Lecture 6: Numerical Equivalence

In Lecture 5 we constructed the category of pure motives and the functors

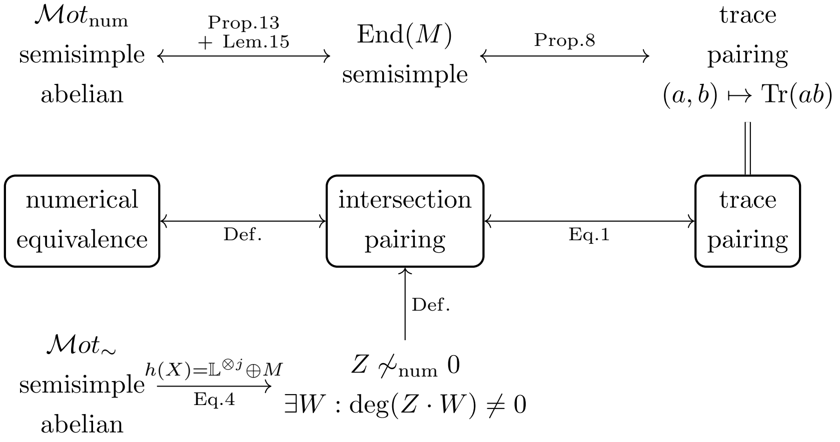
$$\mathcal{M}ot_{\mathrm{rat}} \to \mathcal{M}ot_{\mathrm{hom}} \to \mathcal{M}ot_{\mathrm{num}}.$$

In this lecture we prove Jannsen's theorem [Jan92]: $\mathcal{M}ot_{\mathrm{num}}$ is semisimple abelian, Thm.17, and no other adequate equivalence relation has this property, Thm.16.

The rough idea is as follows. Prop.13 and Lem.15 connect semisimplicity of objects with semisimplicity of endomorphism algebras. For finite-dimensional algebras, semisimplicity can be detected by existence of a nondegenerate "trace" pairing, Prop.8. The Lefschetz trace formula,

$$\begin{aligned} \deg(f \cdot {}^t g) &= \mathrm{Tr}^{\mathcal{M}ot_{\mathrm{hom}}}(g \circ f) \\ &= \mathrm{Tr}^{\mathrm{GrVec}}(H^*(g \circ f)) = \sum_i (-1)^i \, \mathrm{Tr}\big(H^i(g \circ f)\big) \end{aligned} \tag{1}$$

from Lecture 5 connects this trace pairing to the intersection pairing defining numerical equivalence.



The outline of the lecture is as follows. In Section 1 we recall the Jacobson radical, Prop.3, Wedderburn–Artin, Theorem 7, and a trace criterion for semisimplicity, Prop.8. In Section 2 we relate semisimplicity of endomorphism algebras to semisimplicity and abelianness of additive categories. In Section 3 we prove Jannsen's theorem and its converse characterisation of numerical equivalence.

We work throughout with smooth projective varieties over an algebraically closed field k, and use $\mathbb{Q}$-coefficients for cycles.

1 Jacobson radicals and semisimple rings

Throughout this section A denotes an associative ring with unit.

Example 1. The ring $U_2(F) \subseteq Mat_2(F)$ of upper triangular matrices with entries in a field F is an associative ring with unit. There is surjective ring homomorphism

$$U_2(F) \twoheadrightarrow F \times F$$

with nilpotent kernel $J \subseteq U_2(F)$.

In the commutative setting, we would be able to lift the product decomposition from $F \times F$ to the larger ring. Example 1 shows that this breaks in the noncommutative setting. In this section we study when quotienting by a nilpotent ideal results in a nice product decomposition, Thm.7. This generalises to a procedure which makes all objects in a nice additive category become products of simple objects after quotienting by the categorical radical, Appendix A.

Example 2.

1. If K is a field, then the ring of matrices $M_n(K)$ is an associative ring with unit. More generally, if D is a division ring,[1] then $M_n(D)$ is an associative ring with unit.

