Lecture 7: Standard Conjectures

In Lecture 5 we introduced homological and numerical equivalence and stated Grothendieck's conjecture $D(X)$ that hom = num. In this lecture we see the remaining standard conjectures $A(X), B(X)$, and $C(X)$. We will see $I(X)$ in the next lecture. Grothendieck introduced these conjectures as the algebro-geometric input which should make the proof of the Weil conjectures formal [Kle68, pp.360–361], [Mil12, §7, p.12].

We continue to work with smooth projective varieties over an algebraically closed field k, use $\mathbb{Q}$-coefficients for cycles, and fix a Weil cohomology theory[1]

$$H^* : \mathcal{M}ot_{\mathrm{rat}} \to \mathrm{GrVec}_F \,.$$

In this course we have been using the Betti–de Rham realisation. For Betti $F = \mathbb{Q}$ and for (differential geometric) de Rham, $F = \mathbb{R}$. For a smooth projective variety X of dimension d, write $CH^r_{\mathrm{hom}}(X)$ for cycles modulo homological equivalence. Equivalently, $CH^r_{\mathrm{hom}}(X)$ is identified with the image of the cycle class map $\gamma : CH^r(X) \to H^{2r}(X)$.

1 Lefschetz operators and Conjecture A

A *hyperplane* is any inclusion $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$; $(z_0 : \cdots : z_{n-1}) \mapsto (f_0(z) : \cdots : f_n(z))$ such that the f_i are linear.

Exercise 1. Show that all hyperplanes define the same element of $CH^1(\mathbb{P}^n)$.

Let $i : X \hookrightarrow \mathbb{P}^N$ be smooth projective of dimension d, let $[H] \in CH^1(\mathbb{P}^N)$ be the class of a hyperplane and consider it's image $i^*[H] \in CH^1(X)$. Intersecting with $i^*[H]$ defines morphisms $CH^j(X) \to CH^{j+1}(X)$, and more generally, for Y smooth projective, $CH^j(X \times Y) \to CH^{j+1}(X \times Y)$. These are represented by the following composition.

Definition 1. Define $\ell_X \in CH^{d+1}(X \times X) = \hom_{\mathcal{M}ot_{\mathrm{rat}}}(h(X) \otimes \mathbb{L}, h(X))$ as the composition

$$h(X) \otimes \mathbb{L} \xrightarrow{\mathrm{id} \otimes [H]} h(X) \otimes h(\mathbb{P}^N) \xrightarrow{\mathrm{id} \otimes h(i)} \underbrace{h(X) \otimes h(X)}_{\cong h(X \times X)} \xrightarrow{diag.} h(X).$$

in $\mathcal{M}ot_{\mathrm{rat}}$.

Now choose a Weil cohomology theory $H^* : \mathcal{M}ot_{\mathrm{rat}} \to \mathrm{GrVec}_F$ and an isomorphism $\theta : H^2(\mathbb{L}) \cong F[2]$ so that we obtain an induced identification

$$H^*(X \otimes \mathbb{L}) \cong H^*(X) \otimes H^2(\mathbb{L}) \overset{\theta}{\cong} H^{*+2}(X).$$

[1] Recall from Lecture 6 that this means H^* is symmetric monoïdal $\mathbb{Q}$-linear such that for $X \in \mathrm{SmProj}$ irreducible we have $H^{<0}(X) = 0$, $H^0(X) \cong F$, and $H^i(X) \cong H^{2d-i}(X)^\vee$ for all i (via the algebra structure induced by the diagonal $X \to X \times X$). Here $d = \dim X$.

Definition 2. The *Lefschetz operator* (with respect to the choices i and θ) is the composition

$$L : H^i(X) \xrightarrow{H^*(\ell_i)} H^i(X) \otimes H^2(\mathbb{L}) \overset{\theta}{\cong} H^{i+2}(X).$$

In other words, L is cup product with the cohomology class corresponding to the image $\gamma(i^*[H]) \in H^2(X)$ of $i^*[H] \in CH^1(X)$ under the cycle class map $\gamma : CH^1(X) \to H^2(X)$ determined by θ.

