In this lecture we motivate some material that we will see in the course.

1 Cohomology

We begin by considering two cohomology theories: de Rham and ℓ-adic.

1.1 De Rham

Let $k \subseteq \mathbb{C}$ be a subfield of the complex numbers. There exists a functor

$$H^{\bullet}_{dR} : \mathrm{SmProj}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{GrVec}_{\mathbb{C}}$$

from smooth projective k-varieties to the category of graded $\mathbb{C}$-vector spaces called *de Rham cohomology*. We will give more details about this next lecture. For now I just want to note the following.

For each n, there exists a free abelian group $H_{\mathbb{Z}} \in \mathrm{Ab}$, an isomorphism $H_{\mathbb{Z}} \otimes \mathbb{C} \xrightarrow{\sim} H^n_{dR}(X)$, and a decomposition

$$H_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=n} V^{p,q} \tag{1}$$

such that

$$\overline{V^{p,q}} = V^{q,p}. \tag{2}$$

(All this will be defined properly in the next lecture). Here complex conjugation is with respect to the $- \otimes \mathbb{C}$ part.

Definition 1. *A* Hodge structure of weight n *is a free finite rank abelian group $H_{\mathbb{Z}}$, equipped with a decomposition* (1) *which satisfies* (2).

Example 2. Suppose $A/\mathbb{C}$ is an abelian variety of complex dimension g. That is, A is a smooth projective variety equipped with morphisms of varieties

$$mult : A \times A \to A, \qquad inv : A \to A, \qquad unit : \mathrm{Spec}(\mathbb{C}) \to A$$

making A an abelian group. Then there is an isomorphism of complex analytic spaces[1]

$$A(\mathbb{C}) \cong V/\Lambda$$

for some complex vector space V of complex dimension g and subgroup $\mathbb{Z}^{2g} \cong \Lambda \subseteq V$ such that $\Lambda \otimes \mathbb{R} \cong V$ (as $\mathbb{R}$-vector spaces). In fact, notice that $\Lambda \otimes \mathbb{C} \cong V \otimes_{\mathbb{R}} \mathbb{C}$. There is a canonical isomorphism

$$H^1_{dR}(A) \cong (V \otimes_{\mathbb{R}} \mathbb{C})^{\vee}$$

[1] One can take V to be the complex tangent space $T_e A$ at the unit $e \in A$, and the map $V \to A(\mathbb{C})$ is the exponential map. The isomorphism to H^1_{dR} can be obtained by identifying the *CO*tangent space at e with translation invariant 1-forms.

(here $(-)^\vee := \hom_{\mathbb{C}}(-, \mathbb{C})$ is the dual complex vector space), and the decomposition $H^1_{dR}(A) \cong V^{1,0} \oplus V^{0,1}$ is induced by the decomposition of $V \otimes_{\mathbb{R}} \mathbb{C}$ into the i and $-i$ eigenspaces (multiplication by i on the right) of the map $J : v \otimes a \mapsto iv \otimes a$ (multiplication by i on the left). In particular, we can completely reconstruct the complex analytic space $A(\mathbb{C})$ from the triple $(\Lambda^\vee, H^1_{dR}(A), V^{1,0} \oplus V^{0,1})$ as $A(\mathbb{C}) \cong (V^{1,0})^\vee / \Lambda$.

Example 3. Suppose there is a sequence of closed subvarieties $\varnothing = Y_0 \subset Y_1 \subset \cdots \subset X$ such that $Y_i \setminus Y_{i-1} \cong \mathbb{A}^{n_i}$ for some n_i (e.g., projective space, Grassmannians, ...). Then

$$V^{p,q} \cong \begin{cases} 0 & p \neq q \\ \mathbb{C}^{b_p} & p = q \end{cases}$$

where $b_p = \#\{Y_i : Y_i \setminus Y_{i-1} \cong \mathbb{A}^p\}$.

Exercise 1. Let V be a free abelian group equipped with a decomposition (1) satisfying (2). Show that if $V^{p,q} = 0$ for all $p \neq q$ and n is odd, then $V = 0$.

So the Hodge structure contains a lot of information about the variety. On the other hand, it is "just" linear algebra.