2. If A is an associative ring with unit and $I \subseteq A$ is a two sided ideal then A/I is again an associative ring with unit.

3. If A_1, A_2 are associative rings with unit, then so is $A_1 \times A_2$.

4. If $M \in \mathcal{M}ot_{\mathrm{rat}}$ is any object and $H^* : \mathcal{M}ot_{\mathrm{rat}} \to \mathrm{GrVec}_F$ any Weil cohomology theory, then $\mathrm{End}_{\mathcal{M}ot_{\mathrm{hom}}}(M) \subseteq \mathrm{End}(H^*(M))$ is an associative ring with unit which is finite dimensional over F.

5. We will see below that finite dimensionality in the previous item means there exists a two sided nilpotent ideal $J \subseteq A := \mathrm{End}_{\mathcal{M}ot_{\mathrm{hom}}}(M)$ and an isomorphism

$$A/J \cong \prod_{i=1}^{n} M_{n_i}(D_i)$$

for some finite dimension division algebras D_i over F. Note the finitely many J^i/J^{i+1} are A/J-modules.

6. (Frobenius) The finite-dimensional division algebras over $\mathbb{R}$ are $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$, where $\mathbb{H}$ denotes Hamilton's quaternions [Lam91, Frobenius' Theorem A3.12] .

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[1] A *division ring* is an associative ring with unit such that every nonzero element is invertible.

7. (Cyclic algebras) Let L/K be a cyclic Galois extension of fields of degree n, let σ be a generator of $\mathrm{Gal}(L/K) \cong \mathbb{Z}/n$, and let $a \in K^*$. Consider the associative algebra

   $$D_{(L/K,\sigma,a)} = L \oplus Lu \oplus \cdots \oplus Lu^{n-1},$$

   with multiplication determined by

   $$u^n = a, \qquad ux = x^\sigma u \quad \text{for } x \in L.$$

   This $D_{(L/K,\sigma,a)}$ is a division algebra precisely when the class of a in

   $$K^*/N_{L/K}(L^*)$$

   has order n. Here $N_{L/K}(L^*) = \{bb^\sigma b^{\sigma^2} \dots b^{\sigma^{n-1}} \mid b \in L^*\}$. Every division algebra over a local field or global field F, such as $F = \mathbb{Q}$, is of the form $D_{(L/K,\sigma,a)}$ for some finite extension K/F, some cyclic extension L/K, some generator σ of $\mathrm{Gal}(L/K)$, and some $a \in K^*$ [Pie82, p.276; Thm.17.10, Cor.17.10a; Thm.18.6] .

Note that every simple[2] left A-module M is isomorphic to $A/\mathfrak{m}$ for some maximal left ideal[3] $\mathfrak{m}$ and conversely, every such $A/\mathfrak{m}$ is simple.

Proposition 3 ([Lam91, D.1–D.5])**.** *The following subsets of A all coincide.*

1. *The intersection of all maximal left (resp. right) ideals.*
2. *The set of elements that annihilate every simple left (resp. right) A-module.*
3. $\{a \in A \mid 1 + aA \subseteq A^*\}$ *resp.* $\{a \in A \mid 1 + Aa \subseteq A^*\}$.
4. *The largest two-sided ideal $J \subseteq A$ such that $1 + J \subseteq A^*$.*

Exercise 1. Let J_n denote the set described in Item n. of Proposition 3.

(a) Suppose I and I' are two sided ideals satisfying $1 + I, 1 + I' \subseteq A^*$. Show that $1 + I + I' \subseteq A^*$. Deduce that J_4 is well defined, that is, there does actually exist a unique largest J_4. Hint.[4]

(b) Show that $J_1 = J_2$.

(c) Show that $J_2 \subseteq J_3$. Hint.[5] Hint.[6]

(d) Show that $J_3 \subseteq J_1$. Hint.[7]

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[2] A left module is *simple* if it has no proper nonzero submodules.

