Lecture 8: The motivic t-structure

In the first half of this lecture we see the Standard Conjecture I. This holds in characteristic zero, and assuming it, we have $(A) \overset{(I)}{\Leftrightarrow} (B) \overset{(I)}{\Leftrightarrow} (D) \Rightarrow (C)$.

In the second section we discuss Voevodsky's triangulated category of motives and see Beilinon's theorem that the existence of a motivic t-structure implies all the standard conjectures.

1 The Hodge type conjecture I

For X irreducible of dimension d, note that Poincaré duality is a nondegenerate bilinear form in the middle degree

$$PD : H^d(X) \otimes H^d(X) \to F \tag{1}$$

Using the Lefschetz operator, we get nondegenerate bilinear forms for all $0 \leq k \leq d$.

Definition 1. For $k \leq d$ define Q_k as the composition

$$Q_k : H^k(X) \otimes H^k(X) \xrightarrow{\mathrm{id} \otimes L^{d-k}} H^k(X) \otimes H^{2d-k}(X) \xrightarrow{PD} F \tag{2}$$

where PD is Poincaré Duality.

Remark 2.

1. We are using (real) de Rham cohomology of the underlying smooth real manifold $X(\mathbb{C}) \in \mathrm{SmoothManifolds}_{\mathbb{R}}$ of the complex smooth projective variety X. So classes of $H^k(X) = H^k_{dR}(X(\mathbb{C}), \mathbb{R})$ can be represented by smooth k-forms $\alpha \in A^k(X(\mathbb{C}))$ and Poincaré duality is obtained by integration, [Voi02, §5.3, p.133]. If $\eta \in A^2(X(\mathbb{C}))$ is the a 2-form representing the chosen hyperplane section then bilinear form (2) is

$$\begin{aligned} H^k_{dR}(X(\mathbb{C}), \mathbb{R}) \otimes H^k_{dR}(X(\mathbb{C}), \mathbb{R}) &\to \mathbb{R} \\ (\alpha, \beta) &\mapsto \textstyle\int_{X(\mathbb{C})} \alpha \wedge \eta^{d-k} \wedge \beta. \end{aligned}$$

   The orientation used in integration is induced by the usual orientation of $\mathbb{C} \cong \mathbb{R}^2$, [Voi02, §3.1.3, p.67].

2. By Hard Lefschetz, L^{d-k} is an isomorphism, so the non-degeneracy from Poincaré Duality implies Q_k is also nondegenerate.

3. For k odd, by graded commutativity[1] Q_k is alternating. Over any field, in each even dimension there is a unique non-degenerate alternating bilinear

[1] Recall a graded algebra A^* is *graded commutative* if for all $x \in A^k$, $y \in A^j$ we have $x \cup y = (-1)^{kj} y \cup x$. In particular, if $h \in A^2$ and $x, y \in A^k$ then $x \cup h^{d-k} \cup y = (-1)^{2(d-k)k} x \cup y \cup h^{d-k} = (-1)^{2(d-k)k}(-1)^{k(k+2(d-k))} y \cup h^{d-k} \cup x = (-1)^k y \cup h^{d-k} \cup x$.

form up to isomorphism, [Lan02, Cor.8.2, p.587]. Namely, the one with matrix

$$\begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix} \tag{3}$$

where I_r is the $r \times r$ identity matrix.

4. For k even Q_k is symmetric. For $char.F \neq 2$, there is a bijection between symmetric bilinear forms $b(-,-)$ and quadratic forms q given by $q_b(x) = b(x,x)$ and $b_q(x,y) = \frac{1}{2}\big(q(x+y) - q(x) - q(y)\big)$. If $\tau : H^{2d}(X) \xrightarrow{\sim} F$ is the trace isomorphism used in Poincaré Duality we will write

$$q_k : H^k(X) \to F; \qquad \alpha \mapsto Q_k(\alpha, \alpha) = \tau(\alpha \cup h^{d-k} \cup \alpha). \tag{4}$$

for this quadratic form. Here $h \in H^2(X)$ is the class of the hyperplane section.

Example 3.

1. For $X = \mathbb{P}^n$, write $H^*(\mathbb{P}^n) = F[h]/h^{n+1}$ with $h \in H^2(\mathbb{P}^n)$ the class of a hyperplane so $L(h^i) = h^{i+1}$. In particular, the odd cohomology groups vanish. On $H^{2r}(\mathbb{P}^n)$ with $2r \leq n$, the form is

$$q_{2r} : h^r \mapsto \tau(h^r h^{n-2r} h^r) = \tau(h^n) \overset{\text{[Stacks, 0FGW]}}{=} \deg([H] \dots [H]) = 1.$$

2. If C is an irreducible curve, the only two forms are on $H^0(C)$, where it is always $1 \otimes 1 \mapsto 1$, and on $H^1(C)$, where it is the middle degree pairing (1). Concretely, since this pairing is nondegenerate and alternating $(x \cup y = -y \cup x)$, there is a basis for which the corresponding matrix is:

$$\begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$$

where I_g is the $g \times g$ identity matrix. Topologically, $C(\mathbb{C})$ is a g-fold connected sum $(S^1 \times S^1) \# \dots \# (S^1 \times S^1)$, and such a basis is dual to the standard loops in each torus $S^1 \times S^1$.

