# Classical Algebraic Geometry

This section is just to fix notation and conventions.

**References.**

- [1], Chapter I.
  R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer, 1977.
- [2], Chapters I–II.
  I.R. Shafarevich, *Basic Algebraic Geometry*, Vol. 1, Springer, 3rd ed., 2013.
- [3], Lectures 1–3.
  J. Harris, *Algebraic Geometry: A First Course*, Graduate Texts in Mathematics 133, Springer, 1992.

**Definition 1** (Affine space)**.** *For $k$ an algebraically closed field,* affine space *over $k$ is denoted $\mathbb{A}^n$ or $\mathbb{A}^n_k$ and indicates the set $k^n$ of $n$-tuples of elements of $k$.*

**Definition 2** (Projective space)**.** *For $k$ an algebraically closed field,* projective space *over $k$ is denoted $\mathbb{P}^n$ or $\mathbb{P}^n_k$ and indicates the set of orbits $(k^{n+1} \setminus \{0\})/k^*$ where $k^*$ acts via scalar multiplication $(z_0, \ldots, z_n) \mapsto (\lambda z_0, \ldots, \lambda z_n)$. The $k^*$-orbit of $(z_0, \ldots, z_n)$ is commonly denoted $(z_0 : \cdots : z_n)$.*

**Definition 3** (Standard open covering)**.** *The standard open covering of $\mathbb{P}^n$ consists of the subsets*

$$U_i = \{(z_0 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_n)\} \subseteq \mathbb{P}^n.$$

*That is, those tuples with $z_i$ nonzero. In the above form, each $U_i$ is canonically isomorphic to $k^n$ via*

$$U_i \cong \mathbb{A}^n$$

$$(z_0 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_n) \leftrightarrow (z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$$

**Definition 4** (Projective variety)**.** *A* projective algebraic variety *is a subset of the form*

$$V(f_1, \ldots, f_c) = \{(z_0 : \cdots : z_n) \in \mathbb{P}^n \mid f_1(z) = \cdots = f_c(z) = 0\}$$

*for some set of* homogeneous[1] *polynomials $f_1, \ldots, f_c \in k[z_0, \ldots, z_n]$.*

**Definition 5** (Affine variety)**.** *An* affine algebraic variety *is a subset of the form*

$$V(f_1, \ldots, f_c) = \{(z_1, \ldots, z_n) \in \mathbb{A}^n \mid f_1(z) = \cdots = f_c(z) = 0\}$$

*for some set of (not necessarily homogeneous) polynomials $f_1, \ldots, f_c \in k[z_1, \ldots, z_n]$.*

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[1] I.e., a $k$-linear combination of monomials which all have the same degree, for example $3z_0^2 + 2z_0z_1 - 5z_1^2$.

**Definition 6.** *The term* variety *ambiguously refers to something that is either an affine variety or a projective variety.*

**Exercise 1.** Show that for every projective variety $X \subseteq \mathbb{P}^n$, via the isomorphisms $U_i \cong \mathbb{A}^n$, the intersections $X \cap U_i \subseteq U_i \cong \mathbb{A}^n$ are affine varieties. Conversely, show that for every affine variety $X$ there exists a projective variety $\overline{X}$ such that $X = \overline{X} \cap U_0$.

**Definition 7** ($k$-variety)**.** *Let $k \subseteq \overline{k}$ be a subfield of an algebraically closed field. A variety $X$ is said to be* defined over $k$ *if it is of the form $V(f_1, \dots, f_c)$ for polynomials $f_1, \dots, f_c$ with coefficients in $k$. Sometimes we call these* $k$-varieties *and indicate $X$ is defined over $k$ by the notation $X/k$.*

**Definition 8** ($k$-points)**.** *Given a field extension $k \subseteq L \subseteq \overline{k}$ and a projective, resp. affine, $k$-variety we write*

$$X(L) = \{(z_0 : \cdots : z_n) \in \mathbb{P}^n \mid z_0, \dots, z_n \in L\}$$

$$\textit{resp. } X(L) = \{(z_1, \dots, z_n) \in \mathbb{A}^n \mid z_1, \dots, z_n \in L\}$$

*for the set of* $L$-points*.*

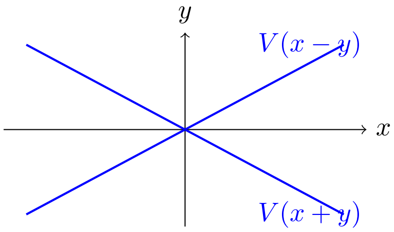
**Example 9.** Consider $X = V(x^2+1) \subseteq \mathbb{A}^1_{\mathbb{C}}$, defined by the polynomial $x^2+1 \in \mathbb{Q}[x]$, so $X$ is a $\mathbb{Q}$-variety. Since $x^2+1 \geq 1 > 0$ for all $x \in \mathbb{R}$, we have $X(\mathbb{R}) = \emptyset$. On the other hand, $X(\mathbb{C}) = \{i, -i\}$, so $X$ has two complex points.

