

Lecture 5: Classical Motives

In this lecture we see the definition of the classical category of motives (with rational equivalence), $\mathcal{M}ot_{\text{rat}}$, Def.2, and define a Weil cohomology theory as a functor $H^* : \mathcal{M}ot_{\text{rat}} \rightarrow \text{GrVec}_F$, Def.8. We observe that adequate equivalence relations are in bijection with tensor ideals of $\mathcal{M}ot_{\text{rat}}$ and use this to define homological equivalence, Exam.17 and numerical equivalence, Def.18.

We continue to work with smooth projective varieties over an algebraically closed field k . Until homological equivalence enters, all Chow groups and groups of cycles have integral coefficients. From homological equivalence on, all cycle groups are with rational coefficients unless otherwise stated.

1 Pure motives

Recall that for $X, Y, Z \in \text{SmProj}$ and $\alpha \in CH^*(X \times Y)$ and $\beta \in CH^*(Y \times Z)$, the *composition* is defined as

$$\beta \circ \alpha = (p_{13})_*(p_{12}^*\alpha \cdot p_{23}^*\beta)$$

where p_{ij} are the various projections

$$\begin{array}{ccccc} & & X \times Y \times Z & & \\ & \swarrow p_{12} & \downarrow p_{13} & \searrow p_{23} & \\ X \times Y & & X \times Z & & Y \times Z \end{array}$$

A cycle class $\pi \in CH^*(X \times X)$ is *idempotent* if

$$\pi \circ \pi = \pi.$$

Example 1. Let $x \in X$ be a closed point. Note that $\{x\} \times X \subseteq X \times X$ is the (transposed) graph of the projection $X \rightarrow \{x\} \rightarrow X$. Since $[{}^t\Gamma_{g \circ f}] = [{}^t\Gamma_f] \circ [{}^t\Gamma_g]$ it follows that

$$[x \times X] \in CH^{\dim X}(X \times X)$$

is idempotent.

Definition 2. The category $\mathcal{M}ot_{\text{rat}}$ of pure motives modulo rational equivalence is defined as follows. An object is a triple

$$(X, p, m)$$

where X is a smooth projective variety, $p \in CH^{\dim X}(X \times X)$ is an idempotent correspondence,¹ and $m \in \mathbb{Z}$. For two objects (X, p, m) and (Y, q, n) , the morphism space is

$$\text{hom}_{\mathcal{M}ot_{\text{rat}}}((X, p, m), (Y, q, n)) = q \circ CH^{\dim X - m + n}(X \times Y) \circ p.$$

Composition is composition of correspondences.

¹If $X = \sqcup X_\lambda$ of different dimensions, then $CH^{\dim X}(X \times X)$ should be interpreted as $\oplus_\lambda CH^{\dim X_\lambda}(X_\lambda \times X) \subseteq CH^*(X \times X)$.

Example 3. For a smooth projective variety X , write

$$h(X) := (X, \text{id}, 0)$$

where id is the class of the diagonal $[\{(x, x) : x \in X\}] \in CH^*(X \times X)$. To a morphism of varieties $f : X \rightarrow Y$ we associate the morphism of motives $h(f) : h(Y) \rightarrow h(X)$ represented by the transposed graph ${}^t\Gamma_f \in CH^{\dim Y}(Y \times X)$. Note that this defines a contravariant functor

$$h : \text{SmProj}^{\text{op}} \rightarrow \text{Mot}_{\text{rat}} \quad (1)$$

Exercise 1. Recall that in an additive category \mathcal{A} , a *categorical sum* of two objects A and B is an object $A \oplus B$ equipped with morphisms

$$A \begin{array}{c} \xrightarrow{\iota_A} \\ \xleftarrow{\pi_A} \end{array} A \oplus B \begin{array}{c} \xrightarrow{\pi_B} \\ \xleftarrow{\iota_B} \end{array} B$$

such that

$$\begin{aligned} \pi_A \iota_A &= \text{id}_A, & \pi_A \iota_B &= 0, \\ \pi_B \iota_B &= \text{id}_B, & \pi_B \iota_A &= 0, \\ \iota_A \pi_A + \iota_B \pi_B &= \text{id}_{A \oplus B}. \end{aligned}$$

1. Show that $(X \sqcup Y, 0, m)$ has a structure of categorical sum for $(X, 0, m)$ and $(Y, 0, m)$. That is, show

$$(X \sqcup Y, 0, m) = (X, 0, m) \oplus (Y, 0, m)$$

for any $m \in \mathbb{Z}$ and $X, Y \in \text{SmProj}$.