[3] For any nonzero $m \in M$ the map $A \to M; a \mapsto am$ is surjective with kernel $\mathfrak{m} = Ann(m)$ a maximal left ideal.

[4] $1 + a + b = (1 + a)(1 + (1 + a)^{-1}b)$ if $(1 + a)^{-1}$ exists.

[5] Suppose $a \in J_2, a \notin J_3$, and $1 + ab \in \mathfrak{m}$ for some $b \in A$ and some maximal left ideal $\mathfrak{m}$

[6] With $\mathfrak{m}$ from the previous hint, show that $1 \in \mathfrak{m}$.

[7] Suppose $Aa + \mathfrak{m} = A$ for some $a \in J_3$ and some maximal left ideal $\mathfrak{m}$.

(e) Show that $J_4 \subseteq J_1$. Hint.[8]

(f) Show that $J_3 \subseteq J_4$. Hint.[9]

(g) Deduce that $J_1 = J_2 = J_3 = J_4$.

Definition 4 ([Lam91, §4, p.53])**.** *The* Jacobson radical $J(A)$ *is the set described by any of the equivalent conditions in Proposition 3.*

Example 5.

(i) If $A = M_n(D)$ is a matrix ring over a division ring D, then $J(A) = 0$.

(ii) If A is a local ring, meaning that it has a unique maximal left ideal $\mathfrak{m}$ (equivalently, a unique maximal right ideal $\mathfrak{m}$), then $J(A) = \mathfrak{m}$.

Recall that a ring is *left Artinian* if every descending chain of left ideals stabilizes. Any finite-dimensional algebra over a field is automatically left Artinian.

Proposition 6 ([Lam91, D.12])**.** *Suppose that A is left Artinian. Then $J(A)$ is a nilpotent two sided (resp. left, right) ideal, and contains every other nilpotent two sided (resp. left, right) ideal.*

Theorem 7 (Wedderburn–Artin; [Lam91, C.3–C.5,D.14])**.** *A left Artinian ring A satisfies $J(A) = 0$ if and only if*

$$A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

for some division rings $D_1, \ldots, D_r$ and positive integers $n_1, \ldots, n_r$. Such a ring is called semisimple.

The criterion we will actually use to show $J(A) = 0$ is the following.

Proposition 8 (Trace criterion)**.** *Let A be a finite-dimensional F-algebra, and let $\tau : A \to F$ be an F-linear map. Suppose that:*

1. *$\tau(f) = 0$ whenever $f \in A$ is nilpotent.*
2. *The associated bilinear form*

$$\begin{aligned} A \times A &\to F \\ (f, g) &\to \tau(fg) \end{aligned}$$

is non-degenerate.

Then $J(A) = 0$. That is, A is semisimple.

Proof. Let $f \in J(A)$. By Proposition 6, f is nilpotent. Since $J(A)$ is a two-sided ideal, $fg \in J(A)$ for every $g \in A$, so fg is also nilpotent. By hypothesis, $\tau(fg) = 0$ for all $g \in A$. Non-degeneracy forces $f = 0$. $\square$

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[8] Suppose $J_4 + \mathfrak{m} = A$ for some maximal ideal $\mathfrak{m}$, and show that $\mathfrak{m}$ contains a unit.

[9] Use $J_1 = J_2 = J_3$.

Exercise 2. Show that if M is an $n \times n$ nilpotent matrix over a field, then $\mathrm{tr}(M) = 0$. Give an example of a matrix with trace zero that is not nilpotent.

Exercise 3. Let A be a finite-dimensional semisimple algebra over a field and let $p \in A$ be an idempotent. Show that pAp is semisimple. Hint.[10]

Exercise 4. Let A be a finite-dimensional semisimple algebra over a field and let $I \trianglelefteq A$ be a two-sided ideal. Show that A/I is semisimple. Hint.[11]

Remark 9. In practice, τ will be the categorical trace in $\mathcal{M}ot_{\mathrm{num}}$. The non-degeneracy of the associated form will follow from the non-degeneracy of the numerical intersection pairing, Thm.**??** in Lecture 5.