**Definition 10** (Reducibility)**.** *Let $X$ be a variety (affine or projective) over an algebraically closed field $k$. We say $X$ is* reducible *if there are distinct non-empty subvarieties $V, W \subsetneq X$ such that $X = V \cup W$. Otherwise $X$ is said to be* irreducible*. If $X = X_1 \cup \cdots \cup X_n$ with each $X_i$ irreducible, the $X_i$ are called the* irreducible components of $X$*.*

**Example 11.** Consider $X = V(x^2 - y^2) \subseteq \mathbb{A}^2_k$. Since $x^2 - y^2 = (x-y)(x+y)$, we have

$$X = V(x-y) \cup V(x+y),$$

the union of the two diagonal lines $\{y = x\}$ and $\{y = -x\}$. Each is isomorphic to $\mathbb{A}^1_k$ and hence irreducible, so these are the two irreducible components of $X$. This reflects the factorisation $x^2-y^2 = (x-y)(x+y)$ in $k[x,y]$; more generally, a hypersurface $V(f) \subseteq \mathbb{A}^n_k$ is reducible if and only if $f$ is reducible in $k[x_1, \dots, x_n]$.



**Definition 12** (Dimension)**.** *Let $X$ be an irreducible variety (affine or projective) over an algebraically closed field $k$. The* dimension *of $X$, denoted* $\dim X$*, is the supremum of lengths $d$ of chains*

$$X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_d = X$$

*of irreducible subvarieties of $X$. We set* $\dim \varnothing = -1$*. For a reducible variety $X = X_1 \cup \cdots \cup X_n$ we set* $\dim X := \max_i \dim X_i$*.*

**Definition 13** (Module of differentials of $\mathbb{A}^n$)**.** *The* module of differentials *of $\mathbb{A}^n_k$ is the free $k[z_1, \dots, z_n]$-module*[2]

$$\Omega_{\mathbb{A}^n}(\mathbb{A}^n) := \bigoplus_{i=1}^{n} k[z_1, \dots, z_n]\, dz_i.$$

*The* differential

$$d : k[z_1, \dots, z_n] \to \Omega_{\mathbb{A}^n}(\mathbb{A}^n)$$

*is the unique $k$-linear map satisfying the Leibniz rule*

$$d(fg) = g df + f dg.$$

**Exercise 2.** Show that $da = 0$ for all $a \in k$.

**Example 14.** Take $f = y^2 - x^3 + x \in k[x, y]$. Then

$$\begin{aligned} df &= \partial_x f\, dx + \partial_y f\, dy \\ &= (-3x^2 + 1)\, dx + 2y\, dy \in \Omega_{\mathbb{A}^2}(\mathbb{A}^2). \end{aligned}$$

**Definition 15** (Module of differentials of $X$)**.** *Let $X \subseteq \mathbb{A}^n_k$ be an affine variety with vanishing ideal $I(X) = \{f \in k[z_1, \dots, z_n] \mid f|_X = 0\}$. The* module of differentials *of $X$ is the quotient*

$$\Omega_X(X) := \frac{\Omega_{\mathbb{A}^n}(\mathbb{A}^n)}{\langle\, df \mid f \in I(X) \,\rangle},$$

*which is naturally a module over the coordinate ring*

$$\mathcal{O}(X) := \frac{k[z_1, \dots, z_n]}{I(X)}.$$

**Remark 16.** There are various notions of differentials in this course that should not be confused. The algebraic module of differentials $\Omega_X(X)$ defined above uses only polynomials with coefficients in the base field $k$. By contrast, in differential geometry one defines a sheaf of differentials on a smooth manifold using infinitely differentiable ($C^\infty$) real functions.

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[2] Here, the $dz_i$ are formal symbols but can be thought of geometrically as coordinate covector fields on $\mathbb{A}^n$: the symbol $dz_i$ represents the covector field dual to the coordinate vector field $\partial_{z_i}$.

