A USER'S GUIDE TO VOEVODSKY CORRESPONDENCES

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Abstract. We discuss the Suslin, Voevodsky theory of relative cycles. The goal is to define and work with categories of finite correspondences over a general base scheme.

Disclaimer. Following Suslin, Voevodsky, all schemes are Noetherian[1] and separated[2]. As we are working diagrammatically, it will be convenient to use the anti-Leibniz notation for composition. We use the symbol * for this, as it is related to pullback. That is,

$$f^*g := g \circ f.$$

There is no new mathematics anywhere in this note. The presentation is taken from [Kel12]. See also [Ivo05] and [CD19] for other accounts of the theory. The reference for the theory of relative cycles is of course [SV00].

1. First definitions

Definition 1.1. For schemes X, Y a *finite correspondence* from X to Y is formal finite sum of roofs

$$\alpha = \sum_{i=1,\dots,N} n_i \left[\begin{tikzcd} & Z_i \arrow[dl, "p_i"'] \arrow[dr, "f_i"] & \\ X & & Y \end{tikzcd} \right].$$

such that

(Clo) each $Z_i \to X \times Y$ is a closed immersion,
(Int) each Z_i is integral,
(Fin) each p_i is finite, and
(Dom) each p_i dominates an irreducible component of X.

If X, Y are S-schemes, and each Z_i is contained in $X \times_S Y \subseteq X \times Y$ then α is called an *finite S-correspondence*. Let's write $\mathrm{Cor}^{pre}(X, Y)$ and $\mathrm{Cor}^{pre}_S(X, Y)$ for the set of all finite correspondences, resp. finite S-correspondences.[3] These are free abelian groups.

Remark 1.2. The conditions (Clo), (Int), and possibly even (Dom) may look unnatural to readers who usually work on the "vector bundle" side of Grothendieck-Riemann-Roch, but one should keep in mind that Voevodsky motives are on the "cycles" side. If X is the spectrum of a field, then α is a zero cycle on the X-scheme

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[1]There are at least two places where the Noetherianity assumption is used. The definitions of $[f]$ and $[{}^tf]$ require finitely many generic points, and the definition of $[{}^tf]$ requires multiplicity to be finite. The former could possibly be avoided by assuming the underlying topological spaces are Noetherian or by generalising finite sums to locally constructible functions. For the latter one could try setting $\infty = 0$ as is done in the definition of proper push-forward of cycles, but probably the most reasonable thing to do is left Kan extend everything from finite type $\mathbb{Z}$-schemes.

[2]Separated is obviously a natural requirement for doing intersection theory.

[3]The "pre" is because not all correspondences have a well-defined pullback. To get a category of correspondences, we restrict our attention to those correspondences with *well defined specialisation*, Def.3.2.

$X \times Y$. In general, α can[4] be thought of as a family of zero cycles and (Dom) is a minimum requirement that this family varies in a reasonable way.

Example 1.3.

(1) (Graph) If $f : X \to Y$ is any morphism of schemes and $X_i \subseteq X$ are the integral components of X, then we write

$$[f] = \left[\begin{array}{ccc} X & & \\ & \searrow^{f} & \\ & & Y \end{array}\right] = \sum_i \left[\begin{array}{ccccc} & & X_i & & \\ & \swarrow & & \searrow & \\ X & & & & Y \end{array}\right].$$

(2) (Transpose) If $p : Y \to X$ is any finite flat morphism, the $Y_i \subseteq Y$ are the integral components of Y with generic points y_i, and $m_i = \frac{\text{length}\, \mathcal{O}_{Y,y_i}}{\text{length}\, \mathcal{O}_{X,fy_i}}$ are the "vertical" multiplicities, then we write

$$[{}^t p] = \left[\begin{array}{ccc} & & Y \\ & {}^{p}\swarrow & \\ X & & \end{array}\right] = \sum_i m_i \left[\begin{array}{ccccc} & & Y_i & & \\ & \swarrow & & \searrow & \\ X & & & & Y \end{array}\right].$$

2. Composition

Wish List 2.1. We would like to define a composition of finite correspondences which satisfies the following three conditions.