2 Semisimple abelian categories

Recall that an endomorphism e is *idempotent* if $e^2 = e$, and an additive category is *idempotent complete* if for every idempotent $e : M \to M$ we have $M \cong M_0 \oplus M_1$ for some M_0, M_1 and e acts as $\left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ via $M \cong M_0 \oplus M_1$.

Convention 10. Through out this section we work with a field F and F-linear additive idempotent complete categories $\mathcal{C}$ such that all hom vector spaces are finite dimensional. We will call such $\mathcal{C}$ *nice*.

Definition 11. *Let $\mathcal{C}$ be an additive category. An object $M \in \mathcal{C}$ is* simple *if $M \neq 0$ and M has no non-trivial direct summands. An object is* semisimple *if it is a direct sum of simple objects. A category $\mathcal{C}$ is* semisimple *if every object is semisimple.*

Exercise 5. Suppose that $\mathcal{C}$ is a nice category, Conv.10. Show that if $\mathrm{End}(M)$ is a division algebra, then M is simple.

Example 12.

(i) The category Vec_F of finite-dimensional vector spaces over a field F is semisimple: every object is a direct sum of copies of the simple object F.

(ii) For a finite group G and a field F of characteristic zero, the category $\mathrm{Rep}_F(G)$ of finite-dimensional representations is semisimple (Maschke's Theorem, [Lam91, §6, F.1; §8]).

(iii) More generally, for a reductive algebraic group G over a field F of characteristic zero, the category $\mathrm{Rep}_F(G)$ of finite-dimensional algebraic representations is semisimple (Weyl's Complete Reducibility Theorem, [Jan87, II.2.9, Rem.; II.5.6, Rem.]).

(iv) In any $\mathcal{C}$ as above, given a collection $\{S_i\}$ of simple objects, the full subcategory of objects isomorphic to finite direct sums $\bigoplus_i S_i^{\oplus n_i}$ is semisimple by construction.

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[10] Use Wedderburn–Artin to reduce to the case $A = M_n(D)$.

[11] Use Wedderburn–Artin.

(v) We will see below that the category $\mathcal{M}ot_{\mathrm{num}}$ is semisimple abelian, Thm.17.

Proposition 13 ([Jan92, p.448])**.** *Let $\mathcal{C}$ be a nice category, Conv.10.*

If $\mathrm{End}_{\mathcal{C}}(M)$ is a semisimple F-algebra, then M is semisimple. If moreover, M is simple then $\mathrm{End}_{\mathcal{C}}(M)$ is a division algebra.

Proof. By Wedderburn–Artin, Thm.7, we have

$$\mathrm{End}(M) \cong \prod_{i=1}^{r} M_{n_i}(D_i)$$

for division algebras D_i. In each factor $M_{n_i}(D_i)$, the diagonal matrix units $e_{11}^{(i)}, \dots, e_{n_i n_i}^{(i)}$ are primitive orthogonal idempotents[12] summing to the identity of that factor. Altogether these give a decomposition

$$\mathrm{id} = e_1 + \cdots + e_n$$

(where $n = \sum n_i$) into primitive orthogonal idempotents. Since $\mathcal{C}$ is idempotent complete, we get a corresponding decomposition of the object

$$M \cong \bigoplus_{k=1}^{n} M_k.$$

In $M_{n_i}(D_i)$, the element e_k is the diagonal matrix $e_k = \mathrm{diag}(0, \dots, 0, 1, 0, \dots, 0)$ for some position j, and $e_k M_{n_i}(D_i) e_k$ consists of matrices with a single nonzero entry in position (j, j), so