Remark 3. The choice of θ only alters L by a scalar, and standard Weil cohomology theories come equipped with a standard choice of $\theta : H^2(\mathbb{P}^1) = H^2(\mathbb{L}) \cong F[2]$. For example, if $H^* = H^*_{dR}(-, \mathbb{R})$ then the standard choice is integration

$$\int_{\mathbb{P}^1(\mathbb{C})}(-) : H^2_{dR}(\mathbb{P}^1(\mathbb{C})) \to \mathbb{R}$$

over $\mathbb{P}^1(\mathbb{C})$ equipped with the orientation $dx \wedge dy$ coming from the complex coordinate $z = x + iy$. For singular cohomology, the standard choice is the morphism induced by the canonical evaluation pairing

$$H^2_{\mathrm{sing}}(\mathbb{P}^1(\mathbb{C}), \mathbb{Q}) \otimes H_2^{\mathrm{sing}}(\mathbb{P}^1(\mathbb{C}), \mathbb{Q}) \to \mathbb{Q}$$

and the fundamental 2-cycle in $H_2^{\mathrm{sing}}(\mathbb{P}^1(\mathbb{C}), \mathbb{Q})$ coming from a triangularisation of $\mathbb{P}^1(\mathbb{C}) \cong S^2$, again, with each triangle oriented by the orientation coming from the complex coordinate.

On the other hand, the choice of embedding $i : X \to \mathbb{P}^N$ can change i^*H more drastically. For example, let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Then the map $\mathbb{Z} \oplus \mathbb{Z} \cong CH^1(\mathbb{P}^1) \oplus CH^1(\mathbb{P}^1) \to CH^1(\mathbb{P}^1 \times \mathbb{P}^1)$ induced by the two projections $p_1, p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \rightrightarrows \mathbb{P}^1$ is an isomorphism, by the projective bundle formula, [Ful98, Theorem 3.3]. The Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$ has hyperplane class

$$i^*[H] = p_1^*[\infty] + p_2^*[\infty]$$

where $[\infty] \in CH^1(\mathbb{P}^1)$ is the class of the point at infinity. However, if we first embed the first factor by the quadratic Veronese map $\mathbb{P}^1 \to \mathbb{P}^2$, $(x : y) \mapsto (x^2 : xy : y^2)$, and then use the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^1 \subseteq \mathbb{P}^5$, the hyperplane class becomes

$$i^*[H] = 2p_1^*[\infty] + p_2^*[\infty].$$

In fact, every

$$ap_1^*[\infty] + bp_2^*[\infty] \qquad \text{with} \qquad a, b > 0$$

is possible to achieve as a hyperplane section i^*H for some $i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^N$.

Theorem 4 (Hard Lefschetz, [Voi02, Theorem 6.25])**.** *Suppose that $H^* = H^*_{dR}$. Then for $X \in \mathrm{SmProj}$ and $0 \leq i \leq d$, the map*

$$L^{d-i} : H^i_{dR}(X) \xrightarrow{\sim} H^{2d-i}_{dR}(X)$$

is an isomorphism.

Remark 5. Hard Lefschetz for de Rham is proved using the Kähler form $\omega \in A^{1,1}(X)$ obtained by restricting the Fubini–Study form along the embedding $X \subseteq \mathbb{P}^N$, using Hodge theory to identify de Rham cohomology with harmonic forms, proving the Kähler identities for L and its adjoint Λ, and applying the representation theory of $\mathfrak{sl}_2$, [Voi02, §§3.3.2, 5.3.1, 6.1.1–6.2.3].

Hard Lefschetz is known for all standard Weil cohomology theories, but proved using different methods. For ℓ-adic cohomology, it is due to Deligne's proof of the Weil conjectures [Del80, Thm. 4.1.1]. For crystalline cohomology, Katz–Messing deduce the corresponding statement from Deligne's ℓ-adic hard Lefschetz theorem [KM74, Cor. 1.1]. (For Betti over $\mathbb{C}$ one can use the comparison with de Rham above).

Example 6.

1. If C is a curve, we have $H^*(C) = H^0(C) \oplus H^1(C) \oplus H^2(C)$ so Hard Lefschetz only has one nontrivial map $L : H^0(C) \to H^2(C)$. Since $\dim H^0 = \dim H^2 = 1$ (assuming C irreducible), Hard Lefschetz says $L(1) = \gamma(i^*[H])$ is nonzero, hence generates $H^2(C)$.

