In this lecture we present some main theorems from algebraic topology, differential topology, symplectic geometry, ... that we will use later. More specifically, we discuss Betti, singular, and de Rham cohomology, the relationship between them, and finish with pure Hodge structures.

We will get back to algebraic geometry next week.

References:

1. Bott&Tu, Differential forms in algebraic topology.
2. Griffiths&Harris, Principles of algebraic geometry.
3. Hatcher, Algebraic topology.
4. Voisin, Hodge theory and complex algebraic geometry I.

1 Course Outline

Here is a more complete map of the course. Recall that last week we had a heuristic about using cohomology to convert algebraic geometry into linear algebra.

$$\text{Alg.Geom.} \xrightarrow{\text{cohomology}} \text{Lin.Alg.}$$

This is a simplification of the more detailed picture below.

$$\begin{tikzcd}
SmProj^{op} \arrow[d] \arrow[rrrr, bend left, "H_{Betti}"] \arrow[rrrr, bend left] \arrow[rrr, bend left, "H_{dR}"'] & & & & & \\
Cycl \arrow[r] & \mathcal{M}ot_{rat} \arrow[r] & \mathcal{M}ot_{hom} \arrow[r, "\text{faithful}"'] \arrow[d] & Hdg \arrow[r, "forget"'] & GrVec \arrow[r, "forget"'] & Vec \\
& & \mathcal{M}ot_{num} \arrow[ur, dotted, "\text{conjecturally factors, conjecturally fully faithful}"'] & & &
\end{tikzcd}$$

1. Lecture 1 was motivation. The goal was to show that correspondences induce very useful morphisms on cohomology. I also prepared notes for a Lecture 0 with some basic notions from algebraic geometry that will appear.
2. This lecture (Lecture 2) has three parts:
    (a) Construct the functor $H_{Betti} : SmProj_{\mathbb{Q}}^{op} \to Top^{op} \to GrVec_{\mathbb{Q}}$.
    (b) Construct the functor $H_{dR} : SmProj_{\mathbb{Q}}^{op} \to Manifolds_{\mathbb{R}}^{op} \to GrVec_{\mathbb{R}}$.
    (c) Observe that $H_{dR} \otimes \mathbb{C} : SmProj_{\mathbb{Q}}^{op} \rightrightarrows Vec_{\mathbb{C}}$ is endowed with a Hodge structure. In other words, a factorisation

$$SmProj^{op} \longrightarrow Hdg$$

3. In Lecture 3, we construct the *graph Cycl* containing $SmProj$ (it is almost a category, but composition is not defined for every "composable" pair of edges). The vertices of $Cycl$ are smooth projective varieties, and edges are cycles. In order to turn $Cycl$ into a category, we use the notion of an adequate equivalence relation. We meet the coarsest one: rational equivalence. The category $\mathcal{M}ot_{rat}$ has the same objects as $SmProj$ and $Cycl$ but the hom groups are cycles *up to rational equivalence.*

$$SmProj \to Cycl \to \mathcal{M}ot_{rat}$$

4. In Lecture 4 we discuss cycle maps for de Rham cohomology. This is equivalent to a factorisation of de Rham cohomology through $\mathcal{M}ot_{rat}$.

$$\mathcal{M}ot_{rat} \to Hdg$$

5. In Lecture 5 we discuss two finer equivalence relations: Homological and Numerical. These give functors which are bijections on objects, and surjections on every *hom* space.

$$\mathcal{M}ot_{rat} \to \mathcal{M}ot_{hom} \to \mathcal{M}ot_{num}.$$

   In this lecture we see Jannsen's proof that $\mathcal{M}ot_{\sim}$ is semisimple abelian if and only if $\sim = num$.

6. In Lecture 6 we discuss the Standard Conjectures. In particular, conjecturally $\mathcal{M}ot_{hom} \to \mathcal{M}ot_{num}$ is an equivalence of categories, and the induced functor

$$\mathcal{M}ot_{num} \to Hdg$$

   is fully faithful. At this stage we have (conjecturally) embedded the category $\mathcal{M}ot_{num}$ made purely from algebraic geometry, into the category Hdg made purely from linear algebra.

7. Lectures 7 $\sim$ 12. The previous lectures will probably all go over time, but if there is time left I currently plan to extend the above picture to include newer categories of motives, namely, Voevodsky's triangulated category of motives.

$$\begin{tikzcd}
SmProj^{op} \arrow[r] \arrow[d] & \mathcal{M}ot_{rat} \arrow[r] \arrow[d, "\text{fully faithful}"] & GrVec \arrow[d] \\
Sm^{op} \arrow[r] & DM_{gm}^{op} \arrow[r] & D^b(Vec)
\end{tikzcd}$$

   There is a theorem of Beilinson that says that *if* there exists a t-structure on DM_{gm}^{op} sufficiently compatible with the realisation functor to $D^b(Vec)$, then all the standard conjectures hold. This would be a nice theorem to end on.

I am also open to requests if there is something in particular you want to see in this course.

2 Topological spaces

Definition 1. *If M is topological space, and R a ring, we define the* Betti cohomology $H^n_{\mathsf{Betti}}(M,R)$ *as the cohomology of the constant sheaf* $\underline{R}$. *That is, we choose an exact*[1] *complex of sheaves on* M

$$0 \to \underline{R} \to I^0 \to I^1 \to \dots$$

with each I^i *injective*[2] *and define*

$$H^n_{\mathsf{Betti}}(M,R) = \frac{\ker(d : I^n(M) \to I^{n+1}(M))}{\operatorname{im}(d : I^{n-1}(M) \to I^n(M))}.$$

Remark 2. Its also possible to calculate cohomology using exact complexes of *flasque*[3] sheaves, (cf. Voisin, Prop.4.34), or *fine*[4] sheaves (cf. Voisin, Prop.4.36).

The singular cohomology is one concrete option of a resolution calculating Betti cohomology. Cf. Voisin, Section.4.3.2.

Definition 3. *Define* $\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : 0 \leq x_i,\ \text{and } \sum x_i = 1\}$. *Then for a topological space* U, *the group of singular cochains is hom set*

$$C^n_{\mathsf{sing}}(U,R) = \hom_{set}(\hom_{cont}(\Delta^n, U), R)$$

Addition in the ring R *makes this set a group.*

Example 4. $C^0_{\mathsf{sing}}(U,R) = \hom_{set}(U,R)$.