1.2 ℓ-adic

Now consider any field k and ℓ a prime different from the characteristic of k. The ℓ-adic cohomology associates to each smooth projective variety X/k a graded $\mathbb{Q}_\ell$-vector space $H^\bullet_\ell(X)$. Any morphism $Y \to X$ induces a graded linear morphism $H^\bullet_{et}(X) \to H^\bullet_{et}(Y)$. Moreover, each $\mathbb{Q}_\ell$-vector space $H^\bullet_{et}(X)$ is equipped with a representation of the group $Gal(\overline{k}/k)$. In particular, any element $\sigma \in Gal(\overline{k}/k)$ induces a morphism $H^n_{et}(X) \to H^n_{et}(X)$ of $\mathbb{Q}_\ell$-vector spaces for all X, n.

Example 4. Let $k = \mathbb{F}_q$. Then the Frobenius $\phi : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q; a \mapsto a^q$ induces morphisms

$$\phi_i : H^i_{et}(X) \to H^i_{et}(X) \tag{3}$$

for each smooth projective $X/\mathbb{F}_q$, and each i. Defining

$$Z(X, t) = \prod_{i=0}^{2 \dim X} \det(\mathrm{id} - \phi_i t)^{(-1)^{i+1}} \in \mathbb{Q}_\ell(t)$$

one can show that[2]

$$\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \frac{t^n}{n} = \log Z(X, t) \in \mathbb{Q}[[t]].$$

[2]Here, log means the operation $1 + t\mathbb{Q}[[t]] \xrightarrow{\sim} t\mathbb{Q}[[t]]$ that sends $(1-g)^{-1}$ to $\sum_{m \geq 1} \frac{g^m}{m}$. Namely, the inverse to $\exp : t\mathbb{Q}[[t]] \xrightarrow{\sim} 1 + t\mathbb{Q}[[t]]$; $f \mapsto \sum_{n \geq 0} \frac{f^n}{n!}$.

That is, the $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-representations $H^0_{et}(X), \ldots, H^{2n}_{et}(X)$ know $|X(\mathbb{F}_{q^m})| = \frac{1}{(m-1)!}(\frac{d^m}{dt^m} \log Z(X,t))(0)$ the number of points X has over every extension of $\mathbb{F}_q$.

Exercise 2 (Advanced)**.** In the situation of Example 3 show that

$$Z(X,t) = \prod_{j=0}^{2\dim X} \frac{1}{(1-q^j t)^{b_j}}.$$

In particular, the eigenvalues of each ϕ_i are integral powers of q.

The Hodge structure (for complex varieties) and the ℓ-adic Galois representation (for $\mathbb{F}_q$-varieties) contains a lot of information about the variety. On the other hand, they are "just" linear algebra.

We have the following general philosophy.

$$\Big(\text{ Algebraic geometry }\Big) \overset{\text{Cohomology}}{\longrightarrow} \Big(\text{ Linear algebra }\Big)$$

If we consider not only morphisms of varieties but "correspondences" of varieties, we can extract even more information out of this linear algebra.

2 Correspondences

Recall that the Riemann(-Weil) Hypothesis states: the eigenvalues of the ϕ_n (from Equation 3) have absolute value $q^{n/2}$ (equivalently, if s is a root or pole of $Z(X, q^{-s})$, then $\Re s = n/2$).

Proposition 5 (Manin, 1968[3])**.** *The Riemann Hypothesis holds for smooth three dimensional projective unirational varieties.*

Proof. Unirational means there exists a birational morphism $W \to \mathbb{P}^3$, and a generically finite morphism $W \to V$ for some (possibly singular) projective W. By Abhyankar's resolution of singularities for threefolds in positive characteristic, we can in fact assume that W is smooth and moreover that $W \to \mathbb{P}^3$ is a sequence $W = W_n \to W_{n-1} \to \cdots \to W_1 \to W_0 = \mathbb{P}^3$ of blowups with smooth centres. For simplicity, we assume that all the centers are geometrically irreducible (changing the base field allows this, and does not affect the calculation).

Step 1, calculate $Z(W,t)$. Let $\nu_n(X) = |X(\mathbb{F}_{q^n})|$. One calculates directly that

$$\nu_n(\mathbb{P}^3) = 1 + q^n + q^{2n} + q^{3n}.$$

[3]NB. In 1968, all Weil conjectures except for the Riemann Hypothesis had been proved using étale cohomology. The Riemann Hypothesis was known at the time in certain cases, and in particular, it was known for curves. The Riemann Hypothesis was proved in full generality by Deligne in 1974.

