Algebraic K-theory

Shane Kelly

August 12, 2025

Abstract

This is a series of five lectures on algebraic K-theory aimed at (advanced) fourth year undergraduates. The goal is to introduce various aspects of algebraic K-theory.

Algebraic K-theory was defined in the late 1950s by Alexander Grothendieck in order to formulate his generalisation of the mid-1800's Riemann–Roch theorem which relates vector bundles on algebraic varieties to algebraic cycles. The foundational idea is to construct a group in which every short exact sequence of vector bundles splits. Since then, algebraic K-theory has developed into a deep and far-reaching subject with applications across algebraic geometry, number theory, and topology.

In the first two lectures we present the classical Riemann–Roch (for smooth projective curves over $\mathbb{C}$) followed by Grothendieck's generalisation (for smooth quasi-projective varieties over an algebraically closed field). In the third lecture we motivate the groups K_1, K_{-1}, K_{-2}, ... surrounding K_0 for rings by the desire to have long exact sequences. The fourth lecture discusses ∞-groupoids, ∞-categories, and K-theory as the universal localising invariant, and in the final lecture, we give a brief survey of some recent advances and open conjectures.

Contents

1 Riemann–Roch **3**
1.1 Riemann–Roch Statement . . . 3
1.2 Affine varieties . . . 3
1.3 Projective varieties . . . 4
1.4 Smooth Complex Projective Curves . . . 5
1.5 Divisors . . . 7
1.6 Riemann–Roch restatement . . . 9

2 Grothendieck–Riemann-Roch **11**
2.1 Statement . . . 11
2.2 Morphisms of varieties . . . 11
2.3 Quasi-coherent $\mathcal{O}_X$-modules . . . 12
2.4 Pushforward . . . 15
2.5 Pullbacks . . . 16
2.6 Cotangent sheaf . . . 17
2.7 Chow groups . . . 18
2.8 Restatement . . . 20

3 $K_{\leq 1}$ **22**
3.1 K_0 . . . 22
3.2 K_1 . . . 24
3.3 $K_{<0}$ . . . 26

4 K-theory as the universal localising invariant **27**
4.1 Simplicial sets . . . 27
4.2 Infinity groupoids . . . 30
4.3 Infinity categories . . . 31
4.4 Limits . . . 33
4.5 Stable infinity categories . . . 35
4.6 Localising invariants . . . 36
4.7 K-theory . . . 37

5 Recent advances; open problems **39**
5.1 Weibel's Conjecture . . . 39
5.2 Prismatic cohomology and K-theory of finite rings . . . 39
5.3 Failure of the telescope conjecture . . . 40
5.4 Stable homotopy groups of spheres . . . 40
5.5 Atiyah-Hirzebruch spectral sequence . . . 41
5.6 Open problems . . . 42

1 Riemann–Roch

In this lecture, the base field is always the complex numbers $\mathbb{C}$.

1.1 Riemann–Roch Statement

The goal for today's talk is to understand the words in the following statement. A good reference is [Ful69]. The theorem is also in [Har77].

Theorem 1.1.1 (Riemann–Roch)**.** *Let X be a smooth projective curve of genus g. Then there exists a divisor K (the canonical divisor) such that for every divisor D on X, we have*

$$\dim H^0(X, \mathcal{O}(D)) - \dim H^0(X, \mathcal{O}(K-D)) = \deg(D) + 1 - g.$$

1.2 Affine varieties

Definition 1.2.1 (Affine Variety over $\mathbb{C}$)**.** An *affine variety* is a subset $X \subseteq \mathbb{C}^n$ of the form

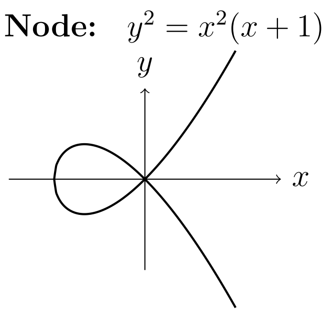
$$X = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \begin{array}{r} f_1(z_1, \ldots, z_n) = 0 \\ f_2(z_1, \ldots, z_n) = 0 \\ \vdots \end{array} \right\}$$

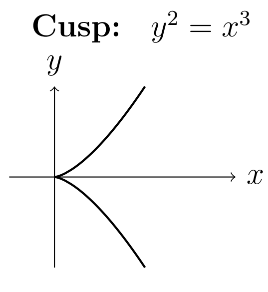
for some collection of polynomials $\{f_i\}_{i \in I} \subseteq \mathbb{C}[x_1, \ldots, x_n]$. We say X is the *zero set* of the f_i.

Remark 1.2.2. Since $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian, we can assume the set $\{f_i\}_{i \in I}$ is finite, but it is convenient to allow infinite sets of polynomials.

Example 1.2.3 (Examples of Affine Varieties)**.**

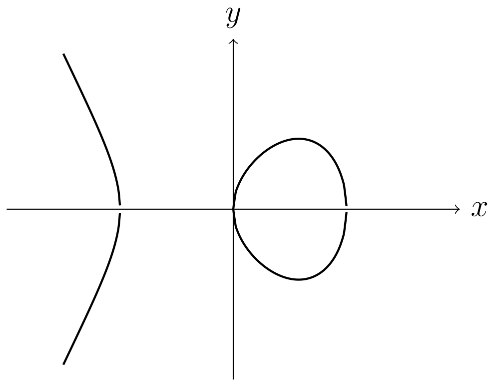
1. *Affine space:* $X = \mathbb{C}^n =: \mathbb{A}^n$ itself (i.e., taking $I = \varnothing$).
2. *Node:* $X = \{(x, y) \in \mathbb{C}^2 : y^2 = x^2(x+1)\}$.
3. *Cusp:* $X = \{(x, y) \in \mathbb{C}^2 : y^2 = x^3\}$.