The inclusions $\mathbb{R}^{n+1} \to \mathbb{R}^{n+2}; (x_0, \dots, x_i, 0, x_{i+1}, \dots, x_n)$ induce maps $\delta_i : \Delta^n \to \Delta^{n+1}$ and, by composition, these induce maps $\delta_i^* : \hom_{cont}(\Delta^{n+1}, U) \to \hom_{cont}(\Delta^n, U)$, and from there, maps

$$d = \sum (-1)^i d_i : C^n_{\mathsf{sing}}(U,R) \to C^{n+1}_{\mathsf{sing}}(U,R).$$

These form a chain complex of sheaves

$$0 \to \underline{R}(-) \to C^0_{\mathsf{sing}}(-,R) \overset{d}{\to} C^1_{\mathsf{sing}}(-,R) \overset{d}{\to} \cdots \overset{d}{\to} C^n_{\mathsf{sing}}(-,R) \overset{d}{\to} \dots \qquad (1)$$

Definition 5. *The* singular cohomology *of* M *is the cohomology of* (1). *I.e.,*

$$H^n_{\mathsf{sing}}(M,R) = \frac{\ker(d : C^n_{\mathsf{sing}}(M,R) \to C^{n+1}_{\mathsf{sing}}(M,R))}{\operatorname{im}(d : C^{n-1}_{\mathsf{sing}}(M,R) \to C^n_{\mathsf{sing}}(M,R))}$$

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[1] Such a sequence of morphisms in an abelian category is called *exact* if $\operatorname{im}(I^{n-1} \to I^n) = \ker(I^n \to I^{n+1})$ for every n.

[2] An object I in an abelian category is called *injective* if $\hom(-,I)$ sends short exact sequences to short exact sequences. It is a theorem that we can always find an exact complex as in the display equation.

[3] Flasque means: (cf. Voisin, Def.4.33) for every inclusion of open subsets $U \subseteq V$ the induced map $F(V) \to F(U)$ is surjective.

[4] Fine means: (cf. Voisin, Def.4.35) F is a sheaf of R-modules where R is a sheaf of rings such that for every open cover $\{U_i \subseteq X\}_{i \in I}$ there exists a partition of unity $f_i, i \in I, \sum f_i = 1$, subordinate to the covering.

Exercise 1. Show that each $C^n_{\mathsf{sing}}(-,R)$ is a flasque sheaf. That is, for every inclusion of open sets $U \subseteq V$ the map $C^n_{\mathsf{sing}}(V,R) \to C^n_{\mathsf{sing}}(U,R)$ is surjective.

Lemma 6. *When M is a locally contractible topology space (e.g., a smooth manifold) the chain complex* (1) *is exact.*

Sketch of proof. This follows from homotopy invariance stated below. $\square$

Corollary 7. *If M is a smooth manifold, then*

$$H^*_{\mathsf{Betti}}(M,R) = H^*_{\mathsf{sing}}(M,R).$$

We have the following properties:

Theorem 8.

1. *(Homotopy invariance) If there is a continuous map $h : M \times [0,1] \to N$ then the map on cohomology induced by $h(-,0)$ is equal to the map induced by $h(-,1)$. In particular, if M is contractible, i.e., there is a continuous map $h : M \times [0,1] \to M$ such that $h(-,0) = \mathrm{id}_M$ and $h(-,1)$ is constant, then $H^n(M,R) = 0$ for $n > 0$ and $H^0(M,R) \cong R$.*

2. *(Mayer-Vietoris) If $U, V \subseteq M$ are two open subspaces then there is a long exact sequence*

$$\cdots \to H^{n-1}_{\mathsf{Betti}}(U \cap V, R) \to H^n_{\mathsf{Betti}}(U \cup V, R) \\ \to H^n_{\mathsf{Betti}}(U,R) \oplus H^n_{\mathsf{Betti}}(V,R) \to H^n_{\mathsf{Betti}}(U \cap V, R) \to \dots$$

3. *(Finiteness) Cf. Bott&Tu, Prop.5.3.2. If M is a compact smooth manifold, then the $H^i(M,\mathbb{Z})$ are finitely generated.*

Sketch of proof.

1. We only prove the "in particular". Any map $\sigma : \Delta^n \to X$ induces a map $\Delta^n \times [0,1] \to X \times [0,1] \xrightarrow{h} X$. Since $h(-,1)$ is constant, this map factors through the projection $\Delta^n \times [0,1] \to \Delta^n \times [0,1]/\Delta^n \times \{1\} \cong \Delta^{n+1}$. So we obtain a map $\sigma' : \Delta^{n+1} \to X$ whose composition with $\Delta^n \cong \Delta^n \times \{0\} \subset \Delta^n \times [0,1]/\Delta^n \times \{1\} \cong \Delta^{n+1}$ is σ (because $h(-,0) = \mathrm{id}_X$). It also has the property that $\sigma' \circ \delta_i = (\sigma \circ \delta_i)'$. Hence, $\mathrm{id} = (- \circ d)' - d \circ (-)'$. So the chain complex $C^*_{\mathsf{sing}}(X,R)$ is exact.

2. Consider the free sheaves $\mathbb{Z}h_{U\cap V}, \mathbb{Z}h_U, \mathbb{Z}h_V, \mathbb{Z}h_{U\cup V}$ represented by $U \cap V, U, V, U \cup V$ respectively. Explicitly, if $W \subseteq M$ is connected then

$$\mathbb{Z}h_X(W) = \begin{cases} \mathbb{Z} & W \subseteq X \\ 0 & W \not\subseteq X \end{cases}$$

and if $W = \sqcup_{i\in I} W_i$ then $\mathbb{Z}h_X(W) = \prod_{i\in I} \mathbb{Z}h_X(W_i)$. In the category of sheaves, the sequence

$$0 \to \mathbb{Z}h_{U\cap V} \to \mathbb{Z}h_U \oplus \mathbb{Z}h_V \to \mathbb{Z}h_{U\cup V} \to 0$$

is exact. For any injective sheaf I the functor $\hom(-, I)$ is exact (by definition). So applying $\hom(-, I^*)$ to this short exact sequence gives a short exact sequence of complexes of abelian groups. Then the long exact sequence in the statement is the long exact sequence associated to this short exact sequence of chain complexes.