(1) (Contravariance) If $\xrightarrow{p}\xrightarrow{q}$ are composable finite flat morphisms then

$$\left[\; {}^{q}\swarrow \;\right]^* \left[\; {}^{p}\swarrow \;\right] = \left[\; {}^{p^*q}\swarrow \;\right].$$

(2) (Pushforward) For any morphism $f : Y \to Y'$ we have

$$\sum n_i \left[\begin{array}{ccccc} & & Z_i & & \\ & \swarrow & & \searrow & \\ X & & & & Y \end{array}\right]^* \left[\begin{array}{ccc} Y & & \\ & \searrow^{f} & \\ & & Y' \end{array}\right] = \sum n_i d_i \left[\begin{array}{ccccc} & & Z_i' & & \\ & \swarrow & & \searrow & \\ X & & & & Y' \end{array}\right]$$

where Z_i' is the image $Z_i' = im(Z_i \subseteq X \times Y \to X \times Y')$ and

$$d_i = \begin{cases} [k(Z_i) : k(Z_i')] & \text{if finite,} \\ 0 & \text{if infinite.} \end{cases}$$

(3) (Flat locus pullback) If $\iota : x \to X$ is any morphism with x the spectrum of a field and $\iota(x)$ is in the flat locus of $\sqcup Z_i \to X_{\text{red}}$ then

$$\left[\begin{array}{ccc} x & & \\ & \searrow & \\ & & X \end{array}\right]^* \sum n_i \left[\begin{array}{ccccc} & & Z_i & & \\ & \swarrow & & \searrow & \\ X & & & & Y \end{array}\right] = \sum n_i \left[\begin{array}{ccc} & & x \times_X Z_i \\ & \swarrow & \\ x & & \end{array}\right]^* \left[\begin{array}{ccc} x \times_X Z_i & & \\ & \searrow & \\ & & Y \end{array}\right].$$

Definition 2.2. For the moment, let's say $\mathcal{C}$ is *a category of finite correspondences* if:

(1) objects of $\mathcal{C}$ are in bijection with (separated Noetherian) schemes,
(2) $\hom_{\mathcal{C}}(X, Y) \subseteq \mathrm{Cor}^{pre}(X, Y)$,
(3) all correspondences of the form (Graph) and (Transpose) from Ex.1.3 are morphisms in $\mathcal{C}$,
(4) Properties (1), (2), (3) from Wish List 2.1 hold in $\mathcal{C}$.

Exercise 2.3. Suppose that $\mathcal{C}$ is a category of finite correspondences. Show the following.

[4]Although, $\alpha \in \mathrm{Cor}^{pre}(X, Y)$ perhaps *should* be thought of as a multi-valued morphism from X to Y.

(1) (Covariance) For any two composable morphisms of schemes $\xrightarrow{f}\xrightarrow{g}$ we have:

$$\begin{bmatrix} \searrow^{f} \end{bmatrix}^* \begin{bmatrix} \searrow^{g} \end{bmatrix} = \begin{bmatrix} \searrow^{f^* g} \end{bmatrix}.$$

(2) (Generically determined) If $\eta = \sqcup \eta_i \subseteq X$ is the disjoint union of the generic points of X, then we have

$$\hom_{\mathcal{C}}(X, Y) \subseteq \hom_{\mathcal{C}}(\eta, Y).$$

(3) (Field Extensions) If L/K is an extension of fields, we have

$$\hom_{\mathcal{C}}(\mathrm{Spec}(K), Y) \subseteq \hom_{\mathcal{C}}(\mathrm{Spec}(L), Y).$$