2. Show that

$$(X, 0, m) = (X, p, m) \oplus (X, \text{id} - p, m). \quad (2)$$

for any $m \in \mathbb{Z}$, $X \in \text{SmProj}$ and idempotent p .

3. Show that for any two (X, p, m) , (Y, q, m) (same m) the categorical sum $(X, p, m) \oplus (Y, q, m)$ exists.

Exercise 2. The *Lefschetz motive* is defined as

$$\mathbb{L} := (\mathbb{P}^0, \text{id}, -1).$$

Recall that for any smooth projective variety X we have an isomorphism of graded rings

$$CH^*(X)[t]/t^2 \xrightarrow{\sim} CH^*(\mathbb{P}^1 \times X)$$

where $t \in CH^1(\mathbb{P}^1 \times X)$ is the class of $\{\infty\} \times X$ and $CH^*(X) \rightarrow CH^*(\mathbb{P}^1 \times X)$ is pullback. Show that

$$\mathbb{L} \cong (\mathbb{P}^1, \text{id} - p, 0) \quad (3)$$

and $h(\mathbb{P}^1)$ is the categorical sum

$$h(\mathbb{P}^1) \cong \mathbb{1} \oplus \mathbb{L}$$

Here $p : \mathbb{P}^1 \rightarrow \{\infty\} \rightarrow \mathbb{P}^1$ is projection to ∞ and $\mathbb{1} := h(\mathbb{P}^0)$.

Definition 4. The tensor product on $\mathcal{M}ot_{\text{rat}}$ is defined as

$$(X, p, m) \otimes (Y, q, n) = (X \times Y, p \times q, m + n).$$

Exercise 3.

1. Show that the object

$$\mathbb{1} = h(\mathbb{P}^0) = (\mathbb{P}^0, \text{id}, 0)$$

is the unit for this tensor product and \mathbb{L} is tensor invertible.

2. Note that for any (X, p, m) we have

$$(X, p, m) \otimes \mathbb{L} \cong (X, p, m-1). \quad (4)$$

Using Eq.(3) show that the categorical sum exists in $\mathcal{M}ot_{\text{rat}}$ for any two objects (X, p, m) , (Y, q, n) , possibly with $n \neq m$.

Remark 5. Combining Eq.(2) and Eq.(4) one sees that every motive (X, p, m) is a twist of a summand of the motive of a smooth projective variety

$$h(X) \otimes \mathbb{L}^{\otimes -m} \cong (X, p, m) \oplus (X, \text{id} - p, m).$$

Exercise 4. Recall that an idempotent $e : M \rightarrow M$ in an additive category is said to be *split* if there exists a direct sum decomposition $M = M_0 \oplus M_1$ such that, with respect to this decomposition, $e = \begin{pmatrix} \text{id}_{M_0} & 0 \\ 0 & 0 \end{pmatrix}$. Show that in $\mathcal{M}ot_{\text{rat}}$ every idempotent is split.

Exercise 5. Recall that an object M in a symmetric monoidal category is called *dualisable* if there exists an object M^\vee and morphisms

$$\text{ev} : M^\vee \otimes M \rightarrow \mathbb{1}, \quad \text{coev} : \mathbb{1} \rightarrow M \otimes M^\vee$$

satisfying the triangle identities.

$$(\text{ev} \otimes \text{id}_{M^\vee}) \circ (\text{id}_{M^\vee} \otimes \text{coev}) = \text{id}_{M^\vee}, \quad (\text{id}_M \otimes \text{ev}) \circ (\text{coev} \otimes \text{id}_M) = \text{id}_M$$

Equivalently, tensoring with M has right adjoint tensoring with M^\vee :

$$\text{hom}(A \otimes M, B) \cong \text{hom}(A, B \otimes M^\vee)$$

functorially in A and B . In this case M^\vee is called the *dual* of M .

Let \mathcal{C} be an additive idempotent complete symmetric monoidal category.

1. Show that all \otimes -invertible objects are dualisable.
2. Show that if M and N are dualisable so is $M \otimes N$.
3. Show that if M and N are dualisable then so is $M \oplus N$.

4. Show that if $M \oplus N$ is dualisable, then so is M .
5. Let X be irreducible of dimension d . Show that there are morphisms making $h(X) \otimes \mathbb{L}^{\otimes -d}$ the dual of $h(X)$. In particular, $h(X)$ is dualisable.
6. Show that every object of $\mathcal{M}ot_{\text{rat}}$ is dualisable. Hint.²

Exercise 6. Recall that the morphism

$$CH^*(X)[t]/t^{n+1} \xrightarrow{\sim} CH^*(\mathbb{P}^n \times X)$$

induced by pullback is an isomorphism of graded rings where $t = [\mathbb{P}^{n-1} \times X]$ is the class of a hyperplane in \mathbb{P}^n . Using this, show that

$$h(\mathbb{P}^n) \cong \mathbb{1} \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^{\otimes n}.$$

Exercise 7. Show that if X is irreducible of dimension d then $h(X)$ decomposes as

$$h(X) \cong \mathbb{1} \oplus M \oplus \mathbb{L}^{\otimes d}$$

for some M .