Next, we take for granted the calculation for curves C of genus g,

$$\nu_n(C) = 1 - \sum_{i=1}^{2g} \eta_i^n + q^n$$

where η_i are some algebraic integers satisfying $|\eta_i| = q^{1/2}$. Now if $Y \to X$ is the blowup of a point x, then we have

$$\begin{aligned}\nu_n(Y) &= \nu_n(X) - \nu_n(x) + \nu_n(\mathbb{P}^2) \\ &= \nu_n(X) - 1 + 1 + q^n + q^{2n} \\ &= \nu_n(X) + q^n + q^{2n}\end{aligned}$$

− − − − draw picture of blowup − − − −

If $Y \to X$ is the blowup of a curve C, then we have

$$\begin{aligned}\nu_n(Y) &= \nu_n(X) - \nu_n(C) + \nu_n(C \times \mathbb{P}^1) \\ &= \nu_n(X) + q^n - q^n \sum_{i=1}^{2g} \eta_i^n + q^{2n}\end{aligned}$$

− − − − draw picture of blowup − − − −

It follows from this[4] that the eigenvalues of ϕ_n acting on $H^n_{et}(W)$ are of the form $1, q, q^2, q^3$, or $(q\eta)^{-1}$ where $|\eta| = q^{1/2}$.

Step 2, the trace morphism. Now consider the morphism $f : W \to V$. It induces a morphism

$$f^* : H^n_{et}(V) \to H^n_{et}(W),$$

but since it's projective and generically finite, it also induces a morphism

$$f_* : H^n_{et}(W) \to H^n_{et}(V).$$

Moreover, we have the relation

$$\tfrac{1}{\deg f} f_* f^* = \mathrm{id}\,.$$

[4] We have $\nu_n(W) = 1 + Nq^n - q^n \sum_{i,j} \eta_{ij}^n + Nq^{2n} + q^{3n}$ where N is the number of blowups and the η_{ij} depend on which curves are blown up. Putting this into $\log Z(W,t) = \sum_{n=1}^{\infty} \nu_n(W) \frac{t^n}{n}$ produces

$$\sum_{n=1}^{\infty} \tfrac{t^n}{n} + N \sum_{n=1}^{\infty} \tfrac{(qt)^n}{n} - \sum_{i,j} \sum_{n=1}^{\infty} \tfrac{(\eta_{ij} qt)^n}{n} + \sum_{n=1}^{\infty} \tfrac{(q^2 t)^n}{n} + \sum_{n=1}^{\infty} \tfrac{(q^3 t)^n}{n}$$

from which it follows that

$$Z(W,t) = \frac{\prod_{i,j}(1 - \eta_{ij} qt)}{(1-t)(1-qt)^N(1-q^2t)^N(1-q^3t)}$$

This means that we have a direct sum decomposition

$$H^n_{et}(W) \cong H^n_{et}(V) \oplus (\text{something else}).$$

We deduce that every eigenvalue of ϕ_n acting on $H^n_{et}(V)$, is an eigenvalue of ϕ_n acting on $H^n_{et}(W)$. So they are also of the form $1, q, q^2, q^3$, or $(q\eta)^{-1}$ where $|\eta| = q^{1/2}$. □

Remark 6. Key ingredients in the above proof were that we don't only have morphisms $f^* : H^n_{et}(X) \to H^n_{et}(Y)$ associated to morphisms of varieties $f : Y \to X$, but we also had morphisms $b_* : H^n_{et}(W_{i+1}) \to H^n_{et}(W_i)$ associated to the blowups $b : W_{i+1} \to W_i$ and a morphism $f_* : H^n_{et}(W) \to H^n_{et}(V)$ associated to the generically finite morphism $f : W \to V$. Manin modifies the above proof to use only the eigenvalues of ϕ_n, and *without* using the numbers ν_n.

Fact 7. Let X, Y be smooth projective over k, and $Z \subset Y \times X$ a closed irreducible subvariety with $\dim Z = \dim Y$. Then for $H^n = H^n_{et}$ or $H^n = H^n_{dR}$ (if $k \subseteq \mathbb{C}$), there is an induced morphism

$$[Z]^* : H^n(X) \to H^n(Y)$$

of Galois representations, resp. Hodge structures.

Example 8.