2. For $\mathbb{P}^n$ we have an isomorphism of graded rings $H^*(\mathbb{P}^n) \cong F[\eta]/\eta^{n+1}$ where η is the class of our hyperplane section so it sits in degree 2. The Hard Lefschetz is just the isomorphisms $L^{n-2i} : F\eta^i \xrightarrow{\sim} F\eta^{n-i}$; $\eta^i \mapsto \eta^{n-i}$.

3. If S is a smooth projective surface, then Hard Lefschetz says that
$$L^2 : H^0(S) \to H^4(S), \qquad \text{and} \qquad L : H^1(S) \to H^3(S)$$
are isomorphisms. Note that since L^2 factors through $H^2(S)$, Hard Lefschetz includes the claim
$$H^2(S) = \operatorname{im}(H^0 \xrightarrow{L} H^2) \oplus \ker(H^2 \xrightarrow{L} H^4).$$

4. If A is an abelian variety of dimension g, then $H^*(A) \cong \bigwedge^* H^1(A)$. For a hyperplane class $h \in H^2(A) \cong \bigwedge^2 H^1(A)$, Hard Lefschetz is the linear algebra statement that wedging with powers of h gives isomorphisms $\bigwedge^i H^1(A) \cong \bigwedge^{2g-i} H^1(A)$ for $i \leq g$. Note that as in the surface case, it also implies decompositions
$$H^i(A) = \bigoplus_{\substack{j \geq 0 \\ m = i-2j}} \operatorname{im}\left(\ker\big(H^m(A) \xrightarrow{L^{g-m+1}} H^{2g-m+2}(A)\big) \xrightarrow{L^j} H^i(A) \right).$$

Conjecture A (Hard Lefschetz for algebraic cycles)**.** For every r with $2r \leq d$, the map
$$L^{d-2r} : CH^r_{\text{hom}}(X) \longrightarrow CH^{d-r}_{\text{hom}}(X)$$
on Chow groups modulo homological equivalence is an isomorphism.

Thus $A(X,h)$ is the assertion that L^{d-2r} is surjective on algebraic classes.

$$\begin{array}{ccc} CH^r_{\mathrm{hom}}(X) & \xrightarrow[\text{surj.?}]{L^{d-2r}} & CH^{d-r}_{\mathrm{hom}}(X) \\ \cap| & & \cap| \\ H^{2r}(X) & \xrightarrow[\cong]{L^{d-2r}} & H^{2d-2r}(X) \end{array}$$

2 $\mathfrak{sl}_2$ representations

Definition 7. An $\mathfrak{sl}_2$-representation is an F-vector space V equipped with three endomorphisms e, f, h satisfying:

$$[h,e] = 2e, \qquad [h,f] = -2f, \qquad [e,f] = h. \tag{1}$$

Here $[x,y] := xy - yx$.

A morphism of representations is a linear map commuting with the three operators. We write $\mathrm{Rep}(\mathfrak{sl}_2)$ for the resulting category of finite-dimensional representations.

Remark 8. The Lie algebra $\mathfrak{sl}_2$ is a basic piece of representation theory. Semisimple groups are built from copies of $\mathbb{G}_m$ and $\mathbb{G}_a$, and the gluing is visible in the SL_2's they generate; see [Spr98, 8.1, 8.2.10, 9.4].

Example 9. The smallest nontrivial representation is

$$e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise 2. For a representation V of $\mathfrak{sl}_2$, the eigenspaces of h are called the *weight spaces*:

$$V(\lambda) = \{v \in V : hv = \lambda v\}.$$

Using the relations (1) show that

$$e\big(V(\lambda)\big) \subseteq V(\lambda{+}2) \qquad \text{and} \qquad V(\lambda) \supseteq f\big(V(\lambda{+}2)\big).$$

Exercise 3. Suppose that $e : V \to V$ is an endomorphism of a vector space with one Jordan block.[2] That is, $e^{d-1} \neq 0$ and $e^d = 0$ where $d = \dim V$. Choose $v \notin \ker e^{d-1}$.