4. *Elliptic curve:* $X = \{(x, y) \in \mathbb{C}^2 : y^2 = x^3 + ax + b\}$ where $4a^3 + 27b^2 \neq 0$.

Elliptic curve: $y^2 = x^3 - x = x(x-1)(x+1)$



5. *Complement of a hypersurface:* Given an affine variety $X = V(\{f_i\}) \subseteq \mathbb{C}^n$ and polynomial g, the complement $U := X \setminus V(g)$ is not a closed subvariety of $\mathbb{C}^n$. However, the affine variety

$$U' = \{(x_1, \ldots, x_n, y) \in \mathbb{C}^{n+1} : f_i(x_1, \ldots, x_n) = 0, g(x_1, \ldots, x_n) \cdot y = 1\}$$

projects bijectively to U. This gives the commutative diagram:

$$\begin{array}{ccc} U' & \longrightarrow & \mathbb{C}^{n+1} \\ \big\downarrow{\scriptstyle \sim} & & \big\downarrow \\ U & \longrightarrow & \mathbb{C}^n \end{array}$$

6. *General linear group:* $\mathrm{GL}_n(\mathbb{C}) = \{A \in \mathrm{Mat}_n(\mathbb{C}) \mid \det(A) \neq 0\}$ is an example of a U as in the previous point. That is,

$$GL_n(\mathbb{C}) \cong \{(A, t) \in \mathrm{Mat}_n(\mathbb{C}) \times \mathbb{C} \mid \det(A) \cdot t = 1\}$$

7. *Intersection:* If $X_1, X_2 \subseteq \mathbb{A}^n$ are affine varieties defined by sets of polynomials $\mathcal{F}_1, \mathcal{F}_2$ respectively, then $X_1 \cap X_2$ is the affine variety defined by $\mathcal{F}_1 \cup \mathcal{F}_2$.

8. *Union:* If $X_1, X_2 \subseteq \mathbb{A}^n$ are affine varieties defined by sets of polynomials $\mathcal{F}_1, \mathcal{F}_2$, then $X_1 \cup X_2$ is the affine variety defined by $\{fg : f \in \mathcal{F}_1, g \in \mathcal{F}_2\}$.

1.3 Projective varieties

Definition 1.3.1 (Complex Projective Space)**.** *Complex projective space* is the set

$$\mathbb{P}^n = \frac{\{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}\}}{\sim}$$

of equivalence classes under the relation

$$(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n), \quad \lambda \in \mathbb{C}^\times.$$

One writes $(z_0 : \cdots : z_n) \in \mathbb{P}^n$ for the equivalence class containing $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$.

Remark 1.3.2. For each $i = 0, \ldots, n$ we have a bijection

$$\mathbb{C}^n \xrightarrow{\sim} U_i := \{(z_0 : \cdots : z_n) \mid z_i \neq 0\}$$
$$(x_1, \ldots, x_n) \mapsto (x_1 : \cdots : x_i : 1 : x_{i+1} : \cdots : x_n)$$

These cover $\mathbb{P}^n$.

$$\mathbb{P}^n = \cup_{i=0}^n U_i.$$

Exercise 1.3.3. Describe the intersections $U_{i_1} \cap \cdots \cap U_{i_j}$ as subsets of $U_0 \cong \mathbb{C}^n$.

Definition 1.3.4 (Projective Variety)**.** A *projective variety* is a subset $X \subseteq \mathbb{P}^n$ such that for each affine chart U_i, the intersection $X \cap U_i$ is an affine variety in $U_i \cong \mathbb{A}^n$.

Example 1.3.5 (Homogeneous Polynomials)**.** If $\mathcal{F}$ is a set of homogeneous polynomials (i.e., polynomials of the form $\sum_{i_0+\cdots+i_n=d} a_{i_1,\ldots,i_k} z_0^{i_0} \ldots z_n^{i_n}$ for some d), then the zero set $V(\mathcal{F}) = \{(z_0 : \cdots : z_n) \in \mathbb{P}^n : f(z_0, \ldots, z_n) = 0 \text{ for all } f \in \mathcal{F}\}$ is a projective variety. In fact, *every* projective variety is of this form.

Example 1.3.6 (Grassmannians)**.** The *Grassmannian* $\mathrm{Gr}(k, n)$ is the variety of k-dimensional subspaces of $\mathbb{C}^n$. For example: $\mathrm{Gr}(2, 4)$ (planes in $\mathbb{C}^4$), which can be embedded in $\mathbb{P}^5$ via Plücker coordinates.

$$\mathrm{Gr}(2, 4) \hookrightarrow \mathbb{P}^5$$
$$\langle v_1, v_2\rangle \mapsto \langle v_1 \wedge v_2\rangle$$

If we use p_{ij} for the coordinate of $\mathbb{P}^5$ corresponding to $e_i \wedge e_j \in \mathbb{C}^4 \wedge \mathbb{C}^4 \cong \mathbb{C}^6$, then the image of $\mathrm{Gr}(2, 4)$ in $\mathbb{P}^5$ is defined by the *Plücker relation*:

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0.$$

Example 1.3.7 (Segre Embedding)**.** $\mathbb{P}^n \times \mathbb{P}^m$ has a structure of projective variety via the *Segre embedding*

$$\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{(n+1)(m+1)-1}$$
$$(x_0 : \cdots : x_n), (y_0 : \cdots : y_m) \mapsto (x_0y_0 : x_0y_1 : \cdots : x_iy_i : \cdots : x_ny_m)$$

The image is defined by the quadratic relations $z_{ij}z_{kl} - z_{il}z_{kj} = 0$ for all i, k and j, l. Consequently, if $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ are projective varieties, then $X \times Y$ can canonically be identified with a subvariety of $\mathbb{P}^{(n+1)(m+1)-1}$.