3. Let A be an abelian variety of dimension g and write $V = H^1(A)$ so

$$H^*(A) \cong \bigwedge\nolimits^* V, \qquad h \in H^2(A) = \bigwedge\nolimits^2 V,$$

[Kle68, Thm.2A8]. Since $h^g \neq 0$, there is a basis $x_1, y_1, \dots, x_g, y_g$ of V such that, [Kle68, pg.377],

$$h = x_1 y_1 + \cdots + x_g y_g$$

and consequently,

$$h^{g-k} = (g-k)! \sum_{1 \leq i_1 < \cdots < i_{g-k} \leq g} x_{i_1} y_{i_1} \dots x_{i_{g-k}} y_{i_{g-k}}$$

In particular, since $H^{2g}(A)$ is generated by $\omega := x_1y_1\dots x_gy_g$, for monomials α,β in the variables x_i,y_i we have

$$Q_k(\alpha,\beta) = \begin{cases} \varepsilon(\alpha,\beta)(g-k)!\tau(\omega) & \text{if } \alpha\beta = \pm x_{i_1}y_{i_1}\cdots x_{i_k}y_{i_k} \text{ for distinct } i_1,\dots,i_k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varepsilon(\alpha,\beta)$ is the sign with which $\alpha\beta\prod_{j\notin\{i_1,\dots,i_k\}} x_jy_j$ reorders to ω.

Example 4 (Hodge–Riemann Theorem)**.** As usual, let $H^r(X) = H^r_{dR}(X(\mathbb{C}),\mathbb{R})$ be the de Rham cohomology of the smooth compact real manifold $X(\mathbb{C})$ underlying the smooth projective variety $X \in \mathrm{SmProj}$ so $F = \mathbb{R}$ and $d = \dim_{\mathbb{C}} X$. Let $h \in H^2(X)$ be the hyperplane class and let $\tau : H^{2d}(X) \to \mathbb{R}$ be integration over $X(\mathbb{C})$ with the orientation induced by its complex structure, [Voi02, §3.1.3, Lem.3.8; §5.3, p.133]. Base changing to complex coefficients we have the Hodge decomposition

$$H^k_{dR}(X(\mathbb{C}),\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X) \tag{5}$$

and complex conjugation acting on the coefficients swaps the spaces $H^{p,q}$ and $H^{q,p}$, namely, if $\omega\otimes a \in H^{p,q}$ then $\omega\otimes\overline{a} \in H^{q,p}$ where $\omega \in A^{p+q}(X(\mathbb{C}),\mathbb{R})$ and $a \in \mathbb{C}$.

Theorem 5 (Hodge–Riemann bilinear relations, [Voi02, Thm.6.32])**.** *On the $\mathbb{C}$-vector space $P^k(X)_{\mathbb{C}} \cap H^{p,q}(X)$ the Hermitian form*

$$(\alpha,\beta) \longmapsto (\sqrt{-1})^{p-q+k(k-1)}Q_k(\alpha,\overline{\beta})$$

is positive definite. That is, $(\sqrt{-1})^{p-q+k(k-1)}Q_k(\alpha,\overline{\alpha}) \in \mathbb{R}_{>0}$ for every nonzero $\alpha \in P^k(X)_{\mathbb{C}} \cap H^{p,q}(X)$.

For algebraic classes in codimension r, we have $k = 2r$ and Hodge type (r,r). The scalar is then $(\sqrt{-1})^{2r(2r-1)} = (-1)^r$. In this case, Thm.5 implies that the symmetric form

$$(\alpha,\beta) \longmapsto (-1)^r\alpha\cup h^{d-2r}\cup\beta \tag{6}$$

is positive definite on

$$P^r_{\mathrm{alg}}(X) = CH^r_{\mathrm{hom}}(X) \cap P^{2r}(X) \subseteq H^{2r}(X), \qquad 2r \le d \tag{7}$$

Recall that the cycle class map is compatible with the various cycle operations, Lec.4, Rem.**??**; Lec.5, Rem.**??**; Lec.5, Ex.**??**, and in particular, the following two squares commute (see also [Stacks, 0FGW] for the RHS square)

$$\begin{tikzcd} CH^r_{\mathrm{hom}}(X)\otimes CH^r_{\mathrm{hom}}(X) \arrow[r, "(\alpha,\beta)\mapsto(-1)^r\alpha\cdot h^{d-2r}\cdot\beta"] \arrow[d, "\gamma\otimes\gamma"'] & CH^d(X) \arrow[r, "\deg"] \arrow[d, "\gamma"] & \mathbb{Q} \arrow[d, phantom, "\cap|"] \\ H^{2r}_{dR}(X)\otimes H^{2r}_{dR}(X) \arrow[r, "(\alpha,\beta)\mapsto(-1)^r\alpha h^{d-2r}\beta"'] & H^{2d}_{dR}(X) \arrow[r, "\tau"'] & \mathbb{R}, \end{tikzcd}$$

where the right inclusion is the canonical one. Consequently, Thm.5 gives the following.