Example 6. Let C be a smooth projective connected curve and choose a point $c \in C$. By Exercise 7, there is a decomposition

$$h(C) \cong \mathbb{1} \oplus h^1(C) \oplus \mathbb{L}.$$

Recall from Lecture 3 that there is an abelian variety³ $J(C)$ sitting in a split exact sequence, [Stacks, Tag 0BA0],

$$0 \rightarrow J(C) \rightarrow CH^1(C) \xrightarrow{\sim} \mathbb{Z} \rightarrow 0.$$

More generally, for $X \in \text{SmProj}$, writing $pr : C \times X \rightarrow X$ for the projection, by [Stacks, Tag 0B9N, Tag 0BA0], we have

$$\text{hom}(X, J(C)) \cong \frac{\ker(CH^1(C \times X) \xrightarrow{pr^*} CH^0(X))}{\text{im}(CH^1(X) \xrightarrow{pr^*} CH^1(C \times X))}.$$

In other words, we have an isomorphism of presheaves on the category SmProj

$$\text{hom}_{\mathcal{M}ot_{\text{rat}}}(h^1(C), h(-)) \cong \text{hom}_{\text{SmProj}}(-, J(C)).$$

In fact, we have more. The following proposition is about $\mathcal{M}ot_{\text{rat}}$ with rational coefficients.

²Use Remark 5

³An *abelian variety* is a smooth projective variety A equipped with morphisms of varieties $e : \mathbb{P}^0 \rightarrow A$, $m : A \times A \rightarrow A$ satisfying the axioms of an abelian group.

Proposition 7 ([Dem71, §5, p.27]). *Let \mathcal{M}_k^1 be the smallest idempotent complete \mathbb{Q} -linear subcategory containing $h^1(C)$ for all smooth projective connected curves C . Then \mathcal{M}_k^1 is equivalent to the category of abelian varieties up to isogeny: the category whose objects are abelian varieties, the group of morphisms from A to B being $\text{hom}(A, B) \otimes \mathbb{Q}$.*

$$\left\langle h^1(C) \mid C \text{ is a curve} \right\rangle \cong \{ \text{abelian varieties up to isogeny} \}$$

Demazure proposes this as motivation to consider all motives as generalised abelian varieties.

La proposition 5 montre [...] aussi que les motifs doivent être considérés comme une généralisation naturelle des variétés abéliennes, celles-ci correspondant aux motifs de degré 1 [Dem71, §5, p.28].

This perspective is echoed by Milne (among others)

In 1968 Grothendieck announced two conjectures, known respectively as the Lefschetz standard conjecture and the Hodge standard conjecture, whose proof would allow one to extend Weil's proof of the Weil conjectures from curves to varieties of arbitrary dimension by replacing the Jacobian of the curve with the motive of the variety [Mil12, §7, p.12].

2 Weil cohomology theories

Definition 8. *Let F be a field of characteristic zero. A Weil cohomology theory with coefficient field F is a symmetric monoidal⁴ additive functor*

$$H^* : \text{Mot}_{\text{rat}} \rightarrow \text{GrVec}_F$$

such that:

1. $H^{<0}(X) = 0$ for all $X \in \text{SmProj}$, and
2. the one dimensional vector space $H^*(\mathbb{L})$ is concentrated in degree two.

Remark 9. The graded vector space $H^*(\mathbb{L})$ is one dimensional because \mathbb{L} is \otimes -invertible in Mot_{rat} . The functor H^* is monoidal so sends invertible objects to invertible objects.

Example 10. De Rham cohomology

$$H_{\text{dR}}^* : \text{SmProj}_{\mathbb{C}}^{\text{op}} \rightarrow \text{GrVec}_{\mathbb{R}}$$

is a Weil cohomology theory.

⁴See Mac Lane [Lan98, Ch. XI, §§1–2] for symmetric monoidal categories and monoidal functors.