1. Show that there exists a unique $h : V \to V$ such that

$$[h,e] = 2e, \qquad \mathrm{tr}(h) = 0,$$

and having $v, ev, e^2v, \ldots, e^{d-1}v$ as eigenvectors.

[2] For example, $V = H^*(\mathbb{P}^{d-1})$ and $e = L$ from Example 6(2).

2. Continuing with the h from the first part, show that there is a unique $f : V \to V$ such that

$$[h, f] = -2f, \qquad [e, f] = h, \qquad \text{and} \qquad f(v) = 0.$$

3. Deduce that there is a unique $\mathfrak{sl}_2$ representation on V extending e such that:

   (a) the eigenvectors of h are $v, ev, e^2v, \ldots, e^{d-1}v$, and

   (b) $f(v) = 0$.

4. Show that in characteristic zero, the condition $f(v) = 0$ in Part 3 is automatic. That is, there is a unique $\mathfrak{sl}_2$ representation on V extending e such that the $v, ev, e^2v, \ldots, e^{d-1}v$ are eigenvectors of h.

Exercise 4. Suppose $char.F = 0$. Determine the operators of $f, h : F^n \to F^n$ corresponding to the $\mathfrak{sl}_2$ representation with

$$e : (x_1, x_2, x_3, x_4, \ldots, x_n) \mapsto (0, x_1, 2x_2, 3x_3, \ldots, (n-1)x_{n-1})$$

and having standard basis vectors as eigenvectors of h. In particular, describe the λ for which $V(\lambda)$ is nonzero.

Exercise 5. Suppose that $char.F = 0$ and that we have an endomorphism of an F-vector space satisfying:

$$e : V \to V; \qquad e^{n-1} \neq 0, \qquad e^n = 0.$$

Consider the filtration

$$F_{i,0} = \ker e^{n-i}, \qquad F_{i,j} = \ker e^{n-i} \cap \operatorname{im} e^j.$$

Choose a splitting of the $j = 0$ level, that is, choose subspaces $V_i \subseteq F_{i,0}$ such that $V_i \cong \frac{F_{i,0}}{F_{i+1,0}+F_{i,1}}$.

1. Show that $e^jV_i = 0$ when $i + j = n$, that $e : e^jV_i \to e^{j+1}V_i$ are isomorphisms for $i + j + 1 < n$ (and $j \geq 0$), and that

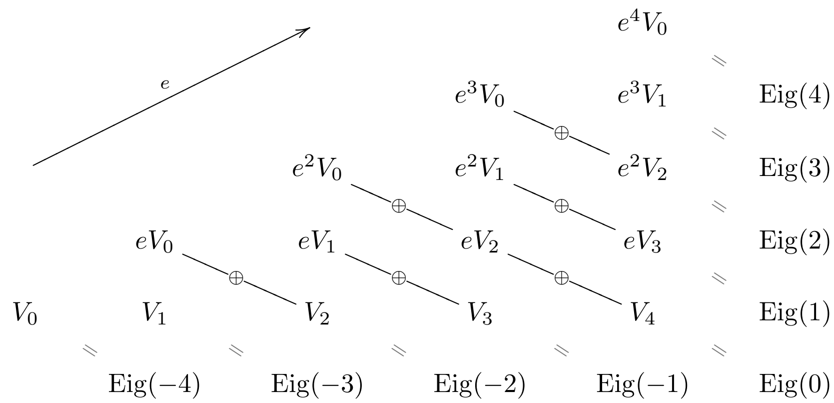
$$V \cong \bigoplus_{i,j} e^jV_i.$$

2. Show that there is a unique $\mathfrak{sl}_2$ action extending e such that the e^jV_i are eigenspaces of h. Hint.[3]

3. Show that in the representation from the previous part, the e^jV_i are precisely the eigenspaces of weight $1-n+i+2j$ in the irreducible summands of dimension $n - i$.

[3] Use Exercise 3.

4. Show that for $v \in e^j V_i$ we have $fev = (j+1)(n-i-j-1)v$.



Theorem 10 ([Hum72, §7.2])**.** *Suppose F has characteristic zero. Then the category* $\mathrm{Rep}(\mathfrak{sl}_2)$ *is semisimple abelian: every representation is a direct sum of irreducible representations. Moreover, there is a unique irreducible representation of each dimension, namely the ones from Exercise 4.*

Exercise 6. Let V be a finite dimensional representation of $\mathfrak{sl}_2$ with operators e, f, h over a field F of characteristic zero.