Corollary 6. *For every r with $2r \le d$, the symmetric bilinear form*

$$P^r_{\mathrm{alg}}(X) \otimes P^r_{\mathrm{alg}}(X) \to \mathbb{Q}, \qquad (\alpha,\beta) \mapsto (-1)^r \deg(\alpha h^{d-2r}\beta),$$

is positive definite. That is $(-1)^r \deg(\alpha h^{d-2r}\alpha) > 0$ for all nonozero $\alpha \in P^r_{\mathrm{alg}}$.

Conjecture I (Standard conjecture of Hodge type, [Kle68, §3])**.** Suppose that $H^* : \mathcal{M}ot_{\mathrm{rat}} \to \mathrm{GrVec}_F$ is a Weil cohomology theory and $\sim_{\mathrm{hom}}$ the corresponding equivalence relation. For every r with $2r \le \dim X$, the symmetric bilinear form

$$P^r_{\mathrm{alg}}(X) \otimes P^r_{\mathrm{alg}}(X) \to \mathbb{Q} \qquad (\alpha,\beta) \mapsto (-1)^r \deg\bigl(\alpha \cdot h^{d-2r} \cdot \beta\bigr) \tag{8}$$

is positive definite on $P^r_{\mathrm{alg}}(X) = CH^r_{\mathrm{hom}}(X) \cap P^{2r}(X) \subseteq H^{2r}(X)$. In other words,

$$(-1)^r \deg L^{d-2r}(\alpha \cdot \alpha) > 0$$

for all nonzero $\alpha \in P^r_{\mathrm{alg}}(X)$.

Remark 7. As noted in Corollary 6, Conjecture I holds for for $H^* = H^*_{\mathrm{dR}}$.

Proposition 8.

$$I(X,h) + A(X,h) \Rightarrow D(X).$$

Proof. We will show that for every nonzero $\alpha \in CH^r_{\mathrm{hom}}(X) \subseteq H^{2r}(X)$ there is $\beta \in CH^{d-r}_{\mathrm{hom}}(X)$ such that $\alpha \cdot \beta \neq 0$. Since $\cdot$ on CH^* corresponds to $\cup$ on H^* it is enough to do this inside H^*.

Using $A(X,h)$ write

$$\alpha \overset{A(X,h)}{=} \sum_{j=0}^{r} L^{r-j}\alpha_j, \qquad \alpha_j \in P^j_{\mathrm{alg}}(X) = \ker L^{d-2j+1}.$$

Define

$$\beta = \sum_{j=0}^{r} (-1)^j L^{d-r-j}\alpha_j \in CH^{d-r}_{\mathrm{hom}}(X).$$

We have

$$\begin{aligned} L^{d-i-j}\alpha_j &= L^{j-i-1}\bigl(L^{d-2j+1}\alpha_j\bigr) = 0, \qquad & i<j \\ L^{d-i-j}\alpha_i &= L^{i-j-1}\bigl(L^{d-2i+1}\alpha_i\bigr) = 0, \qquad & j<i, \end{aligned}$$

so using $L^a\alpha \cup L^b\beta = L^{a+b}(\alpha \cup \beta)$ we obtain

$$\begin{aligned} \deg(\alpha \cup \beta) &= \deg \sum_{i,j=0}^{r} (-1)^j \, L^{d-i-j}\,(\alpha_i \cup \alpha_j) \\ &= \deg \sum_{i=0}^{r} (-1)^i \, L^{d-2i}\,(\alpha_i \cup \alpha_i) \overset{\mathrm{Conj.I}}{>} 0 \end{aligned}$$

since at least one α_i is nonzero. Hence $\alpha \cup \beta \neq 0$ so $\alpha \not\sim_{\mathrm{num}} 0$, and therefore $CH^r_{\mathrm{hom}}(X) \to CH^r_{\mathrm{num}}(X)$ is injective, and therefore a isomorphism. ∎

Remark 9. Since $I(X,h)$ holds in characteristic zero, Cor.6, we get have the following implications:

$$
(char.k = 0): \qquad
\begin{tikzcd}
L\text{ is iso.} & \Lambda\text{ is alg.} & \pi_i\text{ are alg.} & \text{hom=num} \\
(A) \arrow[rrr, Leftrightarrow, bend right, "\text{Prop.}\mathbf{??}\text{,8}"'] & (B) \arrow[l, Rightarrow, "\text{Prop.}\mathbf{??}"'] \arrow[r, Rightarrow, "\text{Prop.}\mathbf{??}"'] & (C) & (D) \arrow[ll, Rightarrow, bend right, "\text{Prop.}\mathbf{??}"']
\end{tikzcd}
$$

or $(A) \Leftrightarrow (B) \Leftrightarrow (D) \Rightarrow (C)$.