(4) (Dominance) If $X' \to X$ is a dominant morphism, we have

$$\hom_{\mathcal{C}}(X, Y) \subseteq \hom_{\mathcal{C}}(X', Y).$$

(5) (Decomposition) If x is the spectrum of a field, then every correspondence from x to Y decomposes as

$$\sum n_i \begin{bmatrix} & z_i & \\ \swarrow & & \searrow \\ x & & Y \end{bmatrix} = \sum n_i \begin{bmatrix} & z_i \\ \swarrow & \\ x & \end{bmatrix}^* \begin{bmatrix} z_i & \\ & \searrow \\ & Y \end{bmatrix}.$$

(6) "Horizontal" nilpotents are invisible

$$\begin{bmatrix} X_{red} & \\ & \searrow \\ & X \end{bmatrix}^* \sum n_i \begin{bmatrix} & Z_i & \\ \swarrow & & \searrow \\ X & & Y \end{bmatrix} = \sum n_i \begin{bmatrix} & Z_i & \\ \swarrow & & \searrow \\ X_{red} & & Y \end{bmatrix}.$$

(7) (Strict transform) If $f : \widetilde{X} \to X$ is a proper birational morphism then

$$\begin{bmatrix} \widetilde{X} & \\ & \searrow \\ & X \end{bmatrix}^* \sum n_i \begin{bmatrix} & Z_i & \\ \swarrow & & \searrow \\ X & & Y \end{bmatrix} = \sum n_i \begin{bmatrix} & \widetilde{Z}_i & \\ \swarrow & & \searrow \\ \widetilde{X} & & Y \end{bmatrix}$$

where the $\widetilde{Z}_i$ are the strict transforms[5] of the Z_i.

(8) (Finite flat base change) If $T \xrightarrow{f} S$ is any morphsm of schemes and $X \xrightarrow{p} S$ any finite flat morphism, we have

$$\begin{bmatrix} T & \\ & \searrow \\ & S \end{bmatrix}^* \begin{bmatrix} & X \\ \swarrow & \\ S & \end{bmatrix} = \begin{bmatrix} & T \times_S X \\ \swarrow & \\ T & \end{bmatrix}^* \begin{bmatrix} T \times_S X & \\ & \searrow \\ & X \end{bmatrix}.$$

(9) (Degree formula). If $Y \to X$ is a finite flat morphism of constant degree d then

$$\begin{bmatrix} & Y \\ \swarrow & \\ X & \end{bmatrix}^* \begin{bmatrix} Y & \\ & \searrow \\ & X \end{bmatrix} = d \cdot \mathrm{id}_X .$$

(10) (Tri1) Suppose that $Y \to X$ is a finite flat morphism, X is integral, $Y_i \subseteq Y$ are the integral components of Y with generic points $\eta_i \in Y_i$ and generic multiplicity $m_i = \mathrm{length}\, \mathcal{O}_{Y,\eta_i}$. Suppose further that each $Y_i \to X$ is still flat. Then we have

$$\begin{bmatrix} & Y \\ \swarrow & \\ X & \end{bmatrix} = \sum m_i \begin{bmatrix} & Y_i \\ \swarrow & \\ X & \end{bmatrix}^* \begin{bmatrix} Y_i & \\ & \searrow \\ & Y \end{bmatrix}.$$

[5]That is, $\widetilde{Z}_i$ is the closure of the generic point of Z_i in $\widetilde{X} \times Y$, where we use birationality of $\widetilde{X} \to X$ to identify the generic fibres of $\widetilde{X} \times Y$ with the generic fibres of $X \times Y$.

(11) (Tri2) Suppose that $Y' \to X$ and $Y \to X$ are finite flat morphisms of integral schemes admitting a factorisation $Y' \to Y \to X$ (here $Y' \to Y$ is not necessarily flat). Then

$$[k(Y') : k(Y)] \begin{bmatrix} & Y \\ \swarrow & \\ X & \end{bmatrix} = \begin{bmatrix} & Y' \\ \swarrow & \\ X & \end{bmatrix}^* \begin{bmatrix} Y' & \\ & \searrow \\ & Y \end{bmatrix}.$$

Remark 2.4. The properties (Tri1) and (Tri2) are exactly the difference between "cycle theoretic" finite correspondences and "vector bundle theoretic" finite correspondences in a precise mathematical sense, cf. [Kel12, Thm.3.7.1], at least for cdh-sheaves.