$$\mathrm{End}_{\mathcal{C}}(M_k) \overset{Ex.6}{=} e_k \,\mathrm{End}_{\mathcal{C}}(M) e_k \cong D_{i(k)}$$

is a division algebra, hence each M_k is simple, Ex.5. ☐

Exercise 6. Suppose $\mathcal{C}$ is an idempotent complete additive category and $e \in \mathrm{End}_{\mathcal{C}}(M)$ an idempotent with corresponding decomposition

$$M = M_0 \oplus M_1.$$

Show that $\mathrm{End}_{\mathcal{C}}(M_0) = e \,\mathrm{End}_{\mathcal{C}}(M) e$.

Lemma 14. *Suppose that $M, N \in \mathcal{C}$ are objects in a nice category, M is simple, $\mathrm{End}(M \oplus N)$ is semisimple, and $f : M \to N$ is nonzero. Then f is the inclusion of a direct summand. That is,*

$$M \oplus M' \cong N$$

for some M'.

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[12] Recall that two idempotents e, f are *orthogonal* if $ef = 0 = fe$. An idempotent e is *primitive* if for every decomposition $e = e_1 + e_2$ into orthogonal idempotents, $e_1 = 0$ or $e_2 = 0$.

Proof. We first show:

($*$) There exists $g : N \to M$ with $gf \neq 0$.

Set $A = \operatorname{End}(M \oplus N)$ and $x = \left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) : M \oplus N \to M \oplus N$ and consider the two sided ideal

$$J = AxA \subseteq A.$$

Since A is semisimple, the Jacobson radical vanishes, Thm.7, so every nonzero two sided ideal satisfies $J^2 \neq 0$, Prop.6. Since $0 \neq x \in J$ we have $J^2 \neq 0$. Since J^2 is linearly generated by $\{axbcxd : a, b, c, d \in A\}$, this means there exists some bc with $0 \neq xbcx = \left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} (bc)_{11} & (bc)_{12} \\ (bc)_{21} & (bc)_{22} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ f(bc)_{12}f & 0 \end{smallmatrix}\right)$. Hence, $g = (bc)_{12} : N \to M$ with $gf \neq 0$.

To finish the proof, since $\mathcal{C}$ is idempotent complete, it suffices to find an idempotent $e : N \to N$ such that $M \xrightarrow{f} N \xrightarrow{e} N$ is f. Let $p \in \operatorname{End}(M \oplus N)$ be the idempotent cutting out the summand M. By Ex.6,

$$\operatorname{End}(M) \cong p \operatorname{End}(M \oplus N) p.$$

Hence $\operatorname{End}(M)$ is semisimple, Ex.3, so $\operatorname{End}(M)$ is a division algebra, Prop.13. Thus $gf \neq 0$ is invertible, and $e = f(gf)^{-1}g$ satisfies $e^2 = e$ and $ef = f$. ☐

Lemma 15 ([Jan92, Lem.2]). *Let $\mathcal{C}$ be a nice category, Conv.10. Suppose that for every object M the F-algebra $\operatorname{End}_{\mathcal{C}}(M)$ is semisimple. Then $\mathcal{C}$ is abelian. In fact,*

$$\mathcal{C} \simeq \bigoplus_{[M]} \operatorname{Vec}_{D_M}, \tag{2}$$

where $[M]$ runs over the isomorphism classes of simple objects and Vec_{D_M} denotes finite-dimensional right vector spaces over the division algebra $D_M = \operatorname{End}(M)$.