3. This follows from the theorem that M admits a "good" covering,[5] [Bott&Tu, Thm.5.1]. That is, covering $\{U_i \to M\}_{i\in I}$ such that all finite intersections $U_{i_1} \cap \cdots \cap U_{i_n}$ are contractible. Since M is compact, we can assume I is finite. Then one can use an induction argument and Mayer-Vietoris. □

Example 9. Define

$$S^m = \{(x_0, \dots, x_m) \in \mathbb{R}^{m+1} \mid x_0^2 + \cdots + x_m^2 = 1\}$$

with the topology induced from $\mathbb{R}^{m+1}$. For $m \geq 1$ we have

$$H^n_{\mathsf{Betti}}(S^m, R) = \begin{cases} R & n = 0, m \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

We prove this by induction on m. Consider $U_+ = S^m \setminus \{(-1, 0, \dots, 0)\}$ and $U_- = S^m \setminus \{(1, 0, \dots, 0)\}$. Notice that both are homeomorphic to $\mathbb{R}^m$, and the intersection is homotopic to S^{m-1}. Then (2) follows by induction from the Mayer-Vietoris sequence, and the homotopies $U_+ \sim \{*\}$, $U_- \sim \{*\}$, $U_+ \cap U_- \sim S^{m-1}$.

Remark 10. Note that the underlying topological space of $\mathbb{A}^n(\mathbb{C}) \setminus \{0\}$ is homeomorphic to $\mathbb{R}^{2n} \setminus \{0\}$, which is homotopic to S^{2n-1} (via $(x, t) \mapsto \frac{x}{(1-t)+t||x||}$).

Exercise 2. Add details to the argument in Example 9. I.e., explicitly give homotopies $U_\pm \sim \{*\}$ and $U_+ \cap U_- \sim S^{m-1}$, write out the long exact sequence, and use the induction hypothesis for $m - 1$ to deduce the result for m. Note that $S^0 = \{-1, 1\} \subseteq \mathbb{R}$ so $H^0_{\mathsf{Betti}}(S^0, R) \cong R \oplus R$, and $H^n_{\mathsf{Betti}}(S^0, R) \cong 0$ for $n > 0$.

Theorem 11.

1. *(Künneth Formula) Cf. Voisin, Thm.11.38. For smooth manifolds M, M', and any field*[6] *K (e.g., $\mathbb{Q}, \mathbb{R}, \mathbb{C}$) there is a canonical isomorphism*

$$H^n_{\mathsf{Betti}}(M \times M', K) \cong \bigoplus_{i+j=n} H^i_{\mathsf{Betti}}(M, K) \otimes_K H^j_{\mathsf{Betti}}(M', K)$$

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[5] Bott&Tu prove this by choosing a Riemannian metric on M (this can be done by gluing together Riemannian metrics on an open covering using partitions of unity) and then observing that every (i) point admits a neighbourhood which is geodesically convex, (ii) any intersection of geodesically convex opens is again geodesically convex, and (iii) geodesically convex opens are contractible. An alternative proof is to use the fact that differentiable manifolds admit triangulations [Munkres, Elementary Differentiable Topology, Chapter II], and that the open stars of the vertices form a good cover.

[6] If $H^*(M, \mathbb{Z}), H^*(M', \mathbb{Z})$ are finitely generated, e.g., if M, M' are compact (cf. Voisin, Rema.4.46) then the result is also true for $K = \mathbb{Z}$.

2. *(Poincaré duality) Cf. Hatcher, Prop.3.38, Voisin, Thm.5.30, Rem.5.31. For X a connected smooth projective complex variety of complex dimension n then there is canonical isomorphism*

$$H^{2n}_{\mathsf{Betti}}(X(\mathbb{C}), R) \cong R.$$

which, when combined with the map

$$H^i_{\mathsf{Betti}}(X) \otimes H^{2n-i}_{\mathsf{Betti}}(X) \overset{Kunneth}{\to} H^{2n}_{\mathsf{Betti}}(X \times X) \overset{diag.}{\to} H^{2n}_{\mathsf{Betti}}(X)$$

induces isomorphisms

$$H^i_{\mathsf{Betti}}(X(\mathbb{C}), R) \cong H^{2n-i}_{\mathsf{Betti}}(X(\mathbb{C}), R)^\vee$$

when R is a field, or when torsion is factored out of $H^*_{\mathsf{Betti}}(X(\mathbb{C}), \mathbb{Z})$.

These isomorphisms are easier to describe for H_{dR} so we omit details here.

Example 12. The cohomology of a torus is:

$$H^n_{\mathsf{Betti}}(\underbrace{S^1 \times \cdots \times S^1}_{m \text{ times}}, R) \cong \bigoplus_{i=1}^{\binom{m}{n}} R.$$

Remark 13. Note that the underlying topological space $E(\mathbb{C})$ of a complex projective elliptic curve E is homeomorphic to $S^1 \times S^1$. More generally, The underlying topological space of a complex abelian variety is homeomorphic to $S^1 \times \cdots \times S^1$.

Exercise 3. Prove the isomorphism of Example 12 by induction using the Künneth Formula starting with the base case $m = 1$, cf. Exercise 9.

Definition 14. *If $U \subseteq M$ is an open immersion of topological spaces, the relative cohomology is defined as*

$$H^n_{\mathsf{Betti}}(M, U; R) = H^{n-1}\Big(Cone(I^\bullet(M) \to I^\bullet(U))\Big)$$

For any injective resolution $\underline{R} \to I^0 \to I^1 \to \dots$ (e.g., $F^\bullet = C^\bullet_{\mathsf{sing}}$).

Remark 15. By definition, there is a long exact sequence

$$\begin{aligned} \cdots \to H^n_{\mathsf{Betti}}(X, U; R) &\to H^n_{\mathsf{Betti}}(X, R) \\ &\to H^n_{\mathsf{Betti}}(U, R) \to H^{n+1}_{\mathsf{Betti}}(X, U; R) \to \dots \end{aligned}$$

Theorem 16 (Thom isomorphism)**.** *Cf. Hatcher, Cor.4D.9, Thm.4D.10, Voisin, Proof of Lem.11.13. Let $Y \subseteq X$ be a closed complex submanifold of (complex) codimension c. Then*

$$H^j_{\mathsf{Betti}}(X, X \setminus Y; \mathbb{Z}) \cong H^{j-2c}_{\mathsf{Betti}}(Y, \mathbb{Z})$$

for $j \geq 2c$.