Remark 1.3.8. One can consider $\mathbb{P}^n$ as a *compactification* of $\mathbb{A}^n \cong U_0$ where we have adjoined one point for every line through the origin in such a way that if a curve approaches that line "at infinity" then it will actually intersect at that new point "at infinity".

For example, consider the affine curves

$$C_1 : x = 0 \text{ (the } y\text{-axis)}\ , \qquad C_2 : xy = 1 \text{ (a hyperbola)}$$

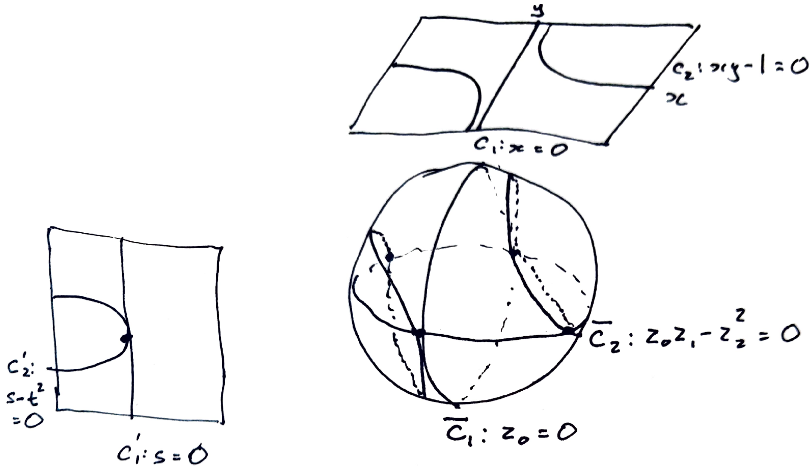
in $\mathbb{C}^2 = \{(x, y)\} \cong \{(x : y : 1)\} = U_0$. These curves do not intersect in the affine plane. However, these curves are the intersection of U_0 with the projective curves

$$\overline{C}_1 : \{(z_0 : z_1 : z_2) \mid z_0 = 0\} \tag{1}$$
$$\overline{C}_2 : \{(z_0 : z_1 : z_2) \mid z_1z_2 = z_0^2\} \tag{2}$$

Intersecting with the chart $U_1 = \{(s : 1 : t)\} \cong \{(s, t)\}$, they become the curves

$$C_1' : s = 0 \text{ (the } t\text{-axis)}\ , \qquad C_2' : t = s^2 \text{ (a quadric)}$$

They intersect at the point $(s, t) = (0, 0) \leftrightarrow (0 : 1 : 0)$, the point at infinity corresponding to the line $\{(0, y) \mid y \in \mathbb{C}\} \subseteq U_0$.



1.4 Smooth Complex Projective Curves

Definition 1.4.1 (Smooth Point)**.** Let $X \subseteq \mathbb{A}^n$ be an affine variety and $x \in X$. We say X is *smooth of dimension* d at x if there exists an open ball $B \ni x$ (in the analytic topology, i.e., $B = \{z \mid ||z - x|| < \varepsilon \text{ for some } \varepsilon > 0\}$) and a biholomorphic map $\phi : B \xrightarrow{\sim} B' \subseteq \mathbb{C}^n$ to an open subset B' of $\mathbb{C}^n$ such that

$$B \cap X = \phi^{-1}\{(z_1, \ldots, z_d, 0, \ldots, 0) \in B'\}.$$

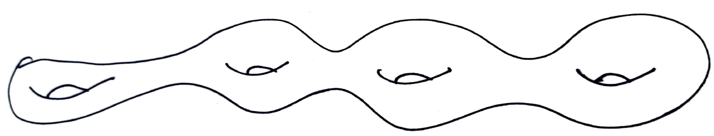
Example 1.4.2 (Non-smooth Points)**.**

1. *Node:* The affine curve $y^2 = x^2(x+1)$ has a node at the origin.
2. *Cusp:* The affine curve $y^2 = x^3$ has a cusp at the origin.

In both cases all other points are smooth.

Definition 1.4.3 (Smooth Complex Projective Curve)**.** A *smooth projective curve* X is a projective variety which is smooth of dimension one at every point.

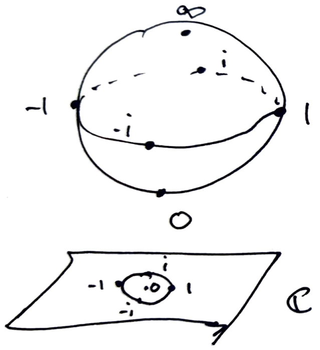
Remark 1.4.4 (Underlying Topological Space)**.** We can consider smooth projective curves as compact real manifolds of real dimension 2. They are automatically oriented, so homeomorphic to a sphere with g handles. This g is called the *genus* of the curve. (There is also a purely algebraic description of genus, namely $\dim H^0(X, \mathcal{O}(K))$ where K is the canonical divisor mentioned in the statement of the Riemann–Roch theorem, and $H^0(X, \mathcal{O}(-))$ is defined below).