2 Motivic t-structure

In Lecture 4 we hinted at the construction of an enlargement of $\mathcal{M}ot_{\mathrm{rat}}$ which accepts motives of smooth but not necessarily projective varieties. This is Voevodsky's category DM_{gm}.

Theorem 10 ([Voe00, Def.2.1.1, Prop.2.1.3, Thm.3.2.6, Thm.4.3.7], [Hub00])**.** *There exists a commutative diagram of functors*

$$
\begin{tikzcd}
\mathrm{SmProj}^{op} \arrow[r] \arrow[d] & \mathcal{M}ot_{\mathrm{rat}} \arrow[r, "H^*"] \arrow[d] & \mathrm{GrVec}_F \arrow[d, equal] \\
\mathrm{Sm}^{op} \arrow[r] & DM_{\mathrm{gm}}^{op} \arrow[r, "\Phi"'] & D(F)
\end{tikzcd}
$$

where DM_{gm} is an idempotent complete $\mathbb{Q}$-linear rigid[2] *tensor triangulated category and Φ is exact.*

For now we treat DM_{gm} as a black box. But in a precise sense, it is the unique extension compatible with Deligne's theory of weights, [Del74, §8], [Bon11].

We will at least need to know what a triangulated category is though.

Definition 11. A *triangle* in an category $\mathcal{T}$ equipped with an endomorphism $[1]: \mathcal{T} \to \mathcal{T}$ is a diagram of the form on the left or the right

$$
A \to B \to C \to A[1]; \qquad
\begin{tikzcd}
A \arrow[rr] & & B \arrow[dl] \\
& C \arrow[ul, "+"] &
\end{tikzcd}
$$

The $+$ indicates that the morphism has target $A[1]$, not A.

Definition 12 ([Stacks, Tag 0145], [Ver96, Chap.II, Def.1.1.1, p.94])**.** A *triangulated category* is an additive category equipped with an automorphism $(-)[1]: \mathcal{T} \to \mathcal{T}$ and a collection of triangles called called *distinguished* or *exact triangles*. These should satisfy the axioms.

[2]This means every object is dualisable.

(TR1) For every object X the triangle on the left is exact. Every morphism $f : A \to B$ can be extended to an exact triangle as on the right. Every diagram isomorphic to an exact triangle is an exact triangle.

$$\begin{tikzcd} X \arrow[rr, "\mathrm{id}"] & & X \arrow[dl] \\ & 0 \arrow[ul, "+"] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl] \\ & C \arrow[ul, "+"] & \end{tikzcd}$$

(TR2) (Rotation) If the triangle on the left is exact, then so are the other two.

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl, "g"] \\ & C \arrow[ul, "h", "+"'] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, "g"] & & C \arrow[dl, "h"] \\ & A[1] \arrow[ul, "-f[1]", "+"'] & \end{tikzcd} \qquad \begin{tikzcd} C[-1] \arrow[rr, "-h[-1]"] & & A \arrow[dl, "f"] \\ & B \arrow[ul, "g", "+"'] & \end{tikzcd}$$

where $[-1]$ is any choice of inverse of $[1]$.

(TR3) If the following diagram has exact rows, then there exists a dotted morphism making the adjacent squares commutative

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dotted] & A[1] \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

(TR4) (Octahedral axiom) Omitted.

A consequence of (TR4) is the following lemma which is often as useful and more easily written.

Lemma 13 ([BBD82, Prop.1.1.11]). *Suppose given a solid commutative diagram as below such that all triangles are exact.*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \arrow[d] \\ A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & A'[1] \arrow[d] \\ A'' \arrow[r, dotted] \arrow[d] & B'' \arrow[r, dotted] \arrow[d] & C'' \arrow[r, dotted] \arrow[d] & A''[1] \arrow[d] \\ A[1] \arrow[r] & B[1] \arrow[r] & C[1] \arrow[r] & A[1][1]. \end{tikzcd}$$

Then the dotted arrows can be chosen so that the third row is also exact, and the new squares are commutative, except for the lower right one which is anti-commutative.

Example 14. The category GrVec_F has a canonical structure of triangulated category with $V[1]^n = V^{n+1}$ and the exact triangles being

$$V \xrightarrow{f} W \to (\operatorname{coker} f) \oplus (\ker f[1]) \to V[1].$$

We write $D(F)$ when thinking of GrVec_F as a triangulated category.