1. Show that projection $V \to V(m) \to V$ to the mth eigenspace of h can be written as a polynomial in h, and a (noncommutative) polynomial in e and f. Hint.[4]

2. Let $V = \bigoplus_{d \geq 0} V_d^{\oplus i_d}$ be the decomposition into irreducibles V_d of dimension d. Show that projection

$$V \to V_d(m)^{\oplus i_d} \to V$$

to the mth eigenspace of h in the irreducibles of dimension d can be written as (noncommutative) polynomial in e and f. Hint.[5] Hint.[6]

3. Let V be a finite dimensional $\mathfrak{sl}_2$ representation and $i \geq 0$. Show that e^i restricts to an isomorphism of weight spaces

$$e^i : \mathrm{Eig}(-i) \xrightarrow{\sim} \mathrm{Eig}(i)$$

and there is a (noncommutative) polynomial $p(e, f)$ in e and f whose restriction to $\mathrm{Eig}(i)$ is the inverse $\mathrm{Eig}(i) \xrightarrow{\sim} \mathrm{Eig}(-i)$.

[4] First show that there is a polynomial in h which kills $V(m)$ and scales each $V(m')$ for $m \neq m'$ by some nonzero scaler $\lambda_{m'}$.

[5] Start with the largest d with $i_d \neq 0$ and then work by induction.

[6] See Remark 2.

3 The dual Lefschetz operator and Conjecture B

Recall that for a surface S we had $H^2(S) = \operatorname{im}(H^0 \xrightarrow{L} H^2) \oplus \ker(H^2 \xrightarrow{L} H^4))$ due to the fact that the isomorphism $L^2 : H^0 \to H^4$ factors through H^2.

Definition 11. For $i \leq d$, the *primitive part* is

$$P^i(X) := \ker\big(H^i(X) \xrightarrow{L^{d-i+1}} H^{2d-i+2}(X)\big).$$

Remark 12. Hard Lefschetz gives the *Lefschetz decomposition*

$$H^i(X) = \bigoplus_{j \geq 0} L^j P^{i-2j}(X),$$

where the sum is over j such that $i - 2j \geq 0$ and $i - 2j \leq d$. In other words, our choice of Lefschetz operator induces a decomposition of cohomology:

$$\begin{array}{cccccc}
 & & & & L^4P^0 & \\
 & & & L^3P^0 & L^3P^1 & H^8 \\
 & & L^2P^0 & L^2P^1 & L^2P^2 & H^7 \\
 & LP^0 & LP^1 & LP^2 & LP^3 & H^6 \\
P^0 & P^1 & P^2 & P^3 & P^4 & H^5 \\
 & H^0 & H^1 & H^2 & H^3 & H^4
\end{array} \tag{2}$$

Notice how diagram (2) resembles the situation in Exercise 5.
In particular, we can extend L to a unique $\mathfrak{sl}_2$ representation with

$$\operatorname{Eig}(\lambda) = H^{d+\lambda}(X).$$

Definition 13 (The operator Λ)**.** The operator

$$\Lambda : H^i(X) \longrightarrow H^{i-2}(X)$$

is defined on each Lefschetz summand by

$$\Lambda(L^j\alpha) = j(\lambda - j + 1)L^{j-1}\alpha \qquad (\alpha \in P^{d-\lambda}(X)),$$

with the convention that $\Lambda(\alpha) = 0$ on primitive classes.

Exercise 7. Show that Λ corresponds to an $\mathfrak{sl}_2$ representation with $e = L$, $f = \Lambda$, $h = [e, f]$.