Remark 11 (Künneth). Part of saying that H^* is monoidal is the requirement that it is equipped with natural isomorphisms

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y).$$

Remark 12 (Cup product and intersection product). For any $X \in \text{SmProj}$, the structure morphism $X \rightarrow \mathbb{P}^0$ and diagonal $\Delta : X \rightarrow X \times X$ induce morphisms

$$\mathbb{1} = h(\mathbb{P}^0) \rightarrow h(X) \quad \text{and} \quad h(X) \otimes h(X) \rightarrow h(X)$$

satisfying the axioms of a commutative monoid object. Consequently, each $H^*(X)$ is not just a graded vector space, but a commutative graded algebra.⁵ Via Künneth, the multiplication on $H^*(X)$ is

$$\begin{aligned} H^*(X) \otimes H^*(X) &\cong H^*(X \times X) \xrightarrow{\Delta^*} H^*(X), \\ \alpha \otimes \beta &\mapsto \alpha \cup \beta. \end{aligned}$$

On Chow groups, the intersection product has the same form:

$$\alpha \cdot \beta = \Delta^*(\alpha \times \beta), \quad \alpha \in CH^i(X), \quad \beta \in CH^j(X).$$

Therefore the cycle class map discussed in Remark 14 sends intersection products to cup products:

$$\begin{aligned} \gamma_X(\alpha \cdot \beta) &= \gamma_X(\Delta^*(\alpha \times \beta)) \\ &= \Delta^* \gamma_{X \times X}(\alpha \times \beta) \\ &= \Delta^*(\gamma_X(\alpha) \otimes \gamma_X(\beta)) \\ &= \gamma_X(\alpha) \cup \gamma_X(\beta) \in H^{2i+2j}(X). \end{aligned}$$

Remark 13 (Poincaré Duality and Finite Dimensionality). Since monoidal functors preserve duals, for X irreducible of dimension d we have $H^*(X)[-2d] \cong H^*(X) \otimes H^*(\mathbb{L})^{\otimes -d} \cong H^*(X \otimes \mathbb{L}^{\otimes -d}) \cong H^*(h(X)^\vee) \cong H^*(X)^\vee$ and hence identifications

$$H^i(X) \xrightarrow{\sim} H^{2d-i}(X)^\vee$$

for all i . Note that this isomorphism implicitly uses a choice of $\theta \in H^2(\mathbb{L})$ to fix an isomorphism $H^{2d}(\mathbb{L}^{\otimes d})[-2d] \cong F$ (since $(-)^\vee = \text{hom}(-, F)$, not $\text{hom}(-, H^{2d}(\mathbb{L}^{\otimes d})[-2d])$).

In particular, the first axiom of Definition 8 implies

$$H^{>2d}(X) = 0$$

and we do not have to assume

$$\dim H^r(X) < \infty.$$

This follows from the fact that dualisable vector spaces are finite dimensional.

⁵The unit element is obtained by applying the unit morphism

$$\mathbb{1}_{\text{GrVec}_F} \rightarrow H^*(\mathbb{1}) = H^*(\mathbb{P}^0) \rightarrow H^*(X)$$

to the distinguished element of $\mathbb{1}_{\text{GrVec}_F}$ determined by the unit constraint of the monoidal category structure on GrVec_F .

Remark 14 (Cycle class map). Via $CH^i(X) = \text{hom}_{\mathcal{M}ot_{\text{rat}}}(\mathbb{L}^{\otimes i}, h(X))$, attached to a choice of $\theta \in H^2(\mathbb{L})$ we obtain a cycle class map

$$\begin{array}{ccc} CH^i(X) & \xrightarrow{\gamma_X} & H^{2i}(X) \\ \parallel & & \uparrow \text{ev}_{\theta \otimes i} \\ \text{hom}(\mathbb{L}^{\otimes i}, h(X)) & \xrightarrow{H^*} & \text{hom}(H^*(L^{\otimes i}), H^*(X)) \end{array}$$

Conversely, a “nice” functor $H^* : \text{SmProj}^{\text{op}} \rightarrow \text{GrVec}_F$ equipped with cycle class maps $\gamma_X : CH^i(X) \rightarrow H^{2i}(X)$ allows one to define a symmetric monoidal functor $\mathcal{M}ot_{\text{rat}} \rightarrow \text{GrVec}_F$ provided the cycle class maps satisfy

$$\begin{aligned} \gamma(f^*\beta) &= f^*\gamma(\beta), & \gamma(f_*\alpha) &= f_*\gamma(\alpha), \\ \gamma(\alpha \cdot \beta) &= \gamma(\alpha) \cup \gamma(\beta), & \gamma([\mathbb{P}^0]) &= 1 \in H^0(\mathbb{P}^0), \end{aligned}$$

where the pushforward formula is for X and Y equidimensional, and f_* is defined via Poincaré Duality as the dual of f^* . More precisely, “nice” means satisfying Poincaré Duality and Künneth.

$$\left\{ \begin{array}{l} \text{Weil coh.theories} \\ H^* : \mathcal{M}ot_{\text{rat}} \rightarrow \text{GrVec}_F \\ \text{with a choice of } \theta \in H^2(\mathbb{L}) \end{array} \right\} \cong \left\{ \begin{array}{l} \text{“nice”} \\ H^* : \text{SmProj}^{\text{op}} \rightarrow \text{GrVec}_F, \\ \gamma : CH^* \rightarrow H^{2*} \end{array} \right\}.$$

Note that θ on the left is the image of the class of $[\{\infty\}] \in CH^1(\mathbb{P}^1) \rightarrow H^2(\mathbb{P}^1)$ on the right. For more details see [Stacks, 0FGS, 0FHA].