Proof. Every object of $\mathcal{C}$ is a finite direct sum of simple objects, Prop.13. So it suffices to prove that for simple M, N we have

$$\hom(M, N) \cong \begin{cases} \operatorname{End}(M) & M \cong N \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Fix simple objects M and N. If $\hom(M, N)$ is nonzero, then by Lemma 14 we have $N \cong M \oplus M'$. But N is simple so $M' = 0$ and $N \cong M$. ☐

3 Jannsen's theorem

Theorem 16 (Jannsen [Jan92, Thm.1(a) ⇒ (c)]). *Let $\sim$ be an adequate equivalence relation on algebraic cycles. If $\mathcal{M}ot_{\sim}$ is semisimple abelian, then $\sim$ is numerical equivalence.*

Proof. Since num is the coarsest nontrivial equivalence relation, Lec.5,Prop.**??**, we get surjections

$$CH^r_\sim(X) \twoheadrightarrow CH^r_{\mathrm{num}}(X).$$

We want to show these are injective. Let $\alpha \in CH^r_\sim(X)$ be in the kernel (so $\deg(\alpha\cdot\beta) = 0$ for all $\beta \in CH^{d-r}_\sim(X)$) and consider the corresponding morphism

$$f : \mathbb{L}^{\otimes r} \to h(X)$$

in $\mathcal{M}ot_\sim$. The object $\mathbb{L}^{\otimes r}$ is simple, since $\mathrm{End}(\mathbb{L}^{\otimes r}) \cong \mathrm{End}(\mathbf{1}) \cong \mathbb{Q}$. In a semisimple abelian category every non-zero morphism from a simple object is the inclusion of a direct summand, Lem.14. Hence, if $\alpha \neq 0$,

$$h(X) = \mathbb{L}^{\otimes r} \oplus M, \qquad \exists M \tag{4}$$

and there exists

$$g : h(X) \to \mathbb{L}^{\otimes r}$$

such that $g \circ f = \mathrm{id}_{\mathbb{L}^{\otimes r}}$. Translating, g and f correspond to $\beta \in CH^{d-r}_\sim(X)$ and $\alpha \in CH^r_\sim(X)$ such that

$$\deg(\alpha \cdot \beta) = 1 \in \mathbb{Q}.$$

This contradicts $\alpha \sim_{\mathrm{num}} 0$, so we must have had $\alpha = 0 \in CH^r_\sim(X)$ all along. □

Theorem 17 (Jannsen [Jan92, Thm.1(c) ⇒ (a)]). *The category $\mathcal{M}ot_{\mathrm{num}}$ is semisimple abelian.*

Proof. Every nice category, Conv.10, with semisimple endomorphism algebras is semisimple abelian, Lem.15, Prop.13. Our category $\mathcal{M}ot_{\mathrm{num}}$ is nice (it is idempotent complete, $\mathbb{Q}$-linear, additive, with finite dimensional hom sets) so it is enough to show that

$$\mathrm{End}_{\mathcal{M}ot_{\mathrm{num}}}(M)$$

is a semisimple $\mathbb{Q}$-algebra for every motive $M = (X, p, m)$. Since

$$\mathrm{End}((X,p,m)) = p\,\mathrm{End}((X,\mathrm{id},m))p$$

it suffices to consider motives of the form $h(X) \otimes \mathbb{L}^{-m}$, Ex.3. Furthermore, the functor $M \mapsto M \otimes \mathbb{L}^{\otimes m}$ is an equivalence, so induces isomorphisms

$$\mathrm{End}_{\mathcal{M}ot_{\mathrm{num}}}(h(X) \otimes \mathbb{L}^{-m}) \cong \mathrm{End}_{\mathcal{M}ot_{\mathrm{num}}}(h(X)).$$

So it suffices to consider motives of the form $h(X)$. For $\mathbb{Q} \subseteq F$ write

$$A_{\sim,F} = \mathrm{End}_{\mathcal{M}ot_\sim}(h(X)) = CH^{\dim X}_\sim(X \times X) \otimes F.$$