Some ideas of a proof (probably omitted from lecture, depending on time). The vague idea is to replace X with a small open neighbourhood of Y in X, then replace this open neighbourhood with the normal bundle to Y in X. In this way, we can assume $Y \to X$ is the zero section of a vector bundle (of real rank $2c$). If the vector bundle is trivial, i.e., $X \cong Y \times \mathbb{R}^{n-2c}$ then Künneth reduces the calculation to the case $Y = \{*\}$, in which case it is straightforward using the long exact sequence of Remark 15 and the homotopy $\mathbb{R}^{2c} \setminus \{0\} \sim S^{2c-1}$. $\square$

3 Smooth manifolds

Differential 1-forms. Given a smooth (real) manifold M let $C^\infty(M)$ denote the ring of infinitely differentiable functions $M \to \mathbb{R}$. To an open subset $U \subseteq \mathbb{R}^n$ we consider the free $C^\infty(U)$-module with generators $dx_1, \dots, dx_n$

$$A^1(U) := \oplus_{i=1}^n C^\infty(U) dx_i.$$

This comes equipped with a morphism of real vector spaces

$$\begin{aligned} d : C^\infty(U) &\to A^1(U); \\ f &\mapsto \Sigma_{i=1}^n \partial_{x_i} f dx_i \end{aligned}$$

satisfying the Leibniz rule $d(fg) = fdg + gdf$.

Differential k-forms. More generally, we define $A^k(U)$ as the kth alternating power

$$A^k(U) = \bigwedge_{C^\infty(U)}^{k} A^1(U).$$

Concretely, $A^k(U)$ is the free $C^\infty(U)$-module with basis

$$\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}.$$

Note that as a special case $(1 \le n)$ we have

$$A^0(U) = C^\infty(U).$$

The cga structure. The sequence $A^0(U), \dots, A^n(U)$ is equipped with a structure of commutative graded algebra where for $f, g \in C^\infty(U), \omega \in A^k(U), \nu \in A^j(U)$, we have

$$(f\omega) \wedge (g\nu) = fg(\omega \wedge \nu); \qquad \omega \wedge \nu = -\nu \wedge \omega.$$

In particular,

$$dx_{i_1} \wedge \dots dx_j \wedge dx_k \dots \wedge dx_{i_k} = -dx_{i_1} \wedge \dots dx_k \wedge dx_j \dots \wedge dx_{i_k}.$$

The cdga structure. There is a unique morphism of $C^\infty(U)$-modules

$$d : A^*(U) \to A^*(U)$$

which restricts to $d : A^0(U) = C^\infty(U) \to A^1(U)$ in degree zero, and satisfies Leibniz

$$d(\omega \wedge \nu) = \omega \wedge d\nu + d\omega \wedge \nu.$$

Functoriality of 1-forms. Continuing with our open $U \subseteq \mathbb{R}^n$, every C^∞ map $\psi : U \to \mathbb{R}^m$ induces a ring homomorphism (in the opposite direction)

$$C^\infty(\mathrm{im}(\psi)) \to C^\infty(U)$$

by composition $f \mapsto f \circ \psi$. It furthermore induces a morphism of $C^\infty(\mathrm{im}(\psi))$-modules

$$A^1(\mathrm{im}(\psi)) \to A^1(U)$$

determined by

$$dx_i \mapsto \sum (\tfrac{\partial}{\partial x_j} \psi_i) dx_j.$$

Note that if $\psi : U \to \psi(U)$ happens to be a diffeomorphism, i.e., there exists a C^∞ inverse $\phi : \psi(U) \to U$, then

$$A^1(\mathrm{im}(\psi)) \to A^1(U)$$

is also an isomorphism of modules.

Gluing 1-forms. Suppose M is a smooth manifold of dimension n, equipped with an open covering $\{U_i \subseteq M\}_{i \in I}$ and charts $\psi_i : U_i \to \mathbb{R}^n$. Notice we have

$$C^\infty(M) = \ker \left(\prod_{i \in I} C^\infty(U_i) \to \prod_{i,j \in I} C^\infty(U_i \cap U_j) \right)$$

where the morphism is difference of the two restrictions

$$C^\infty(U_i) \to C^\infty(U_i \cap U_j)$$

and

$$C^\infty(U_j) \to C^\infty(U_i \cap U_j).$$

We use the same procedure to define $A^1(M)$ as

$$A^1(M) = \ker \left(\prod_{i \in I} A^1(\psi_i(U_i)) \to \prod_{i,j \in I} A^1(\psi_i(U_i \cap U_j)) \right)$$

Here, we are using the difference of the restriction morphism

$$\mathrm{res} : A^1(\psi_i(U_i)) \to A^1(\psi_i(U_i \cap U_j))$$

and the change of chart isomorphism

$$A^1(\psi_j(U_j)) \overset{\mathrm{res}}{\to} A^1(\psi_j(U_i \cap U_j)) \overset{d(\psi_i \circ \psi_j^{-1})}{\cong} A^1(\psi_i(U_i \cap U_j)).$$

Example 17. Consider $S^2 \subset \mathbb{R}^3$ with two charts given by stereographic projection from $(-1,0,0)$ and $(1,0,0)$ respectively, with (s,t) denoting standard coordinates on $\mathbb{R}^2$:

$$\begin{aligned} \psi_+ : U_+ := S^2 \setminus \{(-1,0,0)\} \to \mathbb{R}^2, \qquad & (x_0,x_1,x_2) \mapsto (s,t) := \tfrac{1}{1+x_0}(x_1,x_2), \\ \psi_- : U_- := S^2 \setminus \{(1,0,0)\} \to \mathbb{R}^2, \qquad & (x_0,x_1,x_2) \mapsto (s,t) := \tfrac{1}{1-x_0}(x_1,x_2). \end{aligned}$$