3 Homological equivalence

Definition 15. Let C be a symmetric monoidal additive category. A tensor ideal I in C assigns to every pair of objects M, N a subgroup

$$I(M, N) \subseteq \text{hom}_C(M, N)$$

which is stable under composition on both sides and under tensoring with arbitrary morphisms. That is, for all $L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} P$ and $M' \xrightarrow{g'} N'$ with $g \in I(M, N)$ we have $h \circ g \circ f \in \text{hom}(L, P)$ and $g \otimes g' \in I(M \otimes M', N \otimes N')$.

Proposition 16. There is an isomorphism of posets between adequate equivalence relations and tensor ideals in $\mathcal{M}ot_{\text{rat}}$.

Sketch of proof. The bijection $I \leftrightarrow \sim$ is defined as follows: Given \sim and $M = (X, p, m)$, $N = (Y, q, n)$ define $I_{\sim}(M, N)$ as the image

$$\begin{array}{ccc} q \circ \mathcal{Z}_{\sim}^{\dim X - m + n}(X \times Y) \circ p & \subseteq & q \circ \mathcal{Z}^{\dim X - m + n}(X \times Y) \circ p \\ \downarrow & & \downarrow \\ I_{\sim}(M, N) & \subseteq & \text{hom}(M, N) = CH^{\dim X - m + n}(X \times Y) \end{array}$$

Conversely, given $I(-, -)$ and $X \in \text{SmProj}$ define \sim_I via the preimage

$$\begin{array}{ccc} \mathcal{Z}_{\sim_I}^r(X) & \subseteq & \mathcal{Z}^r(X) \\ \downarrow & & \downarrow \\ I(\mathbb{L}^{\otimes r}, h(X)) & \subseteq & \text{hom}(\mathbb{L}^{\otimes r}, h(X)) = CH^r(X) \end{array}$$

Apart from defining the bijection, there are only a few points which are a little tricky.

1. I_{\sim} is closed under \otimes . Since \otimes is defined in terms of cartesian product, we must show that $\alpha \in I_{\sim}(M, N)$ implies $\alpha \times \beta \in I_{\sim}(M \otimes M', N \otimes N')$ for arbitrary β . This follows from the observation $\alpha \times \beta = pr_1^* \alpha \cdot pr_2^* \beta$.
2. \sim_I satisfies R2. Given $\alpha \in \mathcal{Z}_{\sim_I}^r(X)$ and $\beta \in \mathcal{Z}^s(X)$ in good position we want to know that $\alpha \cdot \beta \in \mathcal{Z}_{\sim_I}^{r+s}(X)$. For this let $\mathbb{L}^{\otimes r} \xrightarrow{a} h(X)$ and $\mathbb{L}^{\otimes s} \xrightarrow{b} h(X)$ be the corresponding morphisms. Then the image of $\alpha \cdot \beta$ is the composition $\mathbb{L}^{\otimes s+r} \xrightarrow{a \otimes b} h(X \times X) \xrightarrow{\text{diag.}} h(X)$.
3. We have $I_{\sim_I} = I$. The obstacle here is that I involves all homs but \sim_I only involves homs of the form $\text{hom}(\mathbb{L}^{\otimes r}, h(X))$. So we need a way to transport information from the latter to the former. The obvious way to do this is via the duality isomorphism.

$$\text{hom}((X, \text{id}, m), (Y, \text{id}, n)) \xrightarrow{\sim} \text{hom}(\mathbb{L}^{\otimes d+n-m}, h(X \times Y)).$$

To check that $I_{\sim_I} = I$ it suffices to check that this isomorphism (and its inverse) can be defined using \otimes and \circ . Then it follows that the isomorphism preserves I . Explicitly, given f on the left, one checks that the image on the right is the composition

$$\begin{aligned} \mathbb{L}^{d+n-m} &\xrightarrow{\text{coev}} \mathbb{L}^{d+n-m} \otimes h(X) \otimes h(X)^\vee \\ &= (X, \text{id}, m) \otimes \mathbb{L}^{\otimes n} \otimes h(X) \\ &\xrightarrow{f \otimes \text{id} \otimes \text{id}} (Y, \text{id}, n) \otimes \mathbb{L}^{\otimes n} \otimes h(X) \\ &= h(X \times Y). \end{aligned}$$

There is a similar description for the inverse.

