December, 2020

Dear Hiroyasu,

Here is a potential setting in which to compare all the categories that are floating around.

Consider $\mathbf{PSh}(\mathbf{Pro}(\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}))$. There are various classes of morphisms of this category we can consider formally inverting.

- (Pro) Morphisms of the form $\varinjlim F(-_\lambda) \to F(\text{``}\varprojlim\text{''} -_\lambda)$ where the right is a presheaf on $\mathbf{Pro}(\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}})$ and the left is the extension of its restriction to $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$.
- (Ad) Admissible blowups.
- (Nis) Nisnevich "equivalences" (i.e., morphisms of presheaves which become isomorphisms after sheafification).
- (Sch) Morphisms of the form $(X^\circ, \varnothing) \to (\overline{X}, X^\infty)$.
- (Prp) Morphisms of the form $\mathcal{U} \to \text{``}\varinjlim\text{''}\mathcal{X}$ where $\mathcal{U} \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$ and $\mathcal{X}$ are the compactifications, i.e., morphisms $\mathcal{U} \to \mathcal{X}$ such that $U^\circ = X^\circ$, $U^\infty = X^\infty|_{\overline{U}}$ and $\overline{X}$ is proper.
- (Log) Morphisms of the form $\mathcal{X} \to \mathcal{Y}$ such that $\overline{X} = \overline{Y}$ and $\mathcal{O}_{\overline{X}} \cap \mathcal{O}^*_{X^\circ} = \mathcal{O}_{\overline{Y}} \cap \mathcal{O}^*_{Y^\circ}$.
- (CI) Morphisms of the form $\overline{\square}_{\mathcal{X}} \to \mathcal{X}$.

It seems reasonable to expect that all categories in play can be obtained by localising at various combinations of these classes. For example, $\mathbf{PSh}(\underline{\mathbf{M}}\mathbf{Cor})$ comes from inverting (Ad), the category $\mathbf{PSh}(\mathbf{Cor})$ comes from inverting (Sch), and $\mathbf{PSh}(\mathbf{MCor})$ should come from inverting (Prp). I admit to not having looked closely (or at all) at Federico's construction, but I guess you get his log presheaves (or something close to it) by inverting (Log). Then of course log cube invariant Nisnevich sheaves would come from inverting (CI) $\cup$ (Log) $\cup$ (Nis).

Since cube invariant Nisnevich sheaves come from inverting (CI) and (Nis), you would get reciprocity sheaves by inverting all morphisms which become isomorphisms under the functor $CI \to RSC$ but there is probably a more class which will create RSC. Potentially the canonical morphisms (of presheaves)

$$(X, \varnothing) \to \text{``}\varinjlim_{\mathcal{X} s.t. X^\circ = X}\text{''}\ H_0^{\overline{\square}}(\mathcal{X}).$$

A careful study of how these classes interact with each other might clear up why various functors are or aren't fully faithful, and which information is more easily accessible in which categories.

Best, Shane