Lecture 4: The Cycle Map

In this lecture we construct the cycle class map

$$\mathcal{Z}^r(X) \to H^{2r}_{\mathrm{dR}}(X)$$

to de Rham cohomology and state the Hodge Conjecture, Conj.9.

We first construct the cycle class map using Poincaré duality, Def.3. After stating the Hodge conjecture, we give a second more modern construction of the cycle class map which automatically gives compatibility with pushforward, pullback, intersection product, etc.

Remark 1. Through-out this lecture we will use the same symbol X for a smooth complex variety and its associated real manifold. Unless otherwise stated, cohomology means the de Rham cohomology of smooth differential forms with real coefficients.

1 Poincaré duals

Recall the statement of Poincaré duality.

Theorem 2 (Poincaré duality, [BT82, (5.4)])**.** *Let X be a smooth projective complex variety of dimension d. The pairing*

$$H^k_{\mathrm{dR}}(X) \times H^{2d-k}_{\mathrm{dR}}(X) \to \mathbb{R}, \qquad (\alpha, \beta) \mapsto \int_X \alpha \cup \beta$$

induces an isomorphism of finite dimensional vector spaces

$$H^k_{\mathrm{dR}}(X) \cong H^{2d-k}_{\mathrm{dR}}(X)^\vee. \tag{1}$$

Via the isomorphism (1), to define a class in $H^{2c}_{\mathrm{dR}}(X)$, it is equivalent to define a class in $H^{2d-2c}_{\mathrm{dR}}(X)^\vee$.

Definition 3 ([BT82, (5.13)])**.** *Let X be a smooth projective complex variety of dimension d, and let $Z \subseteq X$ be a smooth irreducible subvariety of codimension c. The* Poincaré dual *of Z is the class $\gamma(Z) \in H^{2c}_{\mathrm{dR}}(X)$ corresponding to*

$$\int_Z (-|_Z) : H^{2d-2c}_{\mathrm{dR}}(X) \to \mathbb{R}$$

under (1)*. In other words, it is the unique class such that*

$$\int_X \gamma(Z) \cup \beta = \int_Z \beta|_Z$$

for every $\beta \in H^{2d-2c}_{\mathrm{dR}}(X)$.

Remark 4. Defining a class via duality is perhaps not very satisfying if one is trying to make a geometric visualisation. Looking at this duality definition, one sees that $\gamma(Z)$ somehow only sees information along Z, in the cotangent directions $\Omega_X \twoheadrightarrow \Omega_Z$ of Z. In fact it is not unreasonable to visualise $\gamma(Z)$ as a $2c$-form supported[1] on Z which at each point $z \in Z$ is a unit volume form in the conormal directions $\wedge^{2c} C_X Z$ to Z in X (here $C_X Z = \ker(\Omega_X \twoheadrightarrow \Omega_Z)$, [Sta, Tags 01R2 and 06AA]).

Remark 5. Note that Definition 3 is only valid when X is smooth and projective, and Z is also smooth. When Z is not smooth, $\int_Z (-|_Z)$ is interpreted as the integral over the smooth locus Z^{sm} of Z, [Voi02, Thm.11.21]. That is, $\gamma(Z)$ is the unique class such that

$$\int_X \gamma(Z) \cup \beta = \int_{Z^{\mathrm{sm}}} (\beta|_{Z^{\mathrm{sm}}}).$$

Note that it's not obvious that $\int_{Z^{\mathrm{sm}}} (\beta|_{Z^{\mathrm{sm}}})$ converges. One reason is that we can choose a surjective morphism $\tilde{Z} \to Z$ such that $\tilde{Z}$ is smooth and projective, by Hironaka's resolution of singularities, [Hir64], [Voi02, Thm.11.22]. Then $\int_{\tilde{Z}} (\beta|_{\tilde{Z}})$ is finite. But $\tilde{Z} \to Z$ can be chosen to be an isomorphism over Z^{sm}, an open with complement of (real) codimension ≥ 2. Removing closed submanifolds of codimension ≥ 2 does not affect the integral,[2] so $\int_{\tilde{Z}} (\beta|_{\tilde{Z}}) = \int_{Z^{\mathrm{sm}}} (\beta|_{Z^{\mathrm{sm}}})$ is finite.

Example 6. Assume X is connected. If $c = d$, then Z is a point, $H^0_{\mathrm{dR}}(X)$ is the ring of locally constant functions, and $\int_Z (-|_Z) : H^0_{\mathrm{dR}}(X) \to \mathbb{R}$ is evaluation at Z. Consequently, $\gamma(Z) \in H^{2d}_{\mathrm{dR}}(X)$ is precisely any top degree class whose integral over X is 1.