Example 15. There is a canonical triangulated category $D(\mathrm{Ab})$ whose objects are finitely generated graded abelian groups and $[1]$ is the shift functor $(A[1])^n = A^{n+1}$. For abelian groups A, B, C considered as graded concentrated in degree 0 we have

$$\begin{aligned} \hom_{D(\mathrm{Ab})}(A, B) &= \hom_{\mathrm{Ab}}(A, B) \\ \hom_{D(\mathrm{Ab})}(C, A[1]) &= \mathrm{Ext}_{\mathrm{Ab}}(C, A) \end{aligned}$$

For every short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ with corresponding element $\xi \in \mathrm{Ext}_{\mathrm{Ab}}(C, A)$, we have an exact triangle

$$A \xrightarrow{i} B \xrightarrow{p} C \xrightarrow{\xi} A[1].$$

Remark 16. One writes

$$X[n] = X \underbrace{[1] \cdots [1]}_{n \text{ times}}$$

for $n \in \mathbb{Z}$. For an exact triangle $A \to B \to C \to A[1]$ one sees the notation

$$\mathrm{Cone}(A \to B) = \mathrm{cofib}(A \to B) := C; \qquad \mathrm{fib}(B \to C) := A$$

Warning 17. Axiom (TR1) says that $\mathrm{Cone}(A \to B)$ always exists, while Axiom (TR3) says that any two choices admit a morphism between them inducing the identity on A and B. Exer.1 and Yoneda imply that every such morphism is an isomorphism. It need not, however, be unique. More strongly, if an idempotent complete triangulated category admits functorial cones, then the category is semisimple abelian, [Ver96, Chap.II, Prop.1.2.13, p.104] see also [Ste12, Prop.3.1].

Exercise 1. Suppose $A \xrightarrow{f} B \xrightarrow{g} C \to A[1]$ is an exact triangle.

1. Show that $g \circ f = 0$.
2. Show that any morphism $X \xrightarrow{x} B$ such that $g \circ x = 0$ factors through f.
3. Deduce that for any object X, the above exact triangle induces a long exact sequence

$$\cdots \to \hom(X, A) \to \hom(X, B), \to \hom(X, C) \to \hom(X, A[1]) \to \dots$$

of abelian groups.

4. Similarly, deduce that for any object X, the above exact triangle induces a long exact sequence

$$\cdots \to \hom(A[1], X) \to \hom(C, X), \to \hom(B, X) \to \hom(A, X) \to \dots$$

of abelian groups.

Exercise 2. Suppose $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is an exact triangle. Show that the following are equivalent.

1. There exists $B \xrightarrow{r} A$ with $r \circ f = \mathrm{id}_A$.
2. There exists $C \xrightarrow{s} B$ with $g \circ s = \mathrm{id}_C$.
3. $h = 0$.
4. The above exact triangle is isomorphic to the split one

$$A \to A \oplus C \to C \xrightarrow{0} A[1].$$

Definition 18. A functor between triangulated categories is called *exact* if it sends exact triangles to exact triangles and the shift functor to the shift functor.

A *tensor triangulated category* is a triangulated category that is also a symmetric monodial category and such that $- \otimes X$ and $X \otimes -$ is exact for all objects.

A tensor triangulated category is called *rigid* if every object admits a dual.[3]

Example 19. The category of finite dimensional graded vector spaces $D(F)$ is a rigid tensor triangulated category.

Definition 20. A *t-structure* on a triangulated category $\mathcal{T}$ is a full subcategory $\mathcal{T}^{\leq 0}$ such that:

1. For every exact triangle $A \to B \to C \to A[1]$, if $A, C \in \mathcal{T}^{\leq 0}$ then $A[1]$ and B are also in $\mathcal{T}^{\leq 0}$.
2. The inclusion admits a right adjoint

$$\begin{tikzcd} \mathcal{T}^{\leq 0} \arrow[r, phantom, "\subseteq"] & \mathcal{T} \arrow[l, bend left, "(-)^{\leq 0}"] \end{tikzcd}$$

Notation 21. Associated to $\mathcal{T}^{\leq 0}$ we define $\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]$ with corresponding right adjoint $X^{\leq n} = (X[n])^{\leq 0}[-n]$. and

$$\mathcal{T}^{\geq n} = \{Y \mid \hom(X, Y) = 0 \ \forall X \in \mathcal{T}^{\leq n-1}\}.$$

Exercise 3.

[3]A *dual* of an object X is an object $X^\vee$ such that $-\otimes X$ is left adjoint to $X^\vee \otimes -$.

1. Choose an exact triangle of the form $X^{\leq n} \xrightarrow{\eta} X \to \mathrm{Cone}(\eta) \to X^{\leq n}[1]$. Show that $\mathrm{Cone}(\eta) \in \mathcal{T}^{\geq n+1}$. Hint.[4]

2. Show that for any two commutative diagrams

$$\begin{tikzcd} X^{\leq n} \arrow[r, "\eta"] \arrow[d, equal] & X \arrow[r] \arrow[d, equal] & \mathrm{Cone}(\eta) \arrow[r] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & X^{\leq n}[1] \arrow[d, equal] \\ X^{\leq n} \arrow[r, "\eta"] & X \arrow[r] & \mathrm{Cone}(\eta) \arrow[r] & X^{\leq n}[1] \end{tikzcd}$$

we have $f = g$. Hint.[5]

3. Deduce that there exist functors $(-)^{\geq n}$ and exact triangles

$$X^{\leq n} \xrightarrow{\eta} X \xrightarrow{\varepsilon} X^{\geq n+1} \to X^{\leq n}[1]$$

functorial in X.