By Theorem **??** in Lecture 5, $A_{\mathrm{num},\mathbb{Q}}$ is finite-dimensional over $\mathbb{Q}$. So $A_{\mathrm{num},\mathbb{Q}}$ is left Artinian, so by Wedderburn–Artin, Thm.7, to know that $A_{\mathrm{num},\mathbb{Q}}$ is semisimple it suffices to show that the Jacobson radical vanishes:

$$J(A_{\mathrm{num},\mathbb{Q}}) \overset{?}{=} 0.$$

Choose a Weil cohomology theory $H^* : \mathcal{M}ot_{\mathrm{rat}} \to \mathrm{GrVec}_F$, for example de Rham cohomology[13] with $F = \mathbb{R}$. Since $J(A_{\mathrm{num},\mathbb{Q}})$ is a nilpotent two-sided ideal, so is $J(A_{\mathrm{num},\mathbb{Q}}) \otimes_{\mathbb{Q}} F$ so

$$J(A_{\mathrm{num},\mathbb{Q}}) \subseteq J(A_{\mathrm{num},\mathbb{Q}}) \otimes_{\mathbb{Q}} F \overset{Prop.6}{\subseteq} J(A_{\mathrm{num},F})$$

so it suffices to show that $J(A_{\mathrm{num},F})$ vanishes. Consider the surjection

$$\Phi : A_{\mathrm{hom},F} \twoheadrightarrow A_{\mathrm{num},F}.$$

Since $A_{\mathrm{hom},F}/J(A_{\mathrm{hom},F})$ is semisimple, its quotient $A_{\mathrm{num},F}/\Phi(J(A_{\mathrm{hom},F}))$ is semisimple, Ex.4, hence

$$J(A_{\mathrm{num},F}) \subseteq \Phi(J(A_{\mathrm{hom},F})).$$

Let $\overline{f} \in J(A_{\mathrm{num},F})$, and choose a lift $f \in J(A_{\mathrm{hom},F})$. For every $g \in A_{\mathrm{hom},F}$, the element $g \circ f$ again lies in the two-sided nilpotent ideal $J(A_{\mathrm{hom},F})$, hence is nilpotent. The action of $g \circ f$ on $H^*(X)$ is therefore nilpotent, so each $(g \circ f)^i : H^i(X) \to H^i(X)$ is nilpotent so its Lefschetz trace is zero, Ex.2:

$$\mathrm{Tr}^{\mathrm{GrVec}}(H^*(g \circ f)) = \sum_i (-1)^i \,\mathrm{Tr}\big(H^i(g \circ f)\big) = 0.$$

By Eq.1, discussed in the previous lecture, this trace is the intersection number

$$\deg(f \cdot {}^t g) = \mathrm{Tr}^{\mathcal{M}ot_{\mathrm{hom}}}(g \circ f) = \mathrm{Tr}^{\mathrm{GrVec}}(H^*(g \circ f)).$$

Hence $\deg(f \cdot {}^t g) = 0$ for all $g \in A_{\mathrm{hom}}$. Since transposition is an isomorphism, f pairs trivially with all codimension d correspondences on $X \times X$. By definition of numerical equivalence, this means $f \sim_{\mathrm{num}} 0$, and therefore $\overline{f} = 0$.

Thus $J(A_{\mathrm{num},F}) = 0$, so $J(A_{\mathrm{num},\mathbb{Q}}) = 0$, hence $A_{\mathrm{num},\mathbb{Q}} = \mathrm{End}(M)$ is a semisimple algebra, so $M \in \mathcal{M}ot_{\mathrm{num}}$ is a semisimple object.