We have $\psi_\pm(U_+ \cap U_-) = \mathbb{R}^2 \setminus \{0\}$, and one computes that the transition map

$$\phi := \psi_+ \circ \psi_-^{-1} : \mathbb{R}^2 \setminus \{0\} \xrightarrow{\sim} \mathbb{R}^2 \setminus \{0\}, \qquad (s,t) \mapsto \tfrac{1}{s^2+t^2}(s,t)$$

is an involution ($\phi \circ \phi = \mathrm{id}$). Its pullback on 1-forms is

$$\begin{aligned} \phi^*(ds) &= \tfrac{t^2-s^2}{(s^2+t^2)^2}\, ds - \tfrac{2st}{(s^2+t^2)^2}\, dt, \\ \phi^*(dt) &= -\tfrac{2st}{(s^2+t^2)^2}\, ds + \tfrac{s^2-t^2}{(s^2+t^2)^2}\, dt. \end{aligned}$$

By the gluing construction, a 1-form on S^2 is a pair $(\omega_+, \omega_-) \in A^1(\mathbb{R}^2) \times A^1(\mathbb{R}^2)$ satisfying

$$\omega_+\big|_{\mathbb{R}^2 \setminus \{0\}} = \phi^*\Big(\omega_-\big|_{\mathbb{R}^2 \setminus \{0\}}\Big).$$

For a nontrivial example, consider $\omega = dx_0\big|_{S^2}$. In ψ_+ coordinates $x_0 = \frac{1-\rho^2}{1+\rho^2}$, while in ψ_- coordinates $x_0 = \frac{\rho^2-1}{1+\rho^2}$ (where $\rho^2 = s^2 + t^2$), so the two chart expressions have opposite sign:

$$\omega_+ = \tfrac{-4}{(1+\rho^2)^2}(s\,ds + t\,dt), \qquad \omega_- = \tfrac{4}{(1+\rho^2)^2}(s\,ds + t\,dt).$$

The compatibility $\omega_+ = \phi^*\omega_-$ is non-trivial: one computes $\phi^*(s\,ds + t\,dt) = -\frac{1}{\rho^4}(s\,ds + t\,dt)$, so

$$\phi^*\omega_- = \frac{4\rho^4}{(1+\rho^2)^2} \cdot \Big(-\frac{1}{\rho^4}\Big)(s\,ds + t\,dt) = \frac{-4}{(1+\rho^2)^2}(s\,ds + t\,dt) = \omega_+.$$

Exercise 4. Show that the above definition of $A^1(M)$ does not depend on the choice of open covering or charts.

Gluing cdga's. Now we extend everything above to manifolds. For $U \subseteq \mathbb{R}^n$, our C^∞ map $U \to \mathbb{R}^m$ extends to a morphism of cdgas $A^*(\psi(U)) \to A^*(U)$ which is an isomorphism if there is a C^∞ inverse and we can use these isomorphisms to define

$$A^k(M) = \ker\left(\prod_{i \in I} A^k(\psi_i(U_i)) \to \prod_{i,j \in I} A^k(\psi_i(U_i \cap U_j))\right)$$

where we use the difference of the restriction morphism

$$\mathrm{res} : A^k(\psi_i(U_i)) \to A^k(\psi_i(U_i \cap U_j))$$

and the change of chart isomorphism

$$A^k(\psi_j(U_j)) \overset{\mathrm{res}}{\to} A^k(\psi_j(U_i \cap U_j)) \overset{d(\psi_i \circ \psi_j^{-1})}{\cong} A^k(\psi_i(U_i \cap U_j)).$$

Exercise 5. Suppose that $U, V \subseteq \mathbb{R}^n$ are opens and $\psi : U \to V$ is C^∞. Show that the induced map $A^n(V) \to A^n(U)$ sends $dx_1 \wedge \cdots \wedge dx_n$ to the n-form $\det[\frac{\partial \psi_j}{\partial x_i}] dx_1 \wedge \cdots \wedge dx_n$.

Exercise 6. Show that the cdga structure on the $A^*(\psi_i(U_i))$ induce a structure of cdga on $A^*(M)$.

Exercise 7. Show that the cdga $A^*(M)$ is functorial for morphisms of smooth manifolds. That is, if $M \to N$ is a C^∞ morphism of smooth manifolds, then there is an induced morphism of cdgas $A^*(N) \to A^*(M)$.

In particular, if a manifold M is embedded as a smooth submanifold $M \subseteq \mathbb{R}^N$ for some N, then we obtain a morphism of cdga's

$$A^*(\mathbb{R}^N) \to A^*(M).$$

Definition 18. *The* de Rham cohomology *of a smooth manifold M is the cohomology of the above chain complex*

$$H^k_{dR}(M, \mathbb{R}) = \frac{\ker(d : A^k(M) \to A^{k+1}(M))}{\operatorname{im}(d : A^{k-1}(M) \to A^k(M))}$$

Example 19 (De Rham cohomology of S^m)**.** For $m \geq 1$,

$$H^k_{dR}(S^m, \mathbb{R}) \cong \begin{cases} \mathbb{R} & k = 0, m \\ 0 & \text{otherwise.} \end{cases}$$

$H^0 = \mathbb{R}$ since S^m is connected. That $H^k = 0$ for $0 < k < m$ follows by induction on m using Mayer-Vietoris applied to $U_\pm = S^m \setminus \{(\mp 1, 0, \ldots, 0)\}$, using $U_\pm \sim \{*\}$ and $U_+ \cap U_- \sim S^{m-1}$ (cf. Example 9). For H^m: every m-form is automatically closed since $A^{m+1}(S^m) = 0$. Stokes Theorem says that when M is a compact m-dimensional manifold (without boundary), for any $\alpha \in A^{m-1}(M)$ we have $\int_M d\alpha = 0$ so to find a generator for $H^m_{dR}(S^m, \mathbb{R})$ it suffices to produce a global m-form ω with $\int_{S^m} \omega \neq 0$. We claim that the restriction of

$$\omega = \sum_{i=0}^{m} (-1)^i x_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m \in A^m(\mathbb{R}^{m+1})$$

to S^m is such an m-form. Since $(-1, 0, \ldots, 0)$ has measure zero, it suffices to show $\int_{S^m \setminus (-1,0,\ldots,0)} \omega \neq 0$. Via the diffeomorphism $\psi_+ : S^m \setminus (-1, 0, \ldots, 0) \xrightarrow{\sim} \mathbb{R}^m$; $(x_0, \ldots, x_m) \mapsto (t_1, \ldots, t_n) = \frac{1}{1+x_0}(x_1, \ldots, x_m)$, our form becomes

$$(\psi_+^{-1})^* \omega = \left(\frac{2}{1 + \rho^2} \right)^m dt_1 \wedge \cdots \wedge dt_m,$$

where $\rho^2 = \sum_{i=1}^m t_i^2$ so $\int_{S^m} \omega = \int_{\mathbb{R}^m} \left(\frac{2}{1+\rho^2} \right)^m dt_1 \cdots dt_m > 0$.