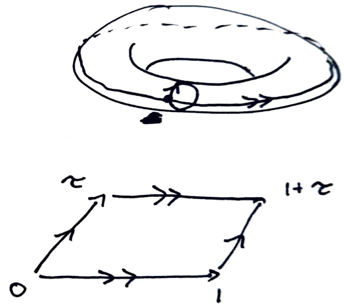


Example 1.4.5 (Genus Examples)**.**

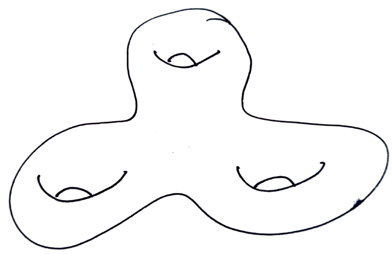
1. *Projective line:* The projective line $\mathbb{P}^1$ is topologically a sphere, hence has genus $g = 0$. Indeed, it is the one point compactification of $\mathbb{C} \cong \mathbb{R}^2$.

2. *Elliptic curve:* A smooth cubic curve in $\mathbb{P}^2$, such as $y^2z = x^3 + axz^2 + bz^3$ with $4a^3 + 27b^2 \neq 0$, is topologically a torus (i.e., the surface of a doughnut, or coffee mug) and has genus $g = 1$. Indeed, every elliptic curve is holomorphic to a quotient abelian group of the form $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with the canonical smooth complex manifold structure, where $\tau \notin \mathbb{R}$.





3. *Higher genus curves:* A smooth curve of degree d in $\mathbb{P}^2$ has genus $g = \frac{(d-1)(d-2)}{2}$. For example, the Klein quartic $x^3y + y^3z + z^3x = 0$ is a smooth degree 4 curve, so has genus $g = \frac{(4-1)(4-2)}{2} = 3$. Topologically it looks like the surface of a fidget spinner.



1.5 Divisors

Definition 1.5.1 (Basic Open)**.** A *basic open* of an affine variety $X \subseteq \mathbb{A}^n$ is a subset of the form

$$D(g) = \{x \in X : g(x) \neq 0\} \subseteq X$$

for some polynomial $g \in \mathbb{C}[x_1, \dots, x_n]$. The basic opens together with inclusion maps form a category which we denote $\mathcal{B}(X)$. If X is projective, then we define $\mathcal{B}(X) = \bigcup_{i=0}^{n} \mathcal{B}(U_i \cap X)$ to be the union of the basic opens of the $n+1$ standard affine varieties associated to X.

Example 1.5.2.

1. If $g = 1$ (or more generally, if g is invertible on X) then $D(g) = X$.

2. If $g = 0$ (or more generally, if g vanishes everywhere on X) then $D(g) = \varnothing$.

3. If $X = \mathbb{A}^1$ and $g = (x - a_1) \dots (x - a_n)$ then $D(g) = X \setminus \{a_1, \dots, a_n\}$. Similarly, every basic open of $\mathbb{P}^1$ is of the form $\mathbb{P}^1 \setminus \{a_1, \dots, a_n\}$ for some nonempty set of points.

4. More generally, if X is a projective or affine curve, then every basic open is of the form $X \setminus \{x_1, \dots, x_n\}$. (But not conversely).

Definition 1.5.3 (Structure Sheaf on Basic Opens)**.** Given an affine variety $X \subseteq \mathbb{C}^n$ and a basic open $U = D(g) \subseteq X$, write

$$\mathcal{O}_X(U) = \left\{ \varphi : U \to \mathbb{C} \mid \varphi = \frac{f}{g^n} \text{ for some } f \in \mathbb{C}[x_1, \ldots, x_n], n \geq 0 \right\}$$

for the set of functions on U of the form f/g^n.

Remark 1.5.4. Note that if f' vanishes on X, then $f/g^n = (f + f')/g^n$ as a function on X. More precisely, one can show that the ring $\mathcal{O}_X(U)$ of functions, is isomorphic to the abstract ring

$$\mathcal{O}_X(U) \cong \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle f_1, \ldots, f_c \rangle}[g^{-1}]$$

where $X = V(f_1, \ldots, f_c)$.

Remark 1.5.5. As U varies, the $\mathcal{O}_X(U)$ define a functor

$$\begin{aligned} \mathcal{B}(X)^{op} &\to \mathcal{R}\text{ing} \\ U &\mapsto \mathcal{O}_X(U). \end{aligned}$$

That is,

0. for every U we have a ring
$$\mathcal{O}_X(U),$$

1. for every inclusion $U' \subseteq U$, restriction gives a ring homomorphism
$$\mathcal{O}_X(U) \to \mathcal{O}_X(U'),$$

2. for every two inclusions $U'' \subseteq U' \subseteq U$ we have a commutative triangle of ring homomorphisms
$$\begin{array}{ccccc} & & \mathcal{O}_X(U') & & \\ & \nearrow & & \searrow & \\ \mathcal{O}_X(U) & & \longrightarrow & & \mathcal{O}_X(U'') \end{array}$$

Definition 1.5.6. Suppose that $X \subseteq \mathbb{A}^n$ is *irreducible*. That is, X is not a union of two distinct nonempty varieties. Then each $\mathcal{O}_X(U) \to \mathcal{O}_X(U')$ (for $U' \neq \varnothing$) is injective, and we can define

$$K_X := \bigcup_{U \neq \varnothing} \mathcal{O}_X(U).$$

Remark 1.5.7. If X is a smooth curve, then each $f \in K_X$ is a meromorphic function on the corresponding smooth complex manifold. In particular, the *order*

$$\mathrm{ord}_x(f)$$

of the pole (or zero) of f at $x \in X$ is well-defined.