The other steps are just inserting the definitions.

Example 17. Let $H^* : \mathcal{M}ot_{\text{rat}} \rightarrow \text{GrVec}_F$ be a Weil cohomology theory. Then

$$I_{\text{hom}}(M, N) = \{f \mid H^*(f) = 0\}$$

is a tensor ideal. It is precisely the tensor ideal making the functor

$$\mathcal{M}ot_{\text{hom}} \rightarrow \text{GrVec}_F$$

faithful. The corresponding equivalence relation is called *homological equivalence*. A priori it depends on H^* . We will sometimes identify

$$CH_{\text{hom}}^*(X) := \mathcal{Z}^*(X) / \mathcal{Z}_{\text{hom}}^*(X)$$

with the image subgroup/vector space

$$CH_{\text{hom}}^*(X) \subseteq H^{2*}(X).$$

4 Numerical Equivalence and Traces

Exercise 8. Let X be smooth projective of dimension d . For $\alpha \in CH^r(X)$ and $\beta \in CH^{d-r}(X)$, let $\tilde{\alpha} : \mathbb{L}^{\otimes r} \rightarrow h(X)$ and $\tilde{\beta} : h(X) \rightarrow \mathbb{L}^{\otimes r}$ be the corresponding morphisms in $\mathcal{M}ot_{\text{rat}}$. Show that the zero-cycle $\alpha \cdot \beta$ has degree equal to the scalar in \mathbb{Q} corresponding to $\tilde{\beta} \circ \tilde{\alpha} \in \text{End}_{\mathcal{M}ot_{\text{rat}}}(\mathbb{L}^{\otimes r}) = \mathbb{Q} \cdot \text{id}_{\mathbb{L}^{\otimes r}}$.

$$\tilde{\beta} \circ \tilde{\alpha} = \text{deg}(\alpha \cdot \beta) \cdot \text{id}_{\mathbb{L}^{\otimes r}}.$$

Definition 18. Let X be smooth projective of dimension d . A cycle $\alpha \in \mathcal{Z}^r(X)$ is numerically equivalent to zero if

$$\text{deg}(\alpha \cdot \beta) = 0$$

for every $\beta \in \mathcal{Z}^{d-r}(X)$. We write $\mathcal{Z}_{\text{num}}^r(X)$ for the subgroup of numerically trivial cycles.

Remark 19. By Exercise 8, $\alpha \in \mathcal{Z}^r(X)$ with image $a \in CH^r(X) = \text{hom}(\mathbb{L}^{\otimes r}, h(X))$ is numerically equivalent to zero if and only if

$$b \circ a = 0 \in \text{hom}(\mathbb{L}^{\otimes r}, \mathbb{L}^{\otimes r})$$

for all $b : h(X) \rightarrow \mathbb{L}^{\otimes r}$. In particular, if one cycle in a rational equivalence class $\alpha \in CH^r(X)$ is numerically equivalent to zero, then all cycles in that class are. In other words, $\mathcal{Z}_{\text{rat}}^r(X) \subseteq \mathcal{Z}_{\text{num}}^r(X)$ and the latter is a subgroup of $\mathcal{Z}^r(X)$.

To give an even more general condition we need some material about traces.

Definition 20. Suppose that \mathcal{C} is an additive symmetric monoidal category. Let $M \in \mathcal{C}$ be an object admitting a dual M^\vee and $f : M \rightarrow M$ any morphism. The trace of f is the composition

$$\text{Tr}(f) : \mathbb{1} \xrightarrow{\text{coev}} M \otimes M^\vee \xrightarrow{f \otimes \text{id}} M \otimes M^\vee \xrightarrow{\text{ev}} \mathbb{1}$$

where the first and last arrows are the unit and counit of the duality.

Exercise 9. If $\mathcal{C} = \text{Vec}$ show that the above notion of trace is the same as the matrix one. That is for a vector space V with isomorphism $V \cong F^n$ and an endomorphism $V \xrightarrow{f} V$ with corresponding matrix $[a_{ij}]$ we have

$$\text{Tr}^{\text{Vec}}(f) = (\sum a_{ii}) \text{id}_F.$$

More generally, if $\mathcal{C} = \text{Vec}$ and $f : V^* \rightarrow V^*$ is a graded morphism show that

$$\text{Tr}^{\text{GrVec}}(f) = \sum_i (-1)^i \text{Tr}^{\text{Vec}}(f^i)$$

where the left hand side is the categorical trace in GrVec and the right hand side uses the ungraded traces from Vec .