When $k = \mathbb{C}$ and $X$ is a smooth variety, these are related but distinct objects. For example, on $\mathbb{A}^n_{\mathbb{C}} = \mathbb{C}^n$, the algebraic module $\Omega_{\mathbb{A}^n}(\mathbb{A}^n) = \bigoplus_{i=1}^n \mathbb{C}[z_1, \dots, z_n]\, dz_i$ is a free $\mathbb{C}[z_1, \dots, z_n]$-module of rank $n$. On the other hand, viewed as a real smooth manifold, $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and the $C^\infty(\mathbb{R}^{2n})$-module of differentials $A^1(\mathbb{R}^{2n})$ is a free $C^\infty(\mathbb{R}^{2n})$-module of rank $2n$.

There is a canonical comparison map

$$\underset{=\mathcal{O}(\mathbb{A}^n_{\mathbb{C}})}{\mathbb{C}[z_1, \dots, z_n]} \to C^\infty(\mathbb{R}^{2n}) \otimes_{\mathbb{R}} \mathbb{C}$$

obtained by considering a polynomial $f$ as a function $\mathbb{R}^{2n} \xrightarrow{\sim} \mathbb{C}^n \xrightarrow{f} \mathbb{C} \xrightarrow{\sim} \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$. This fits into a commutative square

$$\begin{tikzcd}
\Omega_{\mathbb{A}^n_{\mathbb{C}}}(\mathbb{A}^n_{\mathbb{C}}) \arrow[r, "(*)"] & A^1(\mathbb{R}^{2n}) \otimes_{\mathbb{R}} \mathbb{C} \\
\mathcal{O}(\mathbb{A}^n_{\mathbb{C}}) \arrow[u, "d"] \arrow[r] & C^\infty(\mathbb{R}^{2n}) \otimes_{\mathbb{R}} \mathbb{C} \arrow[u, "d"]
\end{tikzcd}$$

which is unique once we require the morphism $(*)$ to be $\mathbb{C}[z_1, \dots, z_n]$-linear.

**Definition 17** (Cotangent space)**.** *Let $X \subseteq \mathbb{A}^n_k$ be an affine variety and $x \in X$ a point, with maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}(X)$. The* cotangent space *of $X$ at $x$ is the $k$-vector space*

$$\Omega_X(x) := \Omega_X(X) \otimes_{\mathcal{O}(X)} k \cong \Omega_X(X)/\mathfrak{m}_x \Omega_X(X).$$

*Concretely, $\Omega_X(x)$ is the quotient of $\bigoplus_{i=1}^n k\, dz_i$ by the subspace spanned by $\{(df)(x) \mid f \in I(X)\}$, where $(df)(x) = \sum_i (\partial_{z_i} f)(x)\, dz_i$,*

$$\Omega_X(x) = \frac{\bigoplus_{i=1}^n k\, dz_i}{\{(df)(x) \mid f \in I(X)\}}.$$

**Exercise 3.** Suppose that $I$ is generated as an ideal by $f_1, \dots, f_c$. Using the Leibniz rule show that $\Omega_X(x) = \frac{\bigoplus_{i=1}^n k\, dz_i}{\langle (df_1)(x), \dots, (df_c)(x) \rangle}$ where $\langle - \rangle$ means the $k$-linear span.

**Example 18.** Let $X = V(y - x^2) \subseteq \mathbb{A}^2_k$, the parabola. Here $I(X) = \langle y - x^2 \rangle$, and $d(y - x^2) = -2x\, dx + dy$. At the point $p = (a, a^2) \in X$ this gives $(df)(p) = -2a\, dx + dy$, so

$$\Omega_X(p) = \frac{k\, dx \oplus k\, dy}{\langle -2a\, dx + dy \rangle} \cong k,$$

generated by $[dx]$, with $[dy] = 2a\, [dx]$.