3. The category of correspondences

Lemma 3.1. *Suppose that $\mathcal{C}, \mathcal{C}'$ are two categories of finite correspondences and $\alpha \in \mathrm{Cor}^{pre}(X, Y)$, $\beta \in \mathrm{Cor}^{pre}(Y, W)$ are in both $\mathcal{C}$ and $\mathcal{C}'$. Then $\alpha^*\beta$ is the same in both $\mathcal{C}$ and $\mathcal{C}'$. That is composition of arbitrary correspondences is uniquely determined by (Flat locus pullback), (Pushforward), and (Contravariance) from Wish List 2.1.*

Proof. By (2) we can assume that $X = \mathrm{Spec}(\Omega)$ is the spectrum of a field, and by (3) that Ω is algebraically closed. By (5) we can then assume that $X \to Y$ is the graph of a morphism. By (7) and Raynaud-Gruson flatification, we can assume that each $Y \leftarrow Z_i$ is flat, where $\beta = \sum n_i[Y \leftarrow Z_i \to W]$. Then (Pullback) from Wish List 2.1 determines the composition. □

In the proof of Lemma 3.1 we used algebraic closures of function fields, and flatification of correspondences to show that composition of arbitrary finite correspondences was unique, assuming the three properties in Wish List 2.1. Composition does not necessarily exist though. We will now address this.

Definition 3.2. Suppose that $\alpha = \sum n_i[X \leftarrow Z_i \to Y] \in \mathrm{Cor}^{pre}(X, Y)$. We say that α is has *well-defined specialisation* if for any proper birational morphism $p : \widetilde{X} \to X$ such that the strict transforms $\widetilde{Z}_i \to \widetilde{X}$ are all flat, any point $x \in X$ and any extension of fields $\Omega/k(x)$ admitting a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\Omega) & \xrightarrow{\phi} & \widetilde{X} \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle p} \\ x & \xrightarrow{\iota} & X \end{array}$$

the following two conditions are satisfied:

(1) there exists a cycle $\beta \in \mathrm{Cor}^{pre}(x, Y)$ such that $\kappa^*\beta = \phi^*(p^*\alpha)$,
(2) the cycle β is independent of the choice of κ, ϕ, p.

Here, κ^*, ϕ^*, p^* are the operations from Ex.2.3(3), (Pullback), and Ex.2.3(7) respectively. The abelian group of correspondences with well-defined specialisation is denoted $\mathrm{Cor}(X, Y)$.

Note that a consequence of Lemma 3.1 is that the collection of categories of finite correspondences is partially ordered.

Theorem 3.3 ([SV00], see also [Ivo05]). *There exists a category of finite correspondences $\mathcal{C}$ containing all others, and we have $\hom_{\mathcal{C}}(X, Y) = \mathrm{Cor}(X, Y)$ for all X, Y.*

4. Finite correspondences with regular source

Theorem 4.1 ([SV00, Cor.3.4.6])**.** *If X is regular then* $\mathrm{Cor}(X,Y) = \mathrm{Cor}^{pre}(X,Y)$.

Theorem 4.2 ([SV00, Thm.3.5.8, Lem.3.5.9])**.** *Suppose X is regular, $x \in X$ is a point, and $[X \leftarrow Z \to Y]$ is a finite correspondence. Then*

$$\begin{bmatrix} x & \\ & \searrow \\ & & X \end{bmatrix}^* \begin{bmatrix} & Z & \\ \swarrow & & \searrow \\ X & & Y \end{bmatrix} = \sum n_i \begin{bmatrix} & z_i & \\ \swarrow & & \searrow \\ x & & Y \end{bmatrix}$$

where $z_i \in x \times_X Z$ are the points in the fibre over x and

$$n_i = \sum_{j=0}^{\dim \mathcal{O}_{X,x}} (-1)^j \operatorname{length}_{\mathcal{O}_{Z,z_i}} (\operatorname{Tor}_j^{\mathcal{O}_{X,x}}(\mathcal{O}_{Z,z_i}, k(x))).$$

5. Relative correspondences

Recall that if X, Y are S-schemes, then a finite correspondence $\sum n_i[X \leftarrow Z_i \to Y]$ is called an *S-correspondence* if each Z_i is a subscheme of $X \times_S Y \subseteq X \times Y$.