Exercise 10. Let

$$\alpha \in CH^{\dim X - r}(X \times Y) = \text{hom}((X, \text{id}, n+r), (Y, \text{id}, n))$$

and

$$\beta \in CH^{\dim Y + r}(Y \times X) = \text{hom}((Y, \text{id}, n), (X, \text{id}, n+r)).$$

1. Show that

$$\text{Tr}^{\mathcal{M}ot}(\beta \circ \alpha) = \text{deg}(\alpha \cdot \beta^t) \cdot \text{id}_{\mathbb{1}}.$$

2. Deduce that cycles in a class $\alpha \in \text{hom}((X, p, n+r), (Y, q, n)) \subseteq CH^{\dim X - r}(X \times Y)$ are numerically equivalent to zero if and only if

$$\text{Tr}^{\mathcal{M}ot}(\beta \circ \alpha) = 0$$

for all $\beta \in \text{hom}((Y, q, n), (X, p, n+r))$.

3. Deduce also that if $H^* : \mathcal{M}ot_{\text{rat}} \rightarrow \text{GrVec}_F$ is any Weil cohomology theory, we have

$$\text{deg}(\alpha \cdot \beta^t) = \text{Tr}^{\text{GrVec}}(H^*(\beta \circ \alpha)) = \sum_i (-1)^i \text{Tr}^{\text{Vec}}(H^i(\beta \circ \alpha)).$$

Proposition 21. *Numerical equivalence is an adequate equivalence relation.*

Proof. It is enough to check the four axioms of Def.?? from Lecture ??.

(R1) This was already observed in Remark 19.

(R2) Let $\alpha \in CH^i(X)$ be rational equivalence class of numerically trivial cycles, and let $\gamma \in CH^j(X)$. For every complementary β we have

$$\text{deg}((\alpha \cdot \gamma) \cdot \beta) = \text{deg}(\alpha \cdot (\gamma \cdot \beta)) = 0.$$

Thus every cycle in the rational equivalence class $\alpha \cdot \gamma$ is numerically trivial.

(R3) By the moving lemma for rational equivalence, choose $\alpha' \sim_{\text{rat}} \alpha$ such that $\alpha' \cdot \beta$ is defined. Since $\mathcal{Z}_{\text{rat}} \subseteq \mathcal{Z}_{\text{num}}$ we have $\alpha' \sim_{\text{num}} \alpha$.

(R4) Let $\alpha \in CH^i(X)$ be a numerically trivial rational equivalence class of cycles and $\beta \in CH^j(X \times Y)$. We want to show that $\beta \circ \alpha$ is also numerically trivial. That is, for every δ of the appropriate codimension we want $\text{deg}((\beta \circ \alpha) \cdot \delta) = 0$.

Write $d = \dim X$, $e = \dim Y$, and write $pr_1, pr_2 : X \times Y \rightarrow X, Y$, $pr : X, Y \rightarrow \mathbb{P}^0$ for the projections. Choose $\delta \in CH^{d+e-i-j}(Y)$. We have

$$\begin{aligned}
\deg((\beta \circ \alpha) \cdot \delta) &= pr_*((pr_2)_*(pr_1^* \alpha \cdot \beta) \cdot \delta) \\
&\stackrel{(PF)}{=} pr_*((pr_2)_*(pr_1^* \alpha \cdot \beta \cdot pr_2^* \delta)) \\
&= pr_*((pr_1)_*(pr_1^* \alpha \cdot \beta \cdot pr_2^* \delta)) \\
&\stackrel{(PF)}{=} pr_*(\alpha \cdot (pr_1)_*(\beta \cdot pr_2^* \delta)) \\
&= \deg(\alpha \cdot ((pr_1)_*(\beta \cdot pr_2^* \delta))) \\
&= 0.
\end{aligned}$$

Here (PF) indicates projection formula, and the last equality holds because $(pr_1)_*(\beta \cdot pr_2^* \delta) \in CH^{d-i}(X)$ and α is numerically trivial. It may be clearer to consider the diagram

$$\begin{array}{ccccc}
\beta & & X \times Y & \xrightarrow{pr_2} & Y & & \delta \\
& & \downarrow pr_1 & & \downarrow pr & & \\
\alpha & & X & \xrightarrow{pr} & \mathbb{P}^0 & &
\end{array}$$

The idea is that both $\alpha \cdot (pr_1)_*(\beta \cdot pr_2^* \delta)$ and $(pr_2)_*(pr_1^* \alpha \cdot \beta) \cdot \delta$ are push forwards of $pr_1^* \alpha \cdot \beta \cdot pr_2^* \delta$.