Definition 1.5.8. A *divisor* on a smooth projective curve X is a finite formal sum of points $D = \sum_{i=1}^{d} n_i x_i$. We write

$$\mathrm{Div}(X) = \{\Sigma_{i=1}^{d} n_i x_i\}$$

for the (free) abelian group of divisors. The *degree* of a divisor is

$$\deg(\Sigma_{i=1}^{d} n_i x_i) = \Sigma_{i=1}^{d} n_i.$$

Example 1.5.9. The divisor associated to a rational function $f \in K_X$ is

$$\mathrm{div}(f) = \sum_{x \in X} \mathrm{ord}_x(f) \cdot x.$$

Definition 1.5.10. Each divisor D determines a functor

$$\begin{aligned} \mathcal{O}(D) : \mathcal{B}(X)^{op} &\to \mathcal{A}\mathrm{b} \\ U &\mapsto \{f \in K_X \mid \mathrm{div}(f) + D \geq 0 \text{ on } U\} \end{aligned}$$

where a divisor $E = \sum_x n_x \cdot x$ satisfies $E \geq 0$ if $n_x \geq 0$ for all x.

Example 1.5.11. We have $\mathcal{O}_X = \mathcal{O}(0)$ where 0 is the zero divisor.

Remark 1.5.12. Note that the assignement

$$\mathcal{K}_X : U \mapsto \begin{cases} K_X & \text{if } U \neq \varnothing \\ 0 & \text{if } U = \varnothing \end{cases}$$

also defines a functor $\mathcal{B}(X)^{op} \to \mathcal{A}\mathrm{b}$ for each each $\mathcal{K}_X(U)$ is a $\mathcal{O}_X(U)$-module and the transition morphisms are compatible with this structure.

Moreover, each $\mathcal{O}(D)(U)$ is a sub-$\mathcal{O}_X(U)$-module of $\mathcal{K}_X(U)$, and the transition morphisms $\mathcal{O}(D)(U) \to \mathcal{O}(D)(U')$ are compatible with this structure. In other words, we have an inclusion of quasi-coherent $\mathcal{O}_X$-modules

$$\mathcal{O}(D) \subseteq \mathcal{K}_X.$$

Remark 1.5.13 (Physical Interpretation)**.** In string theory, Riemann surfaces appear as worldsheets of strings. Line bundles $\mathcal{O}(D)$ on these surfaces can encode various physical properties:

1. Spin structures

2. Gauge field backgrounds

3. D-brane charges in type II string theory

The degree of a line bundle corresponds to quantized charges or fluxes.

1.6 Riemann–Roch restatement

Definition 1.6.1 (Global sections)**.** Given a divisor D on an irreducible smooth curve X we define

$$H^0(X, \mathcal{O}(D)) := \bigcap_{\varnothing \neq U \in \mathcal{B}(X)} \mathcal{O}(D)(U).$$

That is, an element of $H^0(X, \mathcal{O}(D))$ is an element of K_X which belongs to all $\mathcal{O}(D)(U)$.

Remark 1.6.2. We could also have directly defined

$$H^0(X, \mathcal{O}(D)) = \{f \in K_X \mid \operatorname{div}(f) + D \geq 0 \text{ on } X\}$$

but the above definition is warm-up for the definition of $H^0(X, F)$ that we will see next time when F is an arbitrary quasi-coherent $\mathcal{O}_X$-module.

Example 1.6.3. Consider the divisor $D = d \cdot \infty$ on $\mathbb{P}^1$ where $\infty = (1 : 0)$. Then $H^0(\mathbb{P}^1, \mathcal{O}(D))$ is identified with the set $\mathbb{C}[x, y]_d = \{\sum_{i=0}^{d} a_i x^i y^{d-i}\}$ of homogeneous polynomials of degree d. In particular, it is a complex vector space of dimension $d + 1$ (if $d \geq 0$ and 0 otherwise).

Theorem 1.6.4. *If X is a smooth projective curve, then each $H^0(X, \mathcal{O}(D))$ is a finite dimensional $\mathbb{C}$-vector space.*

Theorem 1.6.5 (Riemann–Roch)**.** *Let X be a smooth projective curve of genus g. Then there exists a unique divisor K (the canonical divisor) such that for every divisor D on X, we have*

$$\dim H^0(X, \mathcal{O}(D)) - \dim H^0(X, \mathcal{O}(K - D)) = \deg(D) + 1 - g.$$

Example 1.6.6. For $X = \mathbb{P}^1$, we have $K = -2 \cdot \infty$ and $g = 0$. Then inputting everything we check that for $D = n \cdot \infty$ we have

$$\deg(D) + 1 - g = n + 1.$$

For the left side, we compute the dimensions case by case.
Case $n \geq 0$:

$$\dim H^0(X, \mathcal{O}(D)) = n + 1 \tag{3}$$
$$\dim H^0(X, \mathcal{O}(K - D)) = \dim H^0(X, \mathcal{O}((-n - 2)\infty)) = 0 \tag{4}$$

since $-n - 2 < 0$.
Case $n = -1$:

$$\dim H^0(X, \mathcal{O}(D)) = 0 \tag{5}$$
$$\dim H^0(X, \mathcal{O}(K - D)) = \dim H^0(X, \mathcal{O}(-\infty)) = 0. \tag{6}$$

Case $n \leq -2$:

$$\dim H^0(X, \mathcal{O}(D)) = 0 \tag{7}$$
$$\dim H^0(X, \mathcal{O}(K - D)) = \dim H^0(X, \mathcal{O}((-n - 2)\infty)) = -n - 1. \tag{8}$$

In all cases,

$$\dim H^0(X, \mathcal{O}(D)) - \dim H^0(X, \mathcal{O}(K - D)) = n + 1,$$

confirming Riemann–Roch.