Exercise 5.1. Show the following.

(1) Suppose X, Y are S-schemes, $\alpha \in \mathrm{Cor}(X,Y)$ is a finite correspondence, and $\eta \subseteq X$ is the disjoint union of the generic points of X. Show that $[\eta \to X]^*\alpha$ is an S-correspondence if and only if α is an S-correspondence.
(2) Suppose that L/K is an extension of fields, $\mathrm{Spec}(K)$ and Y are S-schemes and $\alpha \in \mathrm{Cor}(\mathrm{Spec}(K), Y)$ is a finite correspondence. Show that $[\mathrm{Spec}(L) \to \mathrm{Spec}(K)]^*\alpha$ is an S-correspondence if and only if α is an S-correspondence.

Proposition 5.2. *Suppose that X, Y, W are S-schemes, and $\alpha \in \mathrm{Cor}(X,Y)$ and $\beta \in \mathrm{Cor}(Y,W)$ are correspondences with well-defined specialisation. If α is an S-correspondence, the so is $\alpha^*\beta$.*

Proof. By Exercise 5.1, we can assume that X is the spectrum of an algebraically closed field, and $\alpha = [X \to Y]$ for some morphism. Chose a blowup $\widetilde{Y} \to Y$ which flatifies β, since $k(X)$ is algebraically closed there exists a factorisation $X \to \widetilde{Y} \to Y$. That is, we can assume each summand of β is flat over Y. Then it follows from (Flat locus pullback) that the summands of $\alpha^*\beta$ lie in $X \times_S W$. □

Definition 5.3 ([Voe00], [Ivo05])**.** Let S be a separated Noetherian scheme. The category whose objects are smooth S-schemes, and morphism sets are $\mathrm{Cor}_S(X,Y)$ is denoted $\mathrm{SmCor}(S)$.

Remark 5.4. It follows from Thm.4.1 that if S is a field then $\hom_{\mathrm{SmCor}(S)}(X,Y)$ is the free abelian group generated by closed integral subschemes $Z \subseteq X \times_S Y$ such that $X \leftarrow Z$ is finite and dominates an irreducible component of X.

6. Products of correspondences

Write Cor for the category of correspondences from Theorem 3.3. So objects of Cor are in bijection with separated Noetherian schemes, and $\hom_{\mathrm{Cor}}(X,Y) = \mathrm{Cor}(X,Y)$. When considering a scheme X as an object of Cor we write $[X]$.

Wish List 6.1. We would like an additive functor

$$\otimes : \mathrm{Cor} \times \mathrm{Cor} \to \mathrm{Cor}$$

satisfying the following properties.

(1) If $f : X \to Y$ is any morphism of schemes and T is any other scheme, then

$$\left[\begin{array}{ccc} T & & \\ & \searrow^{\mathrm{id}} & \\ & & T \end{array}\right] \otimes \left[\begin{array}{ccc} X & & \\ & \searrow^{f} & \\ & & Y \end{array}\right] = \left[\begin{array}{ccc} T \times X & & \\ & \searrow^{\mathrm{id} \times f} & \\ & & T \times Y \end{array}\right]$$

and similar for $[f] \otimes [\mathrm{id}_T]$.

