in T : T = [-\infty, t] \sqcup [t_1, \infty] \text{ for some } t_1 \in T\}$$

Since T satisfies (Inf) there is a map $\inf : RCSub(S) \to T$. We claim this is an isomorphism.

Injectivity. Suppose $A, B \subseteq S$ are two right closed subsets of S with $\inf A = \inf B$. Since they are right closed, either $A \subseteq B$ or $B \subseteq A$. Suppose $A \subseteq B$. If $A \neq B$ then there is some $b \in B \setminus A$. Since $b \in B \subseteq S$, it has a successor b_1, satisfying $b < b_1$ and $T = [-\infty, b] \sqcup [b_1, \infty]$. If $b_1 \leq A$ then we have a contradiction by $\inf A = \inf B \leq b < b_1 \leq A$. So there is some $a \in A$ with $a < b_1$. But then $b < a < b_1$ gives a contradiction to $T = [-\infty, b] \sqcup [b_1, \infty]$. So we deduce that $A = B$.

Surjectivity. It suffices to see that for any $t \in T$, we have $t = \inf\{s \in S : t \leq s\}$. Proof by contradiction. If t is not this inf, then there is some t' with $t < t' \leq S$. But by (SS) there is a successor pair x_0, x_1 with $t \leq x_0 < x_1 \leq t'$. That is, there is $x_0 \in S$ with $t \leq x_0 < t'$. This contradicts $t' \leq s \in S : t \leq s$. □

Remark 12. The relationship between prime ideals of a valuation ring R and right closed subsets of the toset of commensurate equivalence classes of its value group is completely transparent. The prime corresponding to a right closed subset of T is precisely the preimage under the composition of the two canonical projections

$$\pi : R \setminus R^* \to G_{>0} \cup \{\infty\} \to T \cup \{\infty\}$$

where $G := K^*/R^*$ and $T := G_{>0}/\sim$. Or in other words, the composition above induces an inclusion preserving isomorphism

$$\mathrm{Spec}(R) \stackrel{\sim}{\to} RCSub(T).$$

We can read the topology off from the inherent structure of the toset.

Remark 13. The successor pairs $T_0 \supset T_1$ of $RCSub(T)$ correspond precisely to the quasi-compact opens $\mathrm{Spec}(R_{\pi^{-1}T_1})$ of R, or equivalently, the principal closed subsets $\mathrm{Spec}(R/\pi^{-1}T_0)_{red}$. Indeed, for any $f \in R$ with $\pi(f) \in T_0 \setminus T_1$, we have $\mathrm{Spec}(R_{\pi^{-1}T_1}) = \mathrm{Spec}(R_f)$ and $\mathrm{Spec}(R/\pi^{-1}T_0)_{red} = \mathrm{Spec}(R/f)_{red}$.

Proposition 14. *Let R be a valuation ring and $\mathfrak{p}$ a non-maximal prime. The following are equivalent.*

1. $\mathrm{Spec}(R_\mathfrak{p}) \subseteq \mathrm{Spec}(R)$ *is open.*
2. $R_\mathfrak{p} = R_f$ *for some nonunit* $f \in R$.
3. $\mathfrak{p}$ *is a successor. That is, there exists a prime* $\mathfrak{p}_0 \supset \mathfrak{p}$ *such that there are no primes between* $\mathfrak{p}_0$ *and* $\mathfrak{p}$.

Proof. (1) ⇒ (2). If the quasi-compact subscheme $\mathrm{Spec}(R_\mathfrak{p})$ is open, then it admits a finite cover of the form $\{\mathrm{Spec}(R_{f_i})\}_{i=1}^n$. But being finite, one of the elements has to be maximal.

(2) $\Rightarrow$ (3). If $R_{\mathfrak{p}} = R_f$, then $\mathrm{Spec}(R)$ decomposes into $\mathrm{Spec}(R_f)$ and $\mathrm{Spec}(R/f)$. The latter, begin closed has a generic point.

(3) $\Rightarrow$ (1). If $\mathfrak{p}$ is a successor, with predecessor $\mathfrak{p}_0$, then as a set, we have $\mathrm{Spec}(R) = \mathrm{Spec}(R_{\mathfrak{p}}) \sqcup \mathrm{Spec}(R/\mathfrak{p}_0)$. Since $\mathrm{Spec}(R/\mathfrak{p}_0)$ is closed, $\mathrm{Spec}(R_{\mathfrak{p}})$ must be open. □

Proposition 15. *Let R be a valuation ring and $Z \subseteq \mathrm{Spec}(R)$ a closed subset. The following are equivalent.*

1. *$Z = \mathrm{Spec}(R/f)_{red}$ for some $f \in R$.*
2. *$\mathrm{Spec}(R) \setminus Z$ is quasi-compact.*
3. *The generic point of Z has a successor (in $\mathrm{Spec}(R)$).*

Proof. (1) $\Rightarrow$ (2). If $Z = \mathrm{Spec}(R/f)_{red}$ then $\mathrm{Spec}(R) \setminus Z \cong \mathrm{Spec}(R_f)$.

(2) $\Rightarrow$ (3). If $\mathrm{Spec}(R) \setminus Z$ is quasi-compact then any cover by basic opens $\{\mathrm{Spec}(R_{f_i})\}$ has a finite subcover, and since the primes of R a totally ordered, is equal to one of the $\mathrm{Spec}(R_{f_i})$. Therefore $\mathrm{Spec}(R) \setminus Z$ has a maximal prime.

(3) $\Rightarrow$ (1). If the generic point $\mathfrak{p}_0$ of Z has a successor $\mathfrak{p}_1$, then $Z = \mathrm{Spec}(R/f)_{red}$ for every $f \in \mathfrak{p}_1 \setminus \mathfrak{p}_0$. □

Remark 16. For any toset T there is a canonical injection $T \to RCSub(T)$; $t \mapsto T_{\geq t}$ and its image is intrinsically recognisable as the subtoset of elements of $RCSub(T)$ having an immediate successor

$$T \cong \{T_0 \in RCSub(T) : RCSub(T) = [T, T_0] \sqcup [T_1, \varnothing] \text{ for some } T_1\}.$$

Indeed, for every $t \in T$ we take $T_1 = T_{>t}$, and conversely, a right closed subset T_0 has a successor T_1 if and only if $T_0 \setminus T_1$ is a singleton.

Remark 17. The above suggests that a sensible generalisation of the *rank* of a valuation ring could be the totally ordered set of primes admitting a successor. This would give the valuation ring of Exam.2 the more sensible rank $\mathbb{Q}$ instead of the unwieldy $\{\pm\infty\} \cup (\{0\} \times \mathbb{Q}) \cup (\{1\} \times \mathbb{R})$.