2 Grothendieck–Riemann-Roch

2.1 Statement

Everything in this lecture is over an algebraically closed field $k = \overline{k}$ (e.g., $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{Q}}_p$, $\overline{\mathbb{F}}_p$, $\cup_{n\in\mathbb{N}}\mathbb{C}((t^{1/n}))$, ...). References include [Ful84] and [SGA71]. Note that Fulton's book has an appendix on algebraic geometry.

Theorem 2.1.1 (Grothendieck–Riemann–Roch)**.** *Suppose X is a smooth quasi-projective variety. Then the Chern character induces an isomorphism*

$$\mathrm{ch} : G_0(X)_{\mathbb{Q}} \cong A_*(X)_{\mathbb{Q}}.$$

Moreover, if $X \to Y$ is a projective morphism between smooth quasi-projective varieties, we have

$$\mathrm{ch}(f_*\alpha) \cdot \mathrm{td}(T_Y) = f_*(\mathrm{ch}(\alpha) \cdot \mathrm{td}\, T_X)).$$

Remark 2.1.2. When X is a smooth projective curve and $Y = \mathbb{A}^0$, this recovers the classical Riemann–Roch theorem from Lecture 1.

2.2 Morphisms of varieties

Recall that last time we defined affine varieties $X \subseteq k^n$, projective varieties $X \subseteq \mathbb{P}^n = \frac{k^{n+1}\setminus\{0\}}{k^*}$, and basic opens $U \subseteq X \subseteq \mathbb{C}^n$. We also considered the rings

$$\mathcal{O}_X(U) = \{\phi : U \to k \mid \phi = f/g^n, \text{ for some } f \in k[x_1, \dots, x_n], n \in \mathbb{N}\}$$

where $U = D(g) = \{x \in X \mid g(x) \neq 0\}$.

Definition 2.2.1. A *morphism* of basic opens $U \subseteq X \subseteq \mathbb{A}^n$, $V \subseteq Y \subseteq \mathbb{A}^m$ is a sequence $(\phi_1, \dots, \phi_m) \in \mathcal{O}_X(U)^m$ such that the corresponding morphism $U \to k^m$ factors through $V \subseteq k^m$.

Example 2.2.2.

1. Any inclusion of basic opens is a morphism.

2. If $D(g) \subseteq V(f_1, \dots, f_c) \subseteq \mathbb{A}^n$, then the canonical bijection

$$\underbrace{V(f_1, \dots, f_c, yg{-}1)}_{\subseteq \mathbb{A}^{n+1}} \to \underbrace{D(g)}_{\subseteq V(f_1,\dots,f_c) \subseteq \mathbb{A}^n}$$

   is a morphism of basic opens. It has inverse given by $(x_1, x_2, \dots, x_n, \frac{1}{g}) : D(g) \to k^{n+1}$. That is, in the (big) category of basic opens, we have

$$V(f_1, \dots, f_c, yg{-}1) \cong D(g).$$

3. A composition of morphisms of basic opens is a morphism of basic opens. So we have a "big" category of basic opens. We don't need a notation for this because we won't often use it.

Remark 2.2.3. A morphism of basic opens $\underset{\subseteq X}{U} \to \underset{\subseteq Y}{V}$ induces a ring homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$. In particular, every point $\mathbb{A}^0 \to U$ induces a (surjective) ring homomorphism $\mathcal{O}_X(U) \to k$. In fact, *every* surjection $\mathcal{O}_X(U) \to k$ comes from a point. That is, there is a bijection

$$\hom_{\mathcal{A}\mathrm{lg}_k}(\mathcal{O}_X(U), k) \cong \hom(\mathbb{A}^0, U) \cong U.$$

This holds more generally,

$$\hom_{\mathcal{A}\mathrm{lg}_k}(\mathcal{O}_X(U), \mathcal{O}_Y(V)) \cong \hom(V, U).$$

Definition 2.2.4. A *quasi-projective variety* or just *variety* is a union of basic opens in some projective variety $\overline{X}$.

$$X = \cup_{\lambda \in \Lambda} U_\lambda \subseteq \overline{X}$$

We continue to write $\mathcal{B}(X)$ for the category of *all* basic opens (of $\overline{X}$) contained in X.

Example 2.2.5. The set $\mathbb{A}^2 \setminus \{0\}$ from last lecture is not (isomorphic to) a basic open, nor a projective variety, but it is a quasi-projective variety. Similarly, $\mathbb{P}^n \setminus \{(0{:}0{:}\ldots{:}0{:}1)\}$ is a quasi-projective variety which is neither affine, nor projective.

Remark 2.2.6. One should think of quasi-projective varieties as being covered by basic opens in the same way that a smooth manifold is covered by opens that are homeomorphic to an open in $\mathbb{R}^n$.

Definition 2.2.7. A *morphism* of quasi-projective varieties is a function

$$f : X \to Y$$

such that for every $x \in X$ there exists a commutative diagram

$$\begin{array}{ccc} x \in & U & \longrightarrow & V \\ & \cap\!| & & \cap\!| \\ & X & \longrightarrow & Y \end{array}$$

such that U, V are basic opens and $U \to V$ is a morphism of basic opens. The category of quasi-projective varieties will be denoted $\mathcal{Q}\mathrm{Proj}$.