**Theorem 19.** *Let $X$ be an irreducible affine variety over an algebraically closed field $k$. Then*

$$\dim X = \min_{x \in X} \dim_k \Omega_X(x).$$

**Remark 20.** There are a number of theorems hidden in the above theorem, so it's a little hard to give a precise reference.[3]

If we ignore implementation, the idea is not so difficult. The kernel of the quotient $k^n \cong \Omega_{\mathbb{A}^n}(x) \to \Omega_X(x) \cong k^d$ is generated by $c := n - d$ elements which by our definition of $\Omega_X(x)$ we can lifted to polynomials $f_1, \dots, f_c$ which vanish along $X$. On the other hand, by completing $df_1(x), \dots, df_c(x) \in \Omega_{\mathbb{A}^n}(x)$ to a basis and using the fact that for $x = (a_1, \dots, a_n) \in \mathbb{A}^n$ the vector space $\Omega_{\mathbb{A}^n}(x)$ has basis $d(z_1 - a_1), \dots, d(z_n - a_n)$, we produce (linear) polynomials $f_{c+1}, \dots, f_n$ vanishing at $x$, but *not* on all of $X$. These latter $f_{c+1}, \dots, f_n$ combined with the former $f_1, \dots, f_c$ cut out a sequence of subvarieties

$$\varnothing \subsetneq V(f_1, \dots, f_n) \subsetneq V(f_1, \dots, f_{n-1}) \subsetneq \cdots \subsetneq V(f_1, \dots, f_{c+1}) \subsetneq X \qquad (1)$$

which have $\dim \Omega_{V(f_1, \dots, f_{n-i})}(x) = i$ by the fact that $df_1(x), \dots, df_n(x)$ is a basis for $\Omega_{\mathbb{A}^n}(x)$. So in order to deduce that $\dim \Omega_X(x) = \dim X$ it remains to show that Eq.(1) is a maximal chain of irreducible subvarieties of $X$ (at least locally around $x$ for the Zariski topology). This is also a theorem, [4, 00NQ, 00NP, 0A21].

The reason we use $\min_{x \in X}$ is that for non-smooth points (discussed below) the vector space $\Omega_X(x)$ will have dimension larger than $\dim X$. So the above statement also uses the theorems that $\dim X \leq \dim \Omega_X(x)$ for a general point [4, 00TS],[4] and that there exists at least one point where $\dim X \leq \dim \Omega_X(x)$, i.e., at least one smooth point [4, 056V, 04QP].[5,6]

**Example 21.** Consider the elliptic curve $E = V(f) \subseteq \mathbb{A}^2_k$ where $f = y^2 - x^3 + x$ (assuming $\mathrm{char}(k) \neq 2, 3$, so that $E$ is smooth). At a point $p = (a, b) \in E$ the formal derivative is

$$df = (-3a^2 + 1)\, dx + 2b\, dy.$$

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[3] Dimension theory in general, but specifically the various characterisations of smoothness is something that the Stacks Project does particularly poorly. If you are really interested, I recommend reading [Matsumura, Commutative ring theory] (purely from the algebra perspective) or [Görtz-Wedhorn, Algebraic Geometry I] rather then trying to construct a complete proof from sketch and the Stacks Project references I give in this remark.

[4] To apply this reference as stated we need various bijections. The only one which is not elementary is the Nullstellensatz.

1. For any ring $R$ and prime ideal $\mathfrak{p}$ there is a bijection of posets of ideals $\{\text{ideals } J \subseteq R_{\mathfrak{p}}\} \cong \{\text{ideals } J' \subseteq \mathfrak{p} \subseteq R\}$.
2. For any ideal $I$ in a ring $R$ there is a bijection of posets $\{\text{ideals } J \subseteq R/I\} \cong \{\text{ ideals } I \subseteq J' \subseteq R\}$.
3. (Nullstellensatz) there is an inclusion reversing bijection $\{\text{ subvarieties of } \mathbb{A}^n\} \cong \{\text{ radical ideals of } k[z_1, \dots, z_n]\}; X \mapsto I(X)$.

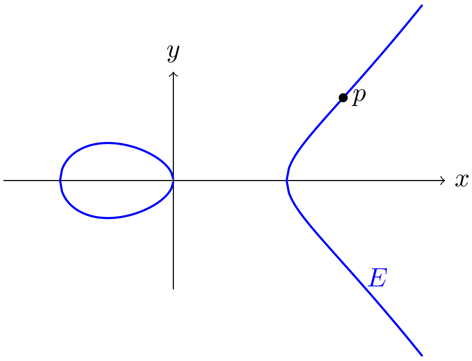
[5] These two stacks project references say that there exists a dense open $U \subseteq X$ where $\dim \Omega_X(x)$ is constant. This will be our definition of smooth.

[6] Also note that since we are working over an algebraically closed field, "geometrically reduced" just means "reduced", and that *all* varieties (considered as schemes) are reduced by definition.