4. Show that the inclusion $\mathcal{T}^{\geq n} \subseteq \mathcal{T}$ admits a left adjoint

$$\begin{tikzcd} \mathcal{T} \arrow[r, bend left, "(-)^{\geq n}"] \arrow[r, phantom, "\supseteq"] & \mathcal{T}^{\geq n} \end{tikzcd}$$

Hint.[6]

Example 22.

1. The subcategory $D(F)^{\leq 0} \subseteq D(F)$ of graded vector spaces concentrated in degrees ≤ 0 defines a t-structure.

2. The subcategory $D(\mathrm{Ab})^{\leq 0} \subseteq D(\mathrm{Ab})$ of graded abelian groups concentrated in degrees ≤ 0 defines a t-structure.

3. For any set of primes S, the subcategory of those graded abelian groups A such that A is concentrated in degrees ≤ 0 and $A[S^{-1}]$ is concentrated in degrees < 0 defines a t-structure.

Definition 23. The *heart* of a t-structure is

$$\mathcal{T}^{\heartsuit} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}.$$

For each n, the *nth cohomology* of an object X with respect to the t-structure τ is

$$\mathcal{H}^0 X := (X^{\leq 0})^{\geq 0}, \qquad \mathcal{H}^n X := \mathcal{H}^0(X[n]) \in \mathcal{T}^{\heartsuit}.$$

[4] Use Exercise 1.
[5] Use $\hom(\mathcal{T}^{\leq n-1}, \mathcal{T}^{\geq n+1}) = 0$.
[6] Use the ε just constructed.

Remark 24. Every exact triangle $A \to B \to C \to A[1]$ gives rise to a long exact cohomology sequence, [BBD82, Thm.1.3.6],

$$\cdots \to \mathcal{H}^n A \to \mathcal{H}^n B \to \mathcal{H}^n C \to \mathcal{H}^{n+1} A \to \dots .$$

Here to understand "exact" we show that $\mathcal{T}^\heartsuit$ is an abelian category, so exactness makes sense in $\mathcal{T}^\heartsuit$.

Proposition 25 ([BBD82, Thm.1.3.6]). *The heart $\mathcal{T}^\heartsuit$ is an abelian category.*

Sketch of proof. For a morphism $f : X \to Y$ in the heart, complete it to an exact triangle $X \xrightarrow{f} Y \to S \to X[1]$. Set $K := S^{\leq -1}[-1]$ and $C := S^{\geq 0}$. Then $K, C \in \mathcal{T}^\heartsuit$ and we obtain maps $K \to S[-1] \to X$ and $Y \to S \to C$. Then one checks that $K = \ker f$, $C = \operatorname{coker} f$, and the axioms of an abelian category are satisfied. The hardest part is showing $\operatorname{coker}(K{\to}X) \to \ker(Y{\to}C)$ is an isomorphism.

Let $I := \operatorname{Cone}(K \to X)$. Then Lemma 13 gives a diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K[1] \arrow[r, equal] \arrow[d] & K[1] \arrow[d] \\
X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, equal] & S \arrow[r] \arrow[d] & X[1] \arrow[d] \\
I \arrow[r, dotted] \arrow[d] & Y \arrow[r, dotted] \arrow[d] & C \arrow[r, dotted] \arrow[d] & I[1] \arrow[d] \\
K[1] \arrow[r] & 0 \arrow[r] & K[2] \arrow[r, equal] & K[2]
\end{tikzcd}$$

with exact rows and columns. In particular the dotted row is an exact triangle, so, after verifying that $I \in \mathcal{T}^\heartsuit$ we have $I \cong \ker(Y \to C)$. $\square$

Remark 26. Suppose that $\mathcal{T}$ is a tensor triangulated category with a t-structure such that

$$\mathcal{T}^{\leq 0} \otimes \mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 0}, \qquad \text{and} \qquad \mathcal{T}^{\geq 0} \otimes \mathcal{T}^{\geq 0} \subseteq \mathcal{T}^{\geq 0}.$$

If $X \in \mathcal{T}^\heartsuit$ is dualisable, then $X^\vee \in \mathcal{T}^\heartsuit$. Indeed, for any $Y \in \mathcal{T}^{\leq -1}$ we have $\hom(Y, X^\vee) = \hom(Y \otimes X, \mathbb{1}) = 0$ because $Y \otimes X \in \mathcal{T}^{\leq -1}$ and $\mathbb{1} \in \mathcal{T}^\heartsuit$. Symmetrically, $\hom(X^\vee, Y) = \hom(\mathbb{1}, X \otimes Y) = 0$ for $Y \in \mathcal{T}^{\geq 1}$. Hence $X^\vee \in \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$.