In other words, a morphism of quasi-projective varieties is a morphism defined by quotients of polynomials.

Example 2.2.8.

1. For $X \in \mathcal{Q}\mathrm{Proj}$ and $\{U_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{B}(X)$ then $U := \cup_\Lambda U_\lambda \in \mathcal{Q}\mathrm{Proj}$ and the inclusion $U \to X$ is a morphism. In this case U is called an *open subvariety* of X.

2. In the previous notation, we also have $Z = X \setminus U \in \mathcal{Q}\mathrm{Proj}$ and $Z \to X$ is a morphism. In this case Z is called a *closed subvariety* of X.

3. For $X, Y \in \mathcal{Q}\mathrm{Proj}$, the product $X \times Y$ has a canonical structure of quasi-projective variety (via the Segre embedding). The two projections $X \leftarrow X \times Y \to Y$ are morphisms.

2.3 Quasi-coherent $\mathcal{O}_X$-modules

Now we have a nice category of quasi-projective varieties. We are going to fix a quasi-projective variety X and study certain families of vector spaces parameterised by X.

Definition 2.3.1 (Quasi-coherent $\mathcal{O}_X$-module)**.** A *quasi-coherent $\mathcal{O}_X$-module* on a quasi-projective variety X is a functor $F : \mathcal{B}(X)^{op} \to \mathcal{A}\mathrm{b}$ such that:

1. Each $F(U)$ is an $\mathcal{O}_X(U)$-module
2. Each restriction map $F(U) \to F(V)$ (for $V \subseteq U$) is a morphism of $\mathcal{O}_X(U)$-modules
3. For every inclusion $V \subseteq U$ of basic opens, the natural map
$$F(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to F(V) \qquad (*)$$
is an isomorphism

A *morphism* of quasi-coherent $\mathcal{O}_X$-modules is a natural transformation $\phi : F \to G$ such that each component $\phi_U : F(U) \to G(U)$ is a morphism of $\mathcal{O}_X(U)$-modules. If each $F(U)$ is a *finitely generated* $\mathcal{O}_X(U)$-module, then we say that F is *coherent.*

Write $\mathcal{Q}\mathrm{Coh}(X)$ and $\mathcal{C}\mathrm{oh}(X)$ for the categories of quasi-coherent and coherent $\mathcal{O}_X$-modules.

Remark 2.3.2. One can check that if $U = U_0 \cup U_1$ with $U, U_0, U_1 \in \mathcal{B}(X)$ then for any quasi-coherent $\mathcal{O}_X$-module F we have $F(U) = F(U_0) \times_{F(U_0 \cap U_1)} F(U_1)$.[1] Consequently, there is a unique sheaf F' on the X (considered as a topological space via open subvarieties) such that $F'|_{\mathcal{B}(X)} = F$. However, I don't want to talk about sheaves in this series of lectures.

Remark 2.3.3. For every point $x \in U$ and $F \in \mathcal{Q}\mathrm{Coh}(X)$ we get an associated k-vector space
$$F_x := F(U) \otimes_{\mathcal{O}_X(U)} k$$
where $\mathcal{O}_X(U) \to k$ is the homomorphism associated to $x \to U$. The condition $(*)$ ensures that this is independent of U. In this way you can/should think of F as a family of vector spaces parameterised by X, at least if F is coherent.

Example 2.3.4 (Examples in $\mathcal{Q}\mathrm{Coh}(X)$)**.**

1. The functor $\mathcal{O}_X$, and more generally the $\mathcal{O}(D)$ (for $D \in \mathrm{Div}(X)$) are in $\mathcal{C}\mathrm{oh}(X)$.
2. The functor $\mathcal{K}_X : U \mapsto \{\begin{smallmatrix} K_X & U \neq \varnothing \\ 0 & U = \varnothing \end{smallmatrix}$ is in $\mathcal{Q}\mathrm{Coh}(X)$ but not in $\mathcal{C}\mathrm{oh}(X)$ in general.
3. On projective space $\mathbb{P}^n$, the $\mathcal{O}(d)$ for $d \in \mathbb{Z}$ are in $\mathcal{C}\mathrm{oh}(\mathbb{P}^n)$. These are defined via the canonical projection $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ as follows: for basic opens $U \subseteq \mathbb{P}^n$, we have
$$\mathcal{O}(d)(U) = \left\{ \phi : \pi^{-1}(U) \to k \;\middle|\; \begin{array}{c} \phi(\lambda x) = \lambda^d \phi(x) \\ \text{for all } \lambda \in k^*, x \in \pi^{-1}(U) \end{array} \right\}$$

[1] Basically, if $U_0 = D(f)$ and $U_1 = D(g)$ then $U = U_0 \cup U_1$ implies that there are $a, b \in \mathcal{O}_X(U)$ with $1 = af + bg$ in $\mathcal{O}_X(U)$. The claim $F(U) = F(U_0) \times_{F(U_0 \cap U_1)} F(U_1)$ follows from $1 = af + bg$ and the condition $(*)$.

4. Direct sums and products: If $\{F_\lambda\}_{\lambda\in\Lambda}$ is a family in $\mathcal{Q}\mathrm{Coh}(X)$, then $\bigoplus_{\lambda\in\Lambda} F_\lambda$ and $\prod_{\lambda\in\Lambda} F_\lambda$ are in $\mathcal{Q}\mathrm{Coh}(X)$ where $(\bigoplus_{\lambda\in\Lambda} F_\lambda)(U) = \bigoplus_{\lambda\in\Lambda} F_\lambda(U)$ and $(\prod_{\lambda\in\Lambda} F_\lambda)(U) = \prod_{\lambda\in\Lambda} F_\lambda(U)$.