Theorem 20. *Let M be a smooth manifold. Then*

$$H^i_{\mathsf{Betti}}(M, \mathbb{R}) \cong H^i_{\mathsf{dR}}(M, \mathbb{R}).$$

Sketch of proof. Poincaré's Lemma (Bott&Tu,§4) says that $H^i_{\mathsf{dR}}(\mathbb{R}^n, \mathbb{R}) = 0$ for $i > 0$ and $H^0_{\mathsf{dR}}(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}$ so we deduce that the sequence of sheaves

$$0 \to \underline{\mathbb{R}} \to A^0(M) \to A^1(M) \to \cdots \to A^n(M) \to 0$$

is exact. Then one shows that each A^k is a "fine sheaf" (cf. Voisin, Def.4.35), and that resolutions of fine sheaves calculate cohomology (cf. Voisin, Prop.4.36, Cor.4.37). □

We deduce from Theorem 20 that de Rham cohomology satisfies:

1. Homotopy Invariance,
2. Mayer-Vietoris,
3. Poncaré Duality,
4. Künneth Formula, and
5. the Thom Isomorphism.

However, de Rham cohomology makes Poincaré duality and Künneth a little easier to state. Recall that an *orientation* on a smooth n-dimensional manifold M is a choice of $\omega \in A^n(M)$ which is nowhere zero.

Example 21. The canonical orientation on S^m is the restriction to S^m of

$$\omega = \sum_{i=0}^{m} (-1)^i x_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m \in A^m(\mathbb{R}^{m+1}).$$

To see this is nowhere zero, note that by symmetry in the coordinates, it suffices to check on the chart $\psi_+ : S^m \setminus (-1, 0, \ldots, 0) \xrightarrow{\sim} \mathbb{R}^m$; $(x_0, \ldots, x_m) \mapsto (t_1, \ldots, t_n) = \frac{1}{1+x_0}(x_1, \ldots, x_m)$, where our form becomes

$$(\psi_+^{-1})^* \omega = \left(\frac{2}{1+\rho^2} \right)^m dt_1 \wedge \cdots \wedge dt_m,$$

where $\rho^2 = \sum_{i=1}^m t_i^2$.

Theorem 22 (Poincaré Duality, Cf. Voisin, Thm.5.30)**.** *Let M be an n-dimensional smooth connected compact orientable manifold. Then $A^n(M) \to \mathbb{R}; \omega \mapsto \int_M \omega$ defines an isomorphism*

$$H^n_{dR}(M, \mathbb{R}) \xrightarrow{\sim} \mathbb{R}.$$

Moreover, the map $A^p(M) \otimes_{\mathbb{R}} A^{n-p}(M) \to A^n(M); (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$ induces a perfect pairing on cohomology. That is, it induces an isomorphism

$$H^p(M, \mathbb{R}) \cong H^{n-p}(M, \mathbb{R})^\vee$$

(which depends on the isomorphism $H^n_{dR}(M, \mathbb{R}) \cong \mathbb{R}$ above).

Theorem 23 (Künneth Formula, Cf. Voisin, §11.3.3, Griffiths&Harris, pg.103,104)**.** *Let M, M' be smooth manifolds. Then the map*

$$A^k(M) \times A^{k'}(M') \to A^{k+k'}(M \times M'); \qquad \alpha \otimes \beta \mapsto pr_1^*\alpha \wedge pr_2^*\beta$$

induces the Künneth Formula

$$H^k_{\mathsf{dR}}(M, \mathbb{R}) \otimes H^k_{\mathsf{dR}}(M', \mathbb{R}) \xrightarrow{\sim} H^{k+k'}_{\mathsf{dR}}(M \times M', \mathbb{R}).$$

Here, $pr_1 : M \times M' \to M, pr_2 : M \times M' \to M'$ are the canonical projections.

4 Complex manifolds

Definition 24. *Consider the canonical identification*

$$\mathbb{C}^\nu \cong \mathbb{R}^{2\nu} : (x_1 + iy_1, \dots, x_\nu + iy_\nu) \leftrightarrow (x_1, y_1, x_2, y_2, \dots, x_\nu, y_\nu).$$

For an open subset $U \subseteq \mathbb{R}^{2\nu}$ we define

$$J : A^1(U) \to A^1(U); \qquad \begin{cases} dx_j \mapsto dy_j \\ dy_j \mapsto -dx_j \end{cases}$$

For any $U \subseteq \mathbb{R}^{2n}$, a smooth function $\psi : U \to \mathbb{R}^{2m}$ is called holomorphic *if $J \circ d\psi = d\psi \circ J$.*

Exercise 8. Show that ψ is holomorphic if and only if the Cauchy-Riemann equations hold. That is, if and only if

$$\partial_{y_i} u_j = -\partial_{x_i} v_j, \qquad \text{and} \qquad \partial_{y_i} v_j = \partial_{x_i} u_j.$$

for each i, j, where $\psi(z) = (u_1(z), v_1(z), \dots, u_m(z), v_m(z))$ and $z = (x_1, y_1, \dots, x_n, y_n)$.

Exercise 9. Show that the morphism $J : A^1(U) \to A^1(U)$ induces a complex vector space structure on $A^1(U)$ by

$$(a + ib)\omega = (a + bJ)\omega.$$

Observe that a smooth function $\psi : U \to \mathbb{R}^{2m}$ is holomorphic if and only if the $\mathbb{R}$-linear morphism $d\psi : A^1(\mathbb{R}^{2m}) \to A^1(U)$ is actually $\mathbb{C}$-linear.

Exercise 10. Show that:

1. A composition of holomorphic functions is holomorphic.
2. A function $U \to \mathbb{R}^{2m}$ is holomorphic if and only if the composition with each projection $U \to \mathbb{R}^{2m} \to \mathbb{R}^2; (x_1, y_1, \dots, x_m, y_m) \mapsto (x_i, y_i)$ is holomorphic.
3. A sum of holomorphic functions $U \to \mathbb{R}^2$ is holomorphic.