1. If $f : Y \to X$ is any morphism, consider the graph $\Gamma_f \subseteq Y \times X$. This defines a morphism $[\Gamma_f]^* : H^n(X) \to H^n(Y)$. In fact, this is just the usual morphism of cohomology f^* defined by f.
2. If $f : W' \to W$ is a blowup, then consider the graph $\Gamma_f \subseteq W \times W'$. This defines a morphism $[\Gamma_f]^* : H^n(W') \to H^n(W)$ (note the direction is backwards to f^*).
3. If $f : W \to V$ is a proper generically finite morphism, consider the graph $\Gamma_f \subseteq V \times W$. This defines a morphism $[\Gamma_f]^* : H^n(W) \to H^n(V)$ (note the direction is backwards to f^*).

In general, it is useful to consider sums of such subvarieties.

Definition 9. *A* cycle of dimension d *on a variety X is a formal sum* $\sum_{i=1}^N n_i Z_i$ *where $n_i \in \mathbb{Z}$ and $Z_i \subseteq X$ is a closed irreducible subvariety of dimension d. The group of cycles is denoted*

$$\mathcal{Z}_d(X) = \{\Sigma_{i=1}^N n_i Z_i\}.$$

Very often, it is more convenient to use codimension. If X is of pure dimension d then

$$\mathcal{Z}^d(X) = \mathcal{Z}_{\dim X - d}(X)$$

The reason I presented Manin's proof is to motivate the following idea:

Cycles define important morphisms of cohomology groups.

The main tool in defining the maps from Fact 7 are cycle class maps.

Fact 10. Let X/k be a smooth projective variety of dimension d, and consider $H^n = H^n_{et}$ or H^n_{dR} (if $k \subseteq \mathbb{C}$). Then there are group morphisms

$$\gamma : \mathcal{Z}^i(X) \to H^{2i}(X)$$

called *cycle class maps.*

Some of the deepest conjectures in algebraic geometry involve cycle class maps. Recall that the de Rham cohomology H^n_{dR} came with an abelian group $H_\mathbb{Z}$, an isomorphism $H_\mathbb{Z} \otimes \mathbb{C} \cong H^n_{dR}$ and a decomposition $H_\mathbb{Z} \otimes \mathbb{C} = \oplus_{p+q=n} V^{p,q}$.

Fact 11.

$$im\left(\gamma : \mathcal{Z}^p(X) \to H^{2p}_{dR}(X)\right) \subseteq V^{p,p}$$

Definition 12.

$$\mathrm{Hdg}^p(X) = \{\eta \in V : \eta \otimes 1 \in V^{p,p}\} = H_\mathbb{Z} \cap V^{p,p}$$

Conjecture 13 (Hodge)**.**

$$\mathcal{Z}^p(X) \otimes \mathbb{Q} \to \mathrm{Hdg}^p(X) \otimes \mathbb{Q}$$

is surjective.

Recall that the ℓ-adic cohomology $H^n_{et}(X)$ is a $\mathbb{Q}_\ell$-vector space, equipped with an action of the absolute Galois group $G = Gal(\overline{k}/k)$ of the base field k. That is, a group homomorphism $G \to Aut(H^n_{et}(X))$. In particular, we can consider the subspace of $H^n_{et}(X)$ on which G is constant.

$$H^n_{et}(X)^G = \{\eta \in H^n_{et}(X) : \sigma(\eta) = \eta \ \forall \ \sigma \in G\}.$$

Conjecture 14 (Tate)**.** Suppose that k is finitely generated over its prime subfield. Then

$$\mathcal{Z}^p(X) \otimes \mathbb{Q}_\ell \to H^{2p}_{et}(X)^G$$

is surjective.

Conjecturally, the kernel of the cycle class maps does not depend on which cohomology theory we chose. In characteristic zero, this follows from a theorem of Artin.

Theorem 15 (Artin)**.** *Suppose $k \subseteq \mathbb{C}$. Then*

$$\ker\left(\mathcal{Z}^p(X) \otimes \mathbb{Q} \to H^{2p}_{dR}(X)\right) = \ker\left(\mathcal{Z}^p(X) \otimes \mathbb{Q} \to H^{2p}_{et}(X)\right)$$

Combining this with the Hodge and Tate conjectures, we get the following.

Conjecture 16. Let X be a smooth projective variety over $k \subseteq \mathbb{C}$ with k finitely generated over $\mathbb{Q}$. Then

$$\mathrm{Hdg}^p(X) \otimes \mathbb{Q}_\ell \cong H^{2p}_{et}(X)^G.$$

Exercise 3. Assuming the Hodge and Tate conjectures, and using Artin's theorem, prove the above conjecture.

So we see that, conjecturally at least:

Cycles are a bridge between Hodge theory and Galois representation theory.