2 Hodge conjecture

Proposition 7. *Let X be a smooth projective complex variety, and let $Z \subseteq X$ be a smooth irreducible subvariety of codimension c. Then*

$$\gamma(Z) \in H^{c,c}(X) \subseteq H^{2c}_{\mathrm{dR}}(X, \mathbb{C}).$$

Proof. Let $d = \dim X$. By definition the class $\gamma(Z) \in H^{2c}_{\mathrm{dR}}(X, \mathbb{C})$ is dual to

$$\xi : \beta \mapsto \int_Z \beta|_Z \qquad \text{for } \beta \in H^{2d-2c}_{\mathrm{dR}}(X, \mathbb{C}) = \bigoplus_{i+j=2d-2c} H^{i,j}(X)$$

Since Poincaré Duality is compatible with the Hodge decomposition, it suffices to show that $\xi \in H^{d-c,d-c}(X)^\vee$. However, this is clear, since ξ factors through

[1] Replacing "$2c$-form supported on Z" with "a bump $2c$-form supported on a small neighbourhood of Z" is more accurate, [BT82, Prop.6.18, Eq.(6.23), Prop.6.24].

[2] This is because they have measure zero.

restriction to
$$\bigoplus_{\substack{i+j=2d-2c \\ 0\le i,j\le \dim Z = d-c}} H^{i,j}(Z)^\vee = H^{d-c,d-c}(Z)^\vee.$$

□

Remark 8. Notice that Definition 3 essentially only used Poincaré Duality. Given a smooth[3] closed subvariety $Z \subseteq X$ the cycle class $\gamma(Z) \in H^{2c}(X)$ is the class which corresponds to the composition $H^{2d-2c}_{\mathrm{dR}}(X) \to H^{2d-2c}_{\mathrm{dR}}(Z) \xrightarrow{\sim} \mathbb{R}$ under the pairing $H^{2c}_{\mathrm{dR}}(X) \cong H^{2d-2c}_{\mathrm{dR}}(X)^\vee$. So it's valid for any cohomology theory with an appropriate structure of Poincaré Duality, for example Betti cohomology, even with other coefficient fields. Since the comparison $H^*_{\mathrm{Betti}}(X,\mathbb{R}) \cong H^*_{\mathrm{dR}}(X)$ is compatible with Poincaré Duality, we get a corresponding commutative diagram

$$\mathcal{Z}^c(X) \longrightarrow H^{2c}_{\mathrm{Betti}}(X,\mathbb{Q}) \longrightarrow H^{2c}_{\mathrm{Betti}}(X,\mathbb{R}) \cong H^{2c}_{\mathrm{dR}}(X).$$

This is all just to say that the de Rham cycle class map (with $\mathbb{R}$-coefficients) factors through the Betti cycle class map (with $\mathbb{Q}$-coefficients). Even with $\mathbb{Z}$ coefficients.

Conjecture 9 (Hodge)**.** Let X be a smooth projective complex variety and $c \in \mathbb{N}$. Then
$$\gamma\colon \mathcal{Z}^c(X)_{\mathbb{Q}} \to H^{c,c}(X) \cap H^{2c}_{\mathrm{Betti}}(X,\mathbb{Q})$$
is surjective. Here the intersection takes place in $H^{2c}_{\mathrm{Betti}}(X,\mathbb{C}) \cong H^{2c}_{\mathrm{dR}}(X,\mathbb{C})$.

3 Finite correspondences

Here is an alternative, more modern way to build the cycle map. In this section F is a field of characteristic zero.

Definition 10 ([Voe00, §2.1])**.** *Let X and Y be smooth varieties with X irreducible. A* finite correspondence *from X to Y is a cycle*

$$\alpha = \sum_i n_i[Z_i] \in \mathcal{Z}^{\dim Y}(X \times Y)$$

such that each $p_i : Z_i \to X$ is finite. That is, the preimage $p_i^{-1}(x)$ of each point $x \in X$ is finite, and the image $p_i(W) \subseteq X$ of every closed $W \subseteq Z_i$ is closed. The subgroup of $\mathcal{Z}(X \times Y)$ generated by such cycles is denoted

$$c(X, Y).$$

We extend this to general smooth X via $c(X_1 \sqcup X_2, Y) = c(X_1, Y) \oplus c(X_2, Y)$.

[3] To get to non-smooth cycles we proposed using resolution of singularities and the fact that integration doesn't see measure zero subsets. Without the comparison $H^*_{\mathrm{Betti}} \cong H^*_{\mathrm{dR}}$, it's not obvious whether or not Betti cohomology would satisfy this. The local cohomology construction discussed in [Voi02, §11.1.2, Lem. 11.13–Def. 11.17] is valid as written for Betti cohomology.