4. Via the multiplication induced by $\mathbb{C} \cong \mathbb{R}^2$, a product of holomorphic functions $U \to \mathbb{R}^2$ is holomorphic.
5. Any polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ defines a holomorphic function $\mathbb{R}^{2n} \cong \mathbb{C}^n \to \mathbb{C} \cong \mathbb{R}^2$.
6. If a function $\phi : U \to \mathbb{R}^{2m}$ admits a smooth inverse $\psi : \phi(U) \to U$, then ϕ is holomorphic if and only if ψ is holomorphic.

Now suppose that X is a smooth complex manifold of dimension n. That is, X is equipped with charts $\{U_i \subseteq X; \phi_i : U_i \to \mathbb{R}^{2n}\}_{i \in I}$ such that the transition maps $\phi_i \circ \phi_j^{-1}$ are holomorphic. Then we obtain an induced automorphism

$$J : A^1 \to A^1$$

of the sheaf A^1. We now define

$$A^1_{\mathbb{C}}(U) := A^1(U) \otimes_{\mathbb{R}} \mathbb{C}.$$

Since $J^2 = -1$, there are two eigenvalues $-i$ and i. Their eigenspaces are denoted $A^{1,0}(U)$, $A^{0,1}(U)$ respectively. Since the transition maps are holomorphic, they preserve the eigenspaces, and we get a decomposition of sheaves

$$A^1_{\mathbb{C}} = A^{1,0} \oplus A^{0,1}.$$

Example 25. Consider $X = \mathbb{R}^{2n}$. Then $A^{1,0}(X)$ (resp. $A^{0,1}(X)$) has basis

$$dz_j := dx_j + i dy_j, \qquad (\text{resp. } \; d\overline{z}_j := dx_j - i dy_j), \qquad j = 1, \ldots, n$$

as a free $C^\infty(\mathbb{R}^{2n})$-module

Exercise 11. Check that $J dz = -i dz$ and $J d\overline{z} = i d\overline{z}$.

The eigenspace decoposition $A^1_{\mathbb{C}} = A^{1,0}(X) \oplus A^{0,1}(X)$ induces a decomposition of $A^n_{\mathbb{C}}(X) := A^n(X) \otimes_{\mathbb{R}} \mathbb{C}$ as

$$A^n_{\mathbb{C}}(X) = \sum_{p+q=n} A^{p,q}(X); \qquad A^{p,q}(X) := \bigwedge^p A^{1,0} \otimes \bigwedge^q A^{0,1}$$

(the tensor and wedge products are over the sheaf C^∞).

Example 26. $A^{p,q}(\mathbb{R}^{2n})$ is spanned by the

$$dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{j_{p+1}} \wedge \cdots \wedge d\overline{z}_{p+q}$$

as a $C^\infty(\mathbb{R}^{2n})$-module

Exercise 12. Let V, W be finitely generated free R-modules for some ring R. Show that there is a canonical isomorphism

$$\bigwedge^n (V \oplus W) \cong \sum_{p+q=n} \left(\bigwedge^p V \otimes \bigwedge^q W \right).$$

In analogy with $dz, d\overline{z}$ we define

$$\partial_z := \tfrac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\overline{z}} := \tfrac{1}{2}(\partial_x + i\partial_y)$$

Exercise 13. For $U \subseteq \mathbb{R}^{2n}$, show that with respect to the basis $dz_j, d\overline{z}_j$ the differential $C^\infty(U) \otimes_{\mathbb{R}} \mathbb{C} \to A^1_{\mathbb{C}}(U)$ becomes $f \mapsto \sum_{j=1}^n (\partial_{z_j} f dz_j + \partial_{\overline{z}_j} f d\overline{z}_j)$.

Theorem 27 (Cf. Voisin, Proof of Prop.6.11)**.** *Let X be a smooth projective variety. Then there is a canonical decomposition*

$$H^k_{\mathsf{dR}}(X(\mathbb{C}), \mathbb{R}) \otimes \mathbb{C} \cong \bigoplus_{p+q=k} H^{p,q}$$

where $H^{p,q}$ is the set of cohomology classes representable by a closed form of type (p,q). Moreover, we have

$$\overline{H^{p,q}} = H^{q,p}$$

where complex conjugation acts via the $- \otimes_{\mathbb{R}} \mathbb{C}$.

Remark 28 (Orientability of smooth complex varieties)**.** The real manifold associated to any smooth projective complex variety is equipped with an orientation. In fact, the real manifold associated to any smooth complex manifold is equipped with an orientation. In particular, we can apply Poincaré duality to any manifold coming from a smooth projective complex variety.

To see this, first note that $dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n \in A^{2n}(\mathbb{R}^{2n})$ is nowhere zero. Now if we have opens $U, V \subseteq \mathbb{R}^{2n}$ and a holomorphic function $\phi : U \to V$, then by the definition of holomorphicity, for every $u \in U$ the matrix $[\frac{\partial \phi_j}{\partial t_i}(u)]$ is in the image of $GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$, see Exercise 10. Here $(t_1, \dots, t_{2n}) = (x_1, y_1, \dots, x_n, y_n)$ and $GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$ is defined by replacing each entry $a + ib \in \mathbb{C}$ with the matrix $[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}]$.

By Exercise 14, the eigenvalues of any matrix in the image of $GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$ come in conjugate pairs $\lambda_1, \overline{\lambda}_1, \dots, \lambda_n, \overline{\lambda}_n$, so in particular, the determinant $\det = \lambda_1 \overline{\lambda}_1 \dots \lambda_n \overline{\lambda}_n = |\lambda_1| \dots |\lambda_n|$ is > 0. By Exercise 5, the induced map $A^{2n}(V) \to A^{2n}(U)$ sends $dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n$ to $\det[\frac{\partial \phi_j}{\partial t_i}] dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n$, and we have just shown that the function $\det[\frac{\partial \phi_j}{\partial t_i}]$ on U is strictly positive everywhere.

So what we now have is: if M is the real manifold associated to a smooth complex manifold, there exists an open covering $U_i \subseteq M$ and nowhere zero forms $\omega_i \in A^{2n}(U_i)$ such that $\omega_i|_{U_i \cap U_j} = f_{ij} \omega_j|_{U_i \cap U_j}$ for some strictly positive $f_{ij} \in C^\infty(U_i \cap U_j)$.