Since $E$ is smooth, $df(p) \neq 0$ at every point, and we have

$$\Omega_E(p) = \frac{k\,dx \oplus k\,dy}{\langle(-3a^2+1)\,dx + 2b\,dy\rangle} \cong k,$$

which is one-dimensional for all $p \in E$. To make this concrete: at the point $p = (0,0)$ (which lies on $E$ since $f(0,0) = 0 - 0 + 0 = 0$) we have $df(0,0) = dx$, so $\Omega_E(0,0)$ is spanned by $[dy]$ with $[dx] = 0$.



**Example 22.** Consider the *nodal cubic* $C = V(f) \subseteq \mathbb{A}^2_k$ where $f = y^2 - x^2(x+1) = y^2 - x^3 - x^2$. The curve $C$ has a node (self-intersection) at the origin. We compute the cotangent space at two points.

*At a smooth point.* Take $p = (a,b) \in C$ with $p \neq (0,0)$. The formal derivative is

$$df = (-3a^2 - 2a)\,dx + 2b\,dy.$$

If $b \neq 0$ then $df(p)$ is nonzero in $k\,dx \oplus k\,dy$, so

$$\Omega_C(p) = \frac{k\,dx \oplus k\,dy}{\langle(-3a^2-2a)\,dx + 2b\,dy\rangle} \cong k$$

is one-dimensional, consistent with $C$ being a curve.

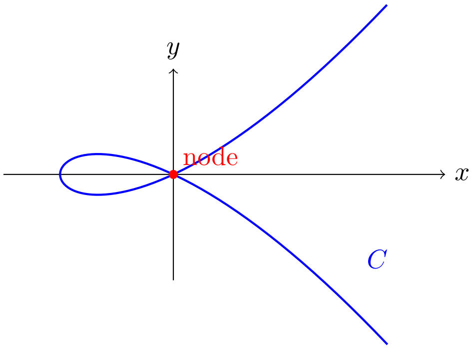
*At the node.* At $p = (0,0)$ we have

$$df(0,0) = 0 \cdot dx + 0 \cdot dy = 0,$$

so the relation imposed on $k\,dx \oplus k\,dy$ is trivial and

$$\Omega_C(0,0) \cong k^2$$

is two-dimensional. This excess dimension reflects the singularity: the two branches of $C$ through the origin each contribute an independent tangent direction.



**Definition 23** (Smoothness)**.** *Suppose that $X$ is an irreducible affine variety of dimension $d$. We say $X$ is* smooth at $x$ *if $\dim_k \Omega_X(x) = d$, and* smooth *if it is smooth at every point. In general we say $X$ is smooth if all irreducible components are smooth.*

*For a projective variety $X \subseteq \mathbb{P}^n$, we say $X$ is* smooth at $x$ *if $x \in U_i$ for some $i$ and $X \cap U_i$ is smooth at $x$ as an affine variety. We say $X$ is* smooth *if it is smooth at every point.*

**Exercise 4.** Show that the definition of smoothness at $x \in X$ is independent of the choice of $i$ with $x \in U_i$.

**Remark 24.** When $k = \mathbb{C}$, the algebraic notion of smoothness coincides with the analytic one. Precisely, if $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ is an affine variety of dimension $d$, then $X$ is smooth at a point $x \in X$ if and only if there exists $\varepsilon > 0$ such that the ball

$$B(x, \varepsilon) := \{z \in \mathbb{C}^n \mid |z - x| < \varepsilon\}$$

intersects $X$ in a smooth submanifold: that is, there is an invertible map $\varphi\colon B(x,\varepsilon) \xrightarrow{\sim} \varphi(B(x,\varepsilon))$, with both $\varphi$ and $\varphi^{-1}$ given by convergent power series, such that $\varphi(x) = 0$ and

$$B(x,\varepsilon) \cap X = \varphi^{-1}(\mathbb{C}^d \times \{0\}^{n-d}).$$

In particular, a smooth $\mathbb{C}$-variety of dimension $d$ carries the natural structure of a complex manifold of complex dimension $d$. For a smooth projective $\mathbb{C}$-variety, the same applies on each affine chart $X \cap U_i$, and these charts glue to give $X$ the structure of a compact complex manifold.

# References

[1] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer, 1977.

[2] I.R. Shafarevich, *Basic Algebraic Geometry*, Vol. 1, Springer, 3rd ed., 2013.

[3] J. Harris, *Algebraic Geometry: A First Course*, Graduate Texts in Mathematics 133, Springer, 1992.

[4] The Stacks Project Authors, *The Stacks Project*, `https://stacks.math.columbia.edu`.