Conjecture B (standard conjecture of Lefschetz type)**.** There exists a cycle class $\alpha \in CH^{d-1}(X \times X)$ whose action on cohomology is Λ. In other words, Λ is in the image of

$$\hom_{\mathcal{M}ot_{\mathrm{rat}}}(h(X), h(X) \otimes \mathbb{L}) \to \hom_{\mathrm{GrVec}_F}(H^{*+2}(X), H^*(X)).$$

Proposition 14. *$B(X)$ implies $A(X)$.*

Proof. This follows from Ex.6(3). □

Exercise 8. Show that Conjecture B implies that there is a decomposition

$$h(X) \cong \bigoplus_{i,j} P^{i,j}(X)$$

in $\mathcal{M}ot_{\mathrm{hom}}$ with $H^*(P^{i,j}(X)) \cong L^i P^j$ via the canonical projectors. Hint.[7]

4 Künneth projectors and Conjecture C

Recall that the *correspondence action* of $\alpha \in H^{2d}(X \times X)$ is

$$H^*(X) \xrightarrow{pr_1^*} H^*(X \times X) \xrightarrow{-\cup\alpha} H^{*+2d}(X \times X) \xrightarrow{pr_{2*}} H^*(X).$$

The Künneth formula gives

$$H^{2d}(X \times X) = \bigoplus_{i+j=2d} H^i(X) \otimes H^j(X).$$

Under the correspondence action, the diagonal class $\gamma(\Delta) \in H^{2d}(X \times X)$ acts as the identity on $H^*(X)$. Its Künneth decomposition[8] $\gamma(\Delta) = \sum_{i+j=2d} \pi_i$ therefore gives projectors

$$\pi_i : H^*(X) \to H^i(X) \to H^*(X)$$

for $0 \leq i \leq 2d$.

Definition 15. The class $\pi_i \in H^{2d}(X \times X)$ is the *ith Künneth projector*. It is characterized by acting as the identity on $H^i(X)$ and as zero on $H^j(X)$ for $j \neq i$ via the correspondence action.

Conjecture C (standard conjecture of Künneth type)**.** Each Künneth projector π_i is algebraic: it lies in the image of

$$\gamma : CH^d(X \times X) \longrightarrow H^{2d}(X \times X).$$

[7] Use Exercise 6.

[8] Here $\pi_i \in H^{2d-i}(X) \otimes H^i(X)$.

Exercise 9. Show that another equivalent formulation of Conjecture C is: there exists a direct sum decomposition

$$h(X) = h^0(X) \oplus \cdots \oplus h^{2d}(X)$$

in $\mathcal{M}ot_{\mathrm{hom}}$ such that $H^*(h^i(X)) \cong H^i(X)$ via the canonical projectors.

Proposition 16. *$B(X) \Rightarrow C(X)$.*

Proof. This follows from Exercise 6. □

5 Numerical equivalence and Conjecture D

We restate Conjecture D here for completeness. Recall from Lecture 6 that homological equivalence is finer than numerical equivalence:

$$\mathcal{Z}^r_{\mathrm{hom}}(X) \subseteq \mathcal{Z}^r_{\mathrm{num}}(X).$$

Conjecture D. For every r and X, homological and numerical equivalence agree:

$$\mathcal{Z}^r_{\mathrm{hom}}(X) = \mathcal{Z}^r_{\mathrm{num}}(X).$$

Equivalently, the functor

$$\mathcal{M}ot_{\mathrm{hom}} \to \mathcal{M}ot_{\mathrm{num}}$$

is an equivalence of categories.

Proposition 17.

$$D(X) \Rightarrow A(X, h).$$

Proof. In the following diagram,

$$\begin{tikzcd}
H^{2r}(X) \arrow[r, "L^{d-2r}", "\cong"'] & H^{2d-2r}(X) \\
CH^r_{\mathrm{hom}}(X) \arrow[u, phantom, "\cup|"] \arrow[r, "\subseteq"] \arrow[d, "\cong"] & CH^{d-r}_{\mathrm{hom}}(X) \arrow[u, phantom, "\cup|"] \arrow[d, "\cong"] \\
CH^r_{\mathrm{num}}(X) \arrow[r] & CH^{d-r}_{\mathrm{num}}(X)
\end{tikzcd}$$

we have

$$\dim_{\mathbb{Q}} CH^r_{\mathrm{num}}(X) = \dim_{\mathbb{Q}} CH^{d-r}_{\mathrm{num}}(X)$$

because the numerical intersection pairing is non-degenerate. □

The proof that D implies B is essentially the following proof of Smirnov. The proof of Thm.18 is short and elementary. It is recommended to have a look at the original paper.