□

Proposition 22. *An adequate equivalence relation is nontrivial⁶ if and only if it is finer than numerical equivalence. That is, if and only if $\mathcal{Z}_{\sim}^*(X) \subseteq \mathcal{Z}_{\text{num}}^*(X)$ for all X .*

Proof. Let \sim be a nontrivial adequate equivalence relation, so in particular, $\mathcal{Z}_{\sim}^0(\mathbb{P}^0) = 0$. Let $p : X \rightarrow \mathbb{P}^0$ be a smooth projective irreducible variety of dimension d . We have

$$\begin{aligned}
\alpha \in \mathcal{Z}_{\sim}^r(X) &\Rightarrow \alpha \cdot \beta \in \mathcal{Z}_{\sim}^d(X) && \forall \beta \in \mathcal{Z}^{d-r}(X) \\
&\Rightarrow p_*(\alpha \cdot \beta) \in \underbrace{\mathcal{Z}_{\sim}^0(\mathbb{P}^0)}_{=0} && \forall \beta \in \mathcal{Z}^{d-r}(X) \\
&\Rightarrow \deg(\alpha \cdot \beta) = p_*(\alpha \cdot \beta) = 0 && \forall \beta \in \mathcal{Z}^{d-r}(X) \\
&\Rightarrow \alpha \in \mathcal{Z}_{\text{num}}^r(X).
\end{aligned}$$

The converse is immediate from the fact that numerical equivalence is itself nontrivial, since $\mathcal{Z}_{\text{num}}^0(\mathbb{P}^0) = 0$. □

Corollary 23. *For any Weil cohomology theory and $X \in \text{SmProj}$ we have*

$$\mathcal{Z}_{\text{rat}}^*(X) \subseteq \mathcal{Z}_{\text{hom}}^*(X) \subseteq \mathcal{Z}_{\text{num}}^*(X).$$

⁶An adequate equivalence relation \sim is *trivial* if every cycle is equivalent to zero.

Corollary 24 (Kleiman [Kle68, Thm. 3.5]). *All hom spaces in $\mathcal{M}ot_{\text{num}}$ are finite dimensional.*

Proof. We have an inclusion

$$\text{hom}_{\mathcal{M}ot_{\text{hom}}}(M, N) \otimes_{\mathbb{Q}} F \subseteq \text{hom}(H^*(M), H^*(N))$$

with finite dimensional target, so the source is finite dimensional. Then we have a surjection

$$\text{hom}_{\mathcal{M}ot_{\text{hom}}}(M, N) \twoheadrightarrow \text{hom}_{\mathcal{M}ot_{\text{num}}}(M, N).$$

with finite dimensional source, so the target is finite dimensional. (Note that a \mathbb{Q} -vector space V is finite dimensional if and only if $V \otimes F$ is finite dimensional.) \square

So at this stage we have $\mathcal{M}ot_{\text{hom}}$ which injects into GrVec_F and $\mathcal{M}ot_{\text{num}}$ which is constructed entirely using algebraic geometry, without any choice of cohomology theory. We will see next lecture that $\mathcal{M}ot_{\text{num}}$ is semi-simple. We would like a single category with both of these properties.

Conjecture D (Grothendieck). The following equivalent statements hold.

1. For all $X \in \text{SmProj}$ we have

$$\mathcal{Z}_{\text{hom}}^*(X) = \mathcal{Z}_{\text{num}}^*(X).$$

2. For all $X \in \text{SmProj}$ we have the intersection pairing

$$CH_{\text{hom}}^i(X) \times CH_{\text{hom}}^{d-i}(X) \rightarrow CH^d(X) \xrightarrow{\text{deg}} \mathbb{Q}$$

is non-degenerate.

3. $\mathcal{M}ot_{\text{hom}} = \mathcal{M}ot_{\text{num}}$.

References

- [Dem71] Michel Demazure. Motifs des variétés algébriques. In *Séminaire Bourbaki, 1969/70*, volume 180 of *Lecture Notes in Mathematics*, pages 19–38. Springer, 1971. Exp. no. 365.
- [Kle68] Steven L. Kleiman. Algebraic cycles and the weil conjectures. In *Dix Exposés sur la Cohomologie des Schémas*. North-Holland, 1968.
- [Lan98] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer, 2 edition, 1998.
- [Mil12] James S. Milne. Motives: Grothendieck’s dream, 2012. <https://www.jmilne.org/math/xnotes/MOT.pdf>.
- [Stacks] Stacks project. <https://stacks.math.columbia.edu>.