(2) If $p : Y \to X$ is any finite flat morphism of schemes and T is any other scheme, then

$$\left[\begin{array}{ccc} & & T \\ & {}^{\mathrm{id}}\swarrow & \\ T & & \end{array}\right] \otimes \left[\begin{array}{ccc} & & Y \\ & {}^{p}\swarrow & \\ X & & \end{array}\right] = \left[\begin{array}{ccc} & & T \times Y \\ & {}^{\mathrm{id} \times p}\swarrow & \\ T \times X & & \end{array}\right]$$

and similar for $[{}^t p] \otimes [\mathrm{id}_T]$.

Exercise 6.2. Suppose that the functor $\otimes$ from Wish List 2.1 exists. Show the following.

(1) For any two schemes X, Y we have $[X] \otimes [Y] = [X \times Y]$.
(2) For any two morphisms f, g we have $[f] \otimes [g] = [f \times g]$.
(3) For any two finite flat morphisms p, q we have $[{}^t p] \otimes [{}^t q] = [{}^t p \times q]$.

Lemma 6.3. *Suppose that $\otimes$, $\otimes'$ are two additive functors satisfying the properties from Wish List 6.1. Then $\otimes = \otimes'$. That is, if $\otimes$ exists, it is unique.*

Proof. Given two finite correspondences $\alpha \in \mathrm{Cor}(X, Y)$ and $\beta \in \mathrm{Cor}(S, T)$, we want to see that $\alpha \otimes \beta = \alpha \otimes' \beta$. Choose blowups $p : \widetilde{X} \to X$ and $q : \widetilde{S} \to S$ which flatify α and β. Since $p \times q : \widetilde{X} \times \widetilde{S} \to X \times S$ is dominant, by Ex.2.3(4) it suffices to show that $[p \times q]^*(\alpha \otimes \beta) = [p \times q]^*(\alpha \otimes' \beta)$. But since $p^*\alpha$ and $q^*\beta$ have flat components, they decompose (cf.Ex.2.3(5)) and the tensor product is determined by linearity and Wish List 6.1. $\square$

Theorem 6.4 ([SV00], [Ivo05])**.** *The functor from Wish List 6.1 exists.*

There is also a relative version.

Theorem 6.5 ([SV00], [Ivo05])**.** *If α, β are S-correspondences then $\alpha \otimes \beta$ is also an S-correspondence. More precisely, if $\alpha \in \mathrm{Cor}_S(A, B)$ and $\beta \in \mathrm{Cor}_S(X, Y)$ then there exists a unique $\gamma \in \mathrm{Cor}_S(A \times_S X, B \times_S Y)$ forming a commutative square*

$$\begin{array}{ccc} A \times_S X & \xrightarrow{\gamma} & B \times_S Y \\ \downarrow & & \downarrow \\ A \times X & \xrightarrow[\alpha \otimes \beta]{} & B \times Y. \end{array}$$

7. Suslin and Voevodsky's notation

In [SV00], Suslin and Voevodsky use $c_{equi}(X/S, 0)$ for what we have written as $\mathrm{Cor}_S(S, X)$. This sits in a series of four families of groups, [SV00, pg.30],

$$z(X/S, r), z_{equi}(X/S, r), c(X/S, r), c_{equi}(X/S, r).$$

The r means the $Z \to S$ should be generically of relative dimension r, the c means $Z \to S$ should be proper, and *equi* means $Z \to S$ should be equidimensional. Before defining $z(X/S, r)$, $z_{equi}(X/S, r)$, $c(X/S, r)$, $c_{equi}(X/S, r)$, Suslin and Voevodsky define

$$Cycl(X/S, r), Cycl_{equi}(X/S, r), PropCycl(X/S, r), PropCycl_{equi}(X/S, r),$$

which have analogous definitions to Def.3.2 but don't require descendability to x. That is, they only require that $\phi^*(p^*\alpha)$ is independent of the choice of ϕ and p, [SV00, Def.3.1.3].