Remark 11. Note that if $\alpha \in c(X,Y)$ and $\beta \in c(Y,Z)$ then the *cycle* (not just the class) $\beta \circ \alpha$ is always defined. It suffices to consider the case $\beta = [B]$ and $\alpha = [A]$. In this case

$$pr_{XY}^{-1}(A) = A \times Z, \qquad pr_{YZ}^{-1}(B) = X \times B.$$

Since $B \to Y$ is finite, $C := (A \times Z) \cap (X \times B) \to A$ is quasi-finite, that is, has finite fibres. Indeed, for each $y_0 \in Y$ there are finitely $z_1, \dots, z_n \in Z$ with $(y_0, z_i) \in B$, so for each $(x_0, y_0) \in A$, the preimage $\{(x_0, y_0, z) \mid (y_0, z) \in B\} \subseteq C$ of (x_0, y_0) is finite. So by [Mum88, §I.8, Thm.2]

$$\dim C \leq \dim A = \dim X.$$

On the other hand, $\operatorname{codim} C = \operatorname{codim}(A \times Z) \cap (X \times B) \leq \operatorname{codim} A \times Z + \operatorname{codim} X \times B = \dim Y + \dim Z$. Thus

$$\dim C \geq \dim X.$$

Therefore $\dim C = \dim X$, so $A \times Z$ and $X \times B$ intersect properly. Hence $\beta \circ \alpha$ is defined.

Definition 12 ([Voe00, §2.1, p.3])**.** *The category* SmCor *has smooth varieties as objects and*

$$\hom_{\mathrm{SmCor}}(X,Y) = c(X,Y)$$

as morphism groups. Composition is the composition of correspondences.

Essentially, a presheaf on SmCor is a presheaf on Sm with "extra" functoriality for transfers. This extra functoriality can be encoded as a descent condition.

Theorem 13 ([Ayo14a, Rem.4.5], [CD19, Th.16.2.18], [Mor06]. See also [Voe10, Thm.4.2], [SV00, Prop.4.2.14].)**.** *Suppose the base field k is of characteristic zero and $K^* : \mathrm{Sm}^{op} \to \mathrm{Cplx}(F)$ is a presheaf of chain complexes of F-vector spaces satisfying:*

1. *cohomological étale descent, and*
2. *$\mathbb{A}^1$-invariance, in the sense that $K^*(X) \to K^*(X \times \mathbb{A}^1)$ is a quasi-isomorphism for all $X \in \mathrm{Sm}$,*

then there exists an extension[4]

$$\tilde{K}^* : \mathrm{Sm}^{op} \to \mathrm{SmCor}^{\mathrm{op}} \dashrightarrow \mathrm{Cplx}(F).$$

The extension is unique up to quasi-isomorphism.[5]

[4] Strictly speaking, the extension exists up to quasi-isomorphism.

[5] Here, "unique up to quasi-isomorphism" means that any two extensions $\tilde{K}^*, \tilde{K}'^* \in \mathrm{PSh}(\mathrm{SmCor}, F)$ are connected by a zig-zag of morphisms $\tilde{K}^* = \tilde{K}_0^* \leftarrow \tilde{K}_1^* \to \cdots \leftarrow \tilde{K}_{n-1}^* \to \tilde{K}_n^* = \tilde{K}'^*$ such that each $\tilde{K}_i^*(X) \to \tilde{K}_{i\pm1}^*(X)$ is a quasi-isomorphism of chain complexes for all $X \in \mathrm{Sm}$.

Remark 14. We don't want to get too distracted talking about cohomological descent, [Con03, Def.6.5, Thm.6.11], here. Briefly, it means for certain diagrams of varieties

$$\dots \Rrightarrow X_2 \rightrightarrows X_1 \to X_0 \to Y$$

the associated morphism $K^*(Y) \to \operatorname{Tot} K^*(X_*)$ is a quasi-isomorphism.

Remark 15. Theorem 13 is actually trying to say that there is an equivalence of categories $DA^{\text{eff}}(k, \mathbb{Q}) \cong DM^{\text{eff}}(k, \mathbb{Q})$ between the étale, $\mathbb{A}^1$-local, non-transfers triangulated category of motives and the Nisnevich, $\mathbb{A}^1$-local, transfers triangulated category of motives (with rational coefficients).

Example 16. Suppose that $f : X \to Y$ is a finite[6] étale[7] Galois[8] morphism of irreducible smooth complex varieties. In this situation the descent condition applied to f says,[9] [CD19, Thm.3.3.23],

$$K^*(Y) \xrightarrow{\sim} K^*(X)^G.$$

In particular, the map $K^*(X) \xrightarrow{\frac{1}{\#G}\sum_{g\in G} g} K^*(X)^G \xleftarrow{\sim} K^*(Y)$ gives an extra covariant functoriality for these morphisms.