Theorem 27 (Beilinson, [Bei10, §1]). *Let $\Phi : DM_{gm}^{op} \to \mathrm{GrVec}_F$ be the realisation functor. Suppose there exists a t-structure μ on DM_{gm}^{op} satisfying the following properties.*

1. *It is nondegenerate:*

$$^{\mu}H^n(M) = 0 \ \forall n \quad \Rightarrow \quad M = 0;$$

2. *The tensor product is t-exact:*

$$A, B \in (DM_{gm}^{op})^{\leq 0} \quad \Rightarrow \quad A \otimes B \in (DM_{gm}^{op})^{\leq 0},$$

$$A, B \in (DM_{gm}^{op})^{\geq 0} \quad \Rightarrow \quad A \otimes B \in (DM_{gm}^{op})^{\geq 0},$$

and

$$\mathbb{1} \in DM_{\mathrm{gm}}^{\heartsuit}.$$

3. *The realisation Φ is t-exact for the standard t-structure on $D(F)$:*

$$\Phi((DM_{gm}^{op})^{\leq 0}) \subseteq D(F)^{\leq 0}, \qquad \Phi((DM_{gm}^{op})^{\geq 0}) \subseteq D(F)^{\geq 0}.$$

Then Conjecture B holds. Consequently, Conjectures A and C hold, and if Conjecture I holds, e.g., if char.$k = 0$, then Conjecture D holds as well.

First we show the following.

Exercise 4. Suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories. Show that the following are equivalent.

1. $\Phi(A) = 0 \Rightarrow A = 0$, for all objects A.
2. $\Phi(f)$ is an isomorphism $\Rightarrow f$ is an isomorphism, for all morphisms f.
3. $\Phi(f) = 0 \Rightarrow f = 0$ for all morphisms f.

Lemma 28 (Faithfulness on the heart, [DM82, Prop. 1.19])**.** *Under the hypotheses of Theorem 27, the functor*

$$\Phi : DM_{gm}^{\heartsuit} \to \mathrm{Vec}_F$$

is faithful. That is, for all objects $DM_{gm}^{\heartsuit}$ we have $\Phi(A) = 0 \Leftrightarrow A = 0$.

Proof. Let $A \in DM_{gm}^{\heartsuit}$ be nonzero. To show that $\Phi(A)$ is nonzero, it suffices to show that

$$\operatorname{coker}(\Phi A \otimes \Phi A^{\vee} \to F) = \Phi(\operatorname{coker}(A \otimes A^{\vee} \to \mathbb{1})) \tag{9}$$

is zero. Since A is nonzero, the evaluation map $\eta : A \otimes A^{\vee} \to \mathbb{1}$ must be nonzero because otherwise $\mathrm{id}_A : A \to A \otimes A^{\vee} \otimes A \to A$ would be zero.

So to show that (9) is zero it suffices to show that the nonzero morphism η is surjective, and for this it suffices to show that all subobjects $I \subseteq \mathbb{1}$ are 0 or $\mathbb{1}$.

Form the following diagram by tensoring the short exact sequence $0 \to I \to \mathbb{1} \to C \to 0$ with $I \subseteq \mathbb{1}$.

$$\begin{tikzcd}
0 \arrow[r] & I \otimes I \arrow[r] \arrow[d] & I \otimes \mathbb{1} \arrow[r, two heads] \arrow[d] \arrow[dr, dashed] & I \otimes C \arrow[r] \arrow[d, shift left] & 0 \\
0 \arrow[r] & I \arrow[r] & \mathbb{1} \arrow[r] & C \arrow[r] \arrow[u, shift left] & 0
\end{tikzcd}$$

Since tensor is exact, both rows are exact and all vertical morphisms are monomorphims. Since the diagonal dashed morphism is zero it follows that $I \otimes C = 0$ so $m : I \otimes I \to I$ is an isomorphism. Consequently, we can form the retraction $\mathbb{1} \to I \otimes I^\vee \xrightarrow{m^{-1}} I \otimes I \otimes I^\vee \to I$ making $I \subseteq \mathbb{1}$ not just a subobject, but a direct summand. However $\mathrm{End}(\mathbb{1}) = \mathbb{Q}$ is a field, so the only direct summands are 0 or $\mathbb{1}$. □

Notation 29. For various reasons, the object $\mathbb{L} \in \mathcal{M}ot_{\mathrm{rat}}$ is denoted $\mathbb{Q}(1)[2]$ in DM_{gm}. That is, $\mathbb{Q}(1)$ is the object such that $\mathbb{Q}(1)[2] = \mathbb{L}$. Since $\mathbb{L}$ is tensor invertible, so is $\mathbb{Q}(1)$ and for $n \in \mathbb{Z}$ one further writes

$$M(n) = M \otimes \mathbb{Q}(1)^{\otimes n}.$$

Remark 30. In the situation of Theorem 27, we have $\mathbb{Q}(j) \in DM_{\mathrm{gm}}^{\heartsuit}$. Indeed, $\Phi({}^{\mu}H^n(\mathbb{Q}(1))) = H^n(\Phi(\mathbb{Q}(1))) = H^n(\Phi(\mathbb{L}[-2])) = 0$ for $n \neq 0$ so $\mathbb{Q}(1) = {}^{\mu}H^0(\mathbb{Q}(1)) \in DM_{\mathrm{gm}}^{\heartsuit}$.