There are two functorialities discussed in [SV00, §3.6]: proper pushforward and flat pullback. As expected these are denoted f_* and f^*. More precisely, if $f : X \to Y$ is a morphism of S-schemes, then what we have written as

$$(-)^*[f] : \mathrm{Cor}_S(S, X) \to \mathrm{Cor}_S(S, Y)$$

is written as

$$f_* : c_{equi}(X/S, 0) \to c_{equi}(Y/S, 0)$$

in [SV00, Cor.3.6.3]. Similarly, if f is flat, then what we have written as

$$[f]^*(-) : \mathrm{Cor}_S(S, Y) \to \mathrm{Cor}_S(S, X)$$

is written as

$$f^* : c_{equi}(Y/S, 0) \to c_{equi}(X/S, 0)$$

in [SV00].

For a morphism $f : T \to S$, Suslin and Voeovdsky write $cycl(f)$ on [SV00, pg.29] for the maps

$$Cycl(X/S, r) \to Cycl(T \times_S X/T, r)$$

that are essentially our $[f]^*(-)$. More precisely, if $pr : T \times_S X \to X$, is the projection, $\alpha \in \mathrm{Cor}_S(S, X) = c_{equi}(X/S, 0)$ then $cycl(f)(\alpha)^*[pr] = [f]^*\alpha$, i.e., $cycl(\alpha)$ is the unique cycle making the square

$$\begin{tikzcd} {[T \times_S X]} \arrow[r, "{[pr]}"] & {[X]} \\ {[T]} \arrow[u, "cycl(\alpha)"] \arrow[r, "{[f]}"'] & {[S]} \arrow[u, "\alpha"'] \end{tikzcd}$$

commutative.

The maps $cycl(f)$ should not be confused with $cycl_X(Z)$ for $Z \subseteq X$ which is Suslin and Voevodsky's version of our $[S \xleftarrow{p} Z]$, [SV00, pg.13]).[6]

In [SV00, §3.7], Suslin and Voevodsky define correspondence homomorphisms $Cor(-,-)$ which are essentially our composition. That is, for $\alpha \in \mathrm{Cor}_X(X, Y) = c_{equi}(Y/X, 0)$ and $\beta \in \mathrm{Cor}_S(S, X) = c_{equi}(X/S, 0)$ what we have written as $\beta^*\alpha$ is written as $Cor(\alpha, \beta)$ in [SV00, pg.49 and Cor.3.7.5].

The Suslin, Voevodsky version of $\otimes$ is at the end of [SV00, §3.7]. They define it in terms of the other operations. More precisely, in our notation, for $\alpha \in \mathrm{Cor}_S(S, X)$ and $\beta \in \mathrm{Cor}_S(S, Y)$ we the cycle $\alpha \otimes \beta \in \mathrm{Cor}_S(S, X \times_S Y)$ is the unique cycle such that $(\alpha \otimes \beta)^* pr_1 = \alpha$ and $(\alpha \otimes \beta)^* pr_2 = \beta$ where pr_i are the projections. (This does not mean Cor has fibre products; if $f : X \to S$ is the structure morphism we almost always have $\alpha^*[f] \neq [\mathrm{id}_S]$).

Us	Suslin, Voevodsky	
$\mathrm{Cor}_S(S, X)$	$c_{equi}(X/S, 0)$	[SV00, pg.30 and Def.3.1.3]
$\alpha^*[f]$	$f_*\alpha$	[SV00, pg.14 and Cor.3.6.3]
$[f]^*\alpha$	$f^*\alpha$	[SV00, pg.13 and Lem.3.6.4]
$[f]^*\alpha$	$pr_* cycl(\alpha)$	[SV00, pg.29]
$[S \leftarrow Z]$	$cycl_Z(Z)$	[SV00, pg.13]
$\beta^*\alpha$	$Cor(\alpha, \beta)$	[SV00, pg.49 and Cor.3.7.5]

[6] Here, there is a small issue. Suslin and Voevodsky's $cycl_X(Z)$ only works properly over generically reduced bases. This is a known issue and is easily fixed by using "vertical" multiplicities as we have done above instead of global multiplicities.

References

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