Example 17. De Rham cohomology $H^*_{\text{dR}}(-)$ has cohomological étale descent.

Indeed, de Rham cohomology agrees with Čech cohomology, [BT82, Thm.14.28, Thm.15.8], and étale maps are locally standard, [Mil80, Chap.I, §3, Thm.3.14], hence local analytic isomorphisms by the inverse function theorem, [Mil80, Chap.I, §3, p.23]. In the finite case this recovers the equivalence between finite étale coverings and analytic coverings, [Mil80, Chap.III, §3, Lem.3.14].

Moreover, it is $\mathbb{A}^1$-invariant, [BT82, Cor.4.1.2.2].

Corollary 18. *Recall that $A^*(X)$ means the complex of smooth real differential forms on the underlying smooth real manifold associated to the smooth complex variety X. This satisfies cohomological étale descent and is $\mathbb{A}^1$-invariant, Exam.17, so it has a unique lift to*

$$A^*(-) : \mathrm{SmCor}^{\mathrm{op}} \to \mathrm{Cplx}(\mathbb{R}).$$

So we have an action of finite correspondences on de Rham cohomology. We can upgrade this to an action of all correspondences (up to rational equivalence) as follows. The following is essentially a moving lemma. It says that every cycle on $X \times Y$ is rationally equivalent to one which is finite over X.

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[6] Finite means all fibres are finite and f sends closed sub-varieties to closed sub-varieties.

[7] Étale means that the induced map on cotangent spaces at each point is an isomorphism. Equivalently, over $\mathbb{C}$ the induced morphism of smooth manifolds is a local homeomorphism.

[8] Galois means that, for $G = \mathrm{Aut}(X/Y) = \{\phi : X \xrightarrow{\sim} X \mid f\phi = f\}$, we have $[k(X) : k(Y)] = \#G$.

[9] If $V \in \mathrm{GrVec}_F$ has an action of a finite group G we write $V^G := \operatorname{im} \frac{1}{\#G}\sum_{g\in G} g$.

Theorem 19 ([Voe00, Proof of Prop.2.1.4], [FV00, Thm.7.1])**.** *Let X and Y be smooth projective irreducible varieties. Then*

$$CH^{\dim Y}(X \times Y) \cong \operatorname{coker}\left(c(X \times \mathbb{A}^1, Y) \xrightarrow{i_0 - i_1} c(X, Y) \right)$$

Here i_a means composition with the inclusion $X \times \{a\} \subseteq X \times \mathbb{A}^1$.

Construction 20. Since projection induces an equivalence $H^*_{\mathrm{dR}}(X) \xrightarrow{\sim} H^*_{\mathrm{dR}}(X \times \mathbb{A}^1)$, [BT82, Cor.4.1.2.2] we obtain a factorisation

$$c(X, Y) \twoheadrightarrow CH^{\dim Y}(X \times Y) \dashrightarrow \hom(H^*_{\mathrm{dR}}(Y), H^*_{\mathrm{dR}}(X)).$$

Inputting $Y = \mathbb{P}^j$ gives

$$CH^j(X \times \mathbb{P}^j) \to \hom(H^*_{\mathrm{dR}}(\mathbb{P}^j), H^*_{\mathrm{dR}}(X)).$$

Composing with $p^* : CH^j(X) \to CH^j(X \times \mathbb{P}^j)$ and evaluating at the top class $t^j \in H^{2j}_{\mathrm{dR}}(\mathbb{P}^j)$ gives

$$\gamma \colon CH^j(X) \to H^{2j}_{\mathrm{dR}}(X).$$

Remark 21. Note that by construction, this cycle class map is compatible with composition. In fact, it is also compatible with cartesian product. This means, in particular, that it is also compatible with pushforward, intersection product, and pullback.

Remark 22. Ayoub and others have used this perspective to study, in particular, Bloch's conjecture on 0-cycles on surfaces, [Ayo17, Prop.2.15], [Ayo14a, Rem.5.9], the motivic Galois group, [Ayo14b, §3.2], [And17], and the period conjectures of Grothendieck and Kontsevich–Zagier. [Ayo14b, §5], [And17], but this "transfer free" setting can also be used for Kimura finiteness for Chow motives, [Kim05, Conj.7.1], the Bloch–Beilinson filtration on Chow groups, [Jan94, Conj.2.1], the Standard Conjectures in characteristic zero, [Bei10, §1], [Gro69].

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