Proof of Theorem 27. First we show Conjecture C. Consider the following claim.

(∗) Suppose that $M \in DM_{\mathrm{gm}}$ has ${}^{\mu}H^n M = 0$ for $n \notin \{d - r, \dots, d + r\}$ with $r > 0$, and $\lambda : M \to M(r)[2r]$ is a morphism such that the composition

$$\sigma : {}^{\mu}H^{d-r}M[r{-}d] \xrightarrow{\eta} M \xrightarrow{\lambda} M(r)[2r] \xrightarrow{\varepsilon} {}^{\mu}H^{d+r}M(r)[r{-}d]$$

is an isomorphism. Then there is a direct sum decomposition

$$M \cong {}^{\mu}H^{d-r}M[r{-}d] \oplus M' \oplus {}^{\mu}H^{d+r}M[-d{-}r] \tag{10}$$

for some M' with

$${}^{\mu}H^n M' = \begin{cases} 0 & n \notin \{d{-}r{+}1, \dots, d{+}r{-}1\} \\ {}^{\mu}H^n M & n \in \{d{-}r{+}1, \dots, d{+}r{-}1\}. \end{cases}$$

Consider the two compositions

$$e : M \xrightarrow{\lambda} M(r)[2r] \xrightarrow{\varepsilon} {}^{\mu}H^{d+r}M(r)[r{-}d] \xrightarrow{\sigma^{-1}} {}^{\mu}H^{d-r}M[r{-}d] \xrightarrow{\eta} M$$

$$\begin{aligned} f : M \xrightarrow{\varepsilon(-r)[-2r]} {}^{\mu}H^{d+r}M[-d{-}r] &\xrightarrow{\sigma^{-1}(-r)[-2r]} {}^{\mu}H^{d-r}M(-r)[-d{-}r] \\ &\xrightarrow{\eta(-r)[-2r]} M(-r)[-2r] \xrightarrow{\lambda(-r)[-2r]} M \end{aligned}$$

These satisfy:

$$e^2 = e; \qquad f^2 = f; \qquad fe = 0.$$

Here the first two identities are just the definition of σ and the latter is because

$$\hom(\underbrace{{}^{\mu}H^{d-r}M[r{-}d]}_{\in DM_{\mathrm{gm}}^{\leq d-r}}, \underbrace{{}^{\mu}H^{d+r}M[-d{-}r]}_{\in DM_{\mathrm{gm}}^{\geq d+r}}) = 0.$$

The idempotent e peels the summand ${}^{\mu}H^{d-r}M[r-d]$ off M and the induced action of f on the complement $\mathrm{im}(1-e)$ peels off the summand ${}^{\mu}H^{d+r}M[-d-r]$.[7] Hence, the decomposition (10).

We apply $(*)$ inductively to $M(X)$ beginning with centre d, radius d, and $\lambda = L^d$. Hard Lefschetz implies that $\Phi(\sigma)$ is an isomorphism so faithfulness on the heart, Lem.28, implies σ is an isomorphism. In this way we obtain a decomposition

$$M(X) \cong \bigoplus_{i=0}^{2d} {}^{\mu}H^i(X)[-i] =: \bigoplus_{i=0}^{2d} h^i(X)$$

with

$$\Phi(h^i(X)) = H^i_{\mathrm{dR}}(X(\mathbb{C}), \mathbb{R}). \tag{11}$$

Since $M(X) \in \mathcal{M}ot_{\mathrm{rat}} \subseteq DM_{\mathrm{gm}}$, so are its direct summands $h^i(X)$, so Conjecture C holds.

Now by construction each $h^i(X)[i]$ lives in the abelian category $DM^{\heartsuit}_{gm}$ so

$${}^{\mu}P^i = \ker\left(h^i(X) \xrightarrow{L^{d-i+1}} h^{2d-i+2}(X)(d-i+1)[2d-2i+2]\right)$$

makes sense as an object of $DM^{\heartsuit}_{gm}[-i]$. Moreover, since Φ is exact, it follows from Eq.(11) that

$$\Phi({}^{\mu}P^i) = P^i(X) \in \mathrm{GrVec}_F \tag{12}$$

and consequently the morphism

$$\bigoplus_{\substack{0\leq r \\ j-d\leq r \\ 0\leq j-2r\leq d}} {}^{\mu}P^{j-2r}(-r)[-2r] \xrightarrow{\oplus L^r} h^j(X) \tag{13}$$

induced by (powers of) L in $DM^{\heartsuit}_{\mathrm{gm}}[-j]$ is sent to an isomorphism under Φ. Since Φ is faithful on $DM^{\heartsuit}_{gm}$ the morphism (13) is an isomorphism in $DM^{\heartsuit}_{gm}$. In particular, we can define

$${}^{\mu}\Lambda : h^j(X) \to h^{j-2}(X)(-1)[-2]$$

in the same way as Definition **??** from the previous lecture. Then we have $\Phi({}^{\mu}\Lambda) = \Lambda$ so Conjecture B holds. □

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