Now choose a partition of unity[7] $\{\rho_i\}$ subordinate to $\{U_i\}$. Then $\omega = \sum_i \rho_i \omega_i$ will be a global nowhere vanishing $2n$-form on M. Indeed, on each U_j we have $\omega = (\sum_i \rho_i f_{ij}) \omega_j$ with $(\sum_i \rho_i f_{ij})$ strictly positive.[8]

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[7] A *partition of unity* subordinate to $\{U_i\}$ is a collection of smooth functions $\rho_i : M \to [0,1]$ with $\mathrm{supp}(\rho_i) \subseteq U_i$ and $\sum_i \rho_i = 1$. Such a partition exists for any open cover of a smooth manifold.

[8] Note that if $u_1, \dots, u_m \in [0,1]$ satisfy $u_1 + \dots + u_m = 1$ then for $v_1, \dots, v_m \in \mathbb{R}_{>0}$ we have $u_1 v_1 + \dots + u_m v_m > 0$.

Exercise 14. Prove the claim in the above remark, that for any matrix in the image of $GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$, the eigenvalues come in conjugate pairs $\lambda_1, \overline{\lambda}_1, \dots, \lambda_n, \overline{\lambda}_n$.

Theorem 29 (Cf. Voisin, Lem.7.30)**.** *Poincaré duality is compatible with the Hodge decomposition, in the sense that it induces isomorphisms*

$$H^{r,s} \cong (H^{d-r,d-s})^*$$

In particular, for $r' + s' = 2d - r - s$, and $(r', s') \neq (d - r, d - s)$, the morphism

$$H^{r,s} \otimes H^{r',s'} \to \mathbb{C}$$

is zero.

Proof. The Poincaré duality pairing on $H^{r,s} \otimes H^{r',s'}$ is $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$. Since $\alpha \wedge \beta$ is of type $(r + r', s + s')$, and X has no nonzero forms of type (p, q) with $p > d$ or $q > d$, we have $\alpha \wedge \beta = 0$ unless $(r + r', s + s') = (d, d)$, i.e., $(r', s') = (d - r, d - s)$. Thus the perfect pairing $H^k \otimes H^{2d-k} \to \mathbb{C}$ of Theorem 22 decomposes as an orthogonal direct sum of pairings $H^{r,s} \otimes H^{d-r,d-s} \to \mathbb{C}$. Since the total pairing is non-degenerate, each component is non-degenerate, giving $H^{r,s} \cong (H^{d-r,d-s})^*$. □

Theorem 30 (Künneth Formula, Cf. Voisin, §11.3.3, Griffiths&Harris, pg.103,104)**.** *Künneth is compatible with the Hodge decomposition in the sense that if X, Y are smooth complex projective varieties, then it induces isomorphisms*

$$H^{r,s}(X(\mathbb{C}) \times Y(\mathbb{C})) \cong \bigoplus_{\substack{p+p'=r,\\ q+q'=s}} H^{p,q}(X(\mathbb{C})) \otimes_{\mathbb{C}} H^{p',q'}(Y(\mathbb{C})).$$

5 Pure Hodge structures

Definition 31. *(Cf. Voisin, Def.7.4) A* pure Hodge structure *of weight k, is a free finitely generated abelian group V, equipped with a decomposition*

$$V \otimes \mathbb{C} \cong \bigoplus_{p+q=k} V^{p,q}$$

satisfying $V^{p,q} = \overline{V^{q,p}}$.

(Cf. Voisin, Def.7.22) A morphism *$(V, V^{p,q}) \to (W, W^{p,q})$ from a Hodge structure of weight n to a Hodge structure of weight m is a morphism $\phi : V \to W$ is of abelian groups such that $\phi \otimes \mathbb{C}(V^{p,q}) \subseteq W^{p+r,q+r}$ where $m = n + 2r$.*

(Cf. Voisin, Def.11.39) The tensor product *$(K, K^{r,s})$ of $(V, V^{p,q})$ and $(W, W^{p',q'})$ is defined to be the Hodge structure of weight $n + m$ whose abelian group is $K = V \otimes W$ and whose decomposition is*

$$K^{r,s} = \bigoplus_{\substack{p+p'=r,\\ q+q'=s}} V^{p,q} \otimes_{\mathbb{C}} W^{p',q'}.$$

Example 32. Putting together all of the above material, we get: If X is a smooth complex projective variety, then each

$$H^k_{\mathsf{Betti}}(X(\mathbb{C}),\mathbb{Z})/\text{torsion}$$

is canonically equipped with a pure Hodge structure of weight k. Moreover, if Y is another smooth complex projective variety, then Künneth induces an isomorphism of pure Hodge structures of weight $k+l$

$$\frac{H^k_{\mathsf{Betti}}(X(\mathbb{C}),\mathbb{Z})}{\text{torsion}} \otimes \frac{H^l_{\mathsf{Betti}}(Y(\mathbb{C}),\mathbb{Z})}{\text{torsion}} \xrightarrow{\sim} \frac{H^{k+l}_{\mathsf{Betti}}((X\times Y)(\mathbb{C}),\mathbb{Z})}{\text{torsion}}.$$

Exercise 15. In this exercise we will classify all pure Hodge structures of weight 1 with $\dim_{\mathbb{C}} V\otimes\mathbb{C} = 2$.

(a) Show that $V \cong \mathbb{Z}^2$ and

$$\dim_{\mathbb{C}} V^{p,q} = \begin{cases} 1 & (p,q) = (n,1-n),(1-n,n) \\ 0 & \text{otherwise} \end{cases}$$

for some $n \geq 1$.

(b) Fix a $\mathbb{Z}$-basis e_1, e_2 for V. Show that $V^{n,1-n} = \{\alpha(\tau e_1 + e_2) \mid \alpha \in \mathbb{C}\}$ for a unique $\tau \in \mathbb{C}\setminus\mathbb{R}$.

(c) Show that two such Hodge structures (with the same $V = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$), corresponding to parameters $\tau, \tau' \in \mathbb{C}\setminus\mathbb{R}$, are isomorphic if and only if $\tau' = \dfrac{a\tau+b}{c\tau+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$.

(d) Conclude that the set of isomorphism classes is in bijection with $\mathfrak{H}/\,\mathrm{SL}_2(\mathbb{Z})$, where $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and $\mathrm{SL}_2(\mathbb{Z})$ acts by Möbius transformations $\tau \mapsto \frac{a\tau+b}{c\tau+d}$.