Thus $J(A_{\mathrm{num},F}) = 0$, hence $J(A_{\mathrm{num},\mathbb{Q}}) = 0$. Therefore $A_{\mathrm{num},\mathbb{Q}} = \mathrm{End}(h(X))$ is semisimple, so $\mathrm{End}(h(X) \otimes \mathbb{L}^{-m})$ is semisimple, so $\mathrm{End}(X, p, m))$ is semisimple for every $M = (X, p, m)$, so the nice category $\mathcal{M}ot_{\mathrm{num}}$ is semisimple. □

A The categorical radical and semisimplicity

Definition 18. *An* ideal $\mathcal{J}$ *in an additive category* $\mathcal{C}$ *is a subgroup*

$$\mathcal{J}(M, N) \subseteq \hom_{\mathcal{C}}(M, N)$$

for every pair of objects, stable under left and right composition by arbitrary morphisms in $\mathcal{C}$*. It is* nilpotent *if there exists* $m \geq 1$ *such that every* m*-fold*

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[13] In this course we have focussed on de Rham, but of course, one could alternatively use ℓ-adic cohomology where $F = \mathbb{Q}_\ell$.

composition of morphisms in $\mathcal{J}$ is zero. The quotient category $\mathcal{C}/\mathcal{J}$ has the same objects as $\mathcal{C}$ and morphism groups

$$\hom_{\mathcal{C}/\mathcal{J}}(M,N) = \hom_{\mathcal{C}}(M,N)/\mathcal{J}(M,N).$$

The radical *of $\mathcal{C}$ is the ideal $\operatorname{rad}_{\mathcal{C}}$ defined by*

$$\operatorname{rad}_{\mathcal{C}}(M,N) = \{f : M \to N \mid \mathrm{id}_M - gf \in \operatorname{End}_{\mathcal{C}}(M)^* \textit{ for all } g : N \to M\}.$$

Theorem 19 (Categorical radical quotient; [Str95, Prop.5, Prop.6, p.146]; [ARS95, I, Prop.3.1])**.** *Let F be a field and $\mathcal{C}$ be an idempotent complete F-linear additive category with finite-dimensional hom-sets. Then the quotient category*

$$\overline{\mathcal{C}} := \mathcal{C}/\operatorname{rad}_{\mathcal{C}}$$

is semisimple: every object of $\overline{\mathcal{C}}$ is a finite direct sum, equivalently a finite product, of simple objects.

Proof. Let $M \in \mathcal{C}$. Since $\operatorname{End}_{\mathcal{C}}(M)$ is finite-dimensional over F, Proposition 6 gives that $J(\operatorname{End}_{\mathcal{C}}(M))$ is nilpotent, and Theorem 7 gives that

$$\operatorname{End}_{\mathcal{C}}(M)/J(\operatorname{End}_{\mathcal{C}}(M))$$

is semisimple. By Proposition 3, the definition of $\operatorname{rad}_{\mathcal{C}}$ gives

$$\operatorname{End}_{\overline{\mathcal{C}}}(M) = \operatorname{End}_{\mathcal{C}}(M)/\operatorname{rad}_{\mathcal{C}}(M,M) = \operatorname{End}_{\mathcal{C}}(M)/J(\operatorname{End}_{\mathcal{C}}(M)).$$

Thus every endomorphism algebra in $\overline{\mathcal{C}}$ is semisimple.

It remains to check that $\overline{\mathcal{C}}$ is idempotent complete. Let $\bar{e} \in \operatorname{End}_{\overline{\mathcal{C}}}(M)$ be an idempotent. Choose a lift $a \in \operatorname{End}_{\mathcal{C}}(M)$. Since $\operatorname{rad}_{\mathcal{C}}(M,M) = J(\operatorname{End}_{\mathcal{C}}(M))$ is nilpotent, idempotents lift modulo this ideal: first lift through square-zero quotients using the correction $a \mapsto a + (1-2a)(a^2 - a)$, then iterate along the powers of the ideal. Thus there is an idempotent $e \in \operatorname{End}_{\mathcal{C}}(M)$ mapping to $\bar{e}$. Since $\mathcal{C}$ is idempotent complete, e splits in $\mathcal{C}$, and its image gives a splitting of $\bar{e}$ in $\overline{\mathcal{C}}$.

Now Proposition 13, applied to $\overline{\mathcal{C}}$, shows that every object of $\overline{\mathcal{C}}$ is a finite direct sum of simple objects. ☐

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