Theorem 18 (Smirnov, [Smi97, Thm.1, pp.204–205])**.** *Let R be a finite dimensional associative algebra containing a field of characteristic zero. Suppose there are $e, f, h \in R$ forming an $\mathfrak{sl}_2$-subalgebra, that is, satisfying*

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

In particular, e, f, h define an $\mathfrak{sl}_2$-representation via

$$R \to \mathrm{End}(R); \qquad x \mapsto [x, -].$$

Suppose that the multiplication of R is compatible with the decomposition into weight spaces

$$R(i)R(j) \subseteq R(i + j).$$

If $A \subseteq R$ is a graded semisimple subalgebra containing 1 and e, then $f, h \in A$.

Remark 19.

1. The relations $[h, e] = 2e$ resp. $[h, f] = -2f$, $[h, h] = 0$ say precisely that $e \in R(2)$, $f \in R(-2)$, $h \in R(0)$.
2. We also have the left multiplication action $L : R \to \mathrm{End}(R)$; $r \mapsto (x \mapsto rx)$ giving a second additional structure of $\mathfrak{sl}_2$-representation on R. By Ex.6(1), the weight space projectors for the L-representation are induced by polynomials $i_\lambda := p_\lambda(h)$ in h. Then $hi_\lambda x = \lambda i_\lambda x$, $1 = \sum_\lambda i_\lambda$, and, since $i_\mu = p_\mu(h)$ commutes with h, we also have $i_\mu h = \mu i_\mu$, hence $x i_\mu h = \mu x i_\mu$. Consequently, $[h, i_\lambda x i_\mu] = (\lambda - \mu) i_\lambda x i_\mu$ and $R(j) = \bigoplus_{j=\lambda-\mu} i_\lambda R i_\mu$.

Proposition 20. *If $D(X \times X)$ holds, then $B(X)$ holds.*

Proof. Let

$$R := \mathrm{End}_{\vec{F}(H^*(X))}, \qquad A := \mathrm{im}\Big(CH^*(X \times X) \to R\Big) = CH^*_{\mathrm{hom}}(X \times X),$$

where $A \to R$ sends a cycle α to the correspondance action $pr_{2*}(\gamma(\alpha) \cup pr_1^*(-))$. The algebra A is graded by cohomological degree. By $D(X \times X)$ and the semisimplicity of numerical motives, A is semisimple. Since $1, L \in A$, Theorem 18 implies that $\Lambda \in A$. $\square$

Remark 21. To summarise, at this point we have:

$$\begin{tikzcd}
L \text{ is iso.} & \Lambda \text{ is alg.} & \pi_i \text{ are alg.} & \text{hom=num} \\
(A) & (B) \arrow[l, Rightarrow, "\text{Prop.14}"'] \arrow[r, Rightarrow, "\text{Prop.16}"'] & (C) & (D) \arrow[ll, Rightarrow, bend right, "\text{Prop.20}"'] \arrow[lll, Rightarrow, bend left, "\text{Prop.17}"]
\end{tikzcd}$$

or $(D) \Rightarrow (B) \Rightarrow (A) \wedge (C)$.

References

[Del80] Pierre Deligne. La conjecture de Weil, II. *Publications Mathématiques de l'IHÉS*, 52:137–252, 1980.

[Ful98] William Fulton. *Intersection Theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 2 edition, 1998.

[Hum72] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*, volume 9 of *Graduate Texts in Mathematics*. Springer, 1972.

[Kle68] Steven L. Kleiman. Algebraic cycles and the weil conjectures. In *Dix Exposés sur la Cohomologie des Schémas*. North-Holland, 1968.

[KM74] Nicholas M. Katz and William Messing. Some consequences of the Riemann hypothesis for varieties over finite fields. *Inventiones Mathematicae*, 23:73–77, 1974.

[Mil12] James S. Milne. Motives: Grothendieck's dream, 2012. `https://www.jmilne.org/math/xnotes/MOT.pdf`.

[Smi97] Oleg N. Smirnov. Graded associative algebras and Grothendieck standard conjectures. *Inventiones Mathematicae*, 128(1):201–206, 1997.

[Spr98] Tonny A. Springer. *Linear Algebraic Groups*, volume 9 of *Progress in Mathematics*. Birkhäuser, 2 edition, 1998.

[Voi02] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2002.