5. Kernels and cokernels: if $\phi : F \to G$ is a morphism in $\mathcal{Q}\mathrm{Coh}(X)$, then $\ker(\phi), \mathrm{coker}(\phi) \in \mathcal{Q}\mathrm{Coh}(X)$ where $(\ker(\phi))(U) = \ker(\phi_U)$ and $(\mathrm{coker}(\phi))(U) = \mathrm{coker}(\phi_U)$.

6. Tensor products and Homs: If $F, G \in \mathcal{Q}\mathrm{Coh}(X)$, then $F \otimes_{\mathcal{O}_X} G \in \mathcal{Q}\mathrm{Coh}(X)$ where $(F \otimes_{\mathcal{O}_X} G)(U) = F(U) \otimes_{\mathcal{O}_X(U)} G(U)$. If $F, G \in \mathcal{Q}\mathrm{Coh}(X)$, then $\mathcal{H}om(F, G) \in \mathcal{Q}\mathrm{Coh}(X)$ where $\mathcal{H}om(F, G)(U) = \hom_{\mathcal{Q}\mathrm{Coh}(U)}(F|_U, G|_U)$.

7. For any closed subvariety $Z \subseteq X$, the ideal sheaf $\mathcal{I}_Z$ defined by $U \mapsto \{f \in \mathcal{O}_X(U) : f|_{Z\cap U} = 0\}$ is in $\mathcal{C}\mathrm{oh}(X)$.

The following proposition follows easily from the definitions.

Proposition 2.3.5. *Let U be a basic open (hence isomorphic to an affine). Then we have equivalences of categories:*

$$\{\ \mathcal{O}_U(U)\text{-modules}\ \} \cong \mathcal{Q}\mathrm{Coh}(U)$$
$$\left\{\begin{matrix}\text{finitely generated} \\ \mathcal{O}_U(U)\text{-modules}\end{matrix}\right\} \cong \mathcal{C}\mathrm{oh}(U)$$

The equivalences are given by:

$$M \mapsto (V \mapsto M \otimes_{\mathcal{O}_U(U)} \mathcal{O}_U(V))$$
$$F(U) \leftarrow\!\shortmid F$$

Definition 2.3.6 (Grothendieck group G_0)**.** Let $X \in \mathcal{Q}\mathrm{Proj}$. The *Grothendieck group*

$$G_0(X) = \frac{\mathbb{Z}[\text{iso. classes of } F \in \mathcal{C}\mathrm{oh}(X)]}{\langle [F] = [F'] + [F''] \mid 0 \to F' \to F \to F'' \to 0\rangle}$$

is the abelian group generated by symbols $[F]$ for $F \in \mathcal{C}\mathrm{oh}(X)$, subject to the relation $[F] = [F'] + [F'']$ whenever there exists a short exact sequence $0 \to F' \xrightarrow{i} F \xrightarrow{p} F'' \to 0$ in $\mathcal{C}\mathrm{oh}(X)$. Here *exact* means that $F' = \ker(i)$ and $F'' = \mathrm{coker}(i)$.

Example 2.3.7 (Examples of Grothendieck groups)**.**

1. **Point:** $G_0(\mathbb{A}^0) \cong \mathbb{Z}$, since $\mathcal{C}\mathrm{oh}(\mathbb{A}^0)$ is equivalent to the category of finite dimensional k-vector spaces.

2. **Affine space:** $G_0(\mathbb{A}^n) \cong \mathbb{Z}$. Since $k[x_1, \dots, x_n]$ has finite global dimension, every $F \in \mathcal{C}\mathrm{oh}(\mathbb{A}^n)$ has a finite free resolution. That is, a sequence of morphisms

$$0 \to \mathcal{O}_{\mathbb{A}^n}^{\oplus r_m} \xrightarrow{d_m} \cdots \xrightarrow{d_2} \mathcal{O}_{\mathbb{A}^n}^{\oplus r_1} \xrightarrow{d_1} \mathcal{O}_{\mathbb{A}^n}^{\oplus r_0} \xrightarrow{d_0} F \to 0,$$

for some r_i such that $\ker(d_i) = \mathrm{im}(d_{i+1})$ for all i. By induction, it follows that $[F] = \sum_i (-1)^i [\mathcal{O}_{\mathbb{A}^n}^{\oplus r_i}] = (\sum_i (-1)^i r_i)\,[\mathcal{O}_{\mathbb{A}^n}]$.

3. **Closed-open decomposition:** If $U \subseteq X$ is open and $Z = X \setminus U$ then there is an exact sequence
$$G_0(Z) \to G_0(X) \to G_0(U) \to 0.$$
This sequence is exact on the left if $Z \subseteq X$ is a *regular embedding*.[2]

4. **Projective space:** $G_0(\mathbb{P}^n) \cong \mathbb{Z}^{\oplus n+1}$ with generators $[\mathcal{O}], [\mathcal{O}(1)], \dots, [\mathcal{O}(n)]$. More generally, if X is a smooth variety then
$$\begin{aligned} G_0(\mathbb{P}^n \times X) &\xleftarrow{\cong} \oplus_{i=0}^n G_0(X) \\ \sum_{i=0}^n [E_i \otimes \mathcal{O}(i)] &\leftarrow\!\shortmid ([E_0], \dots, [E_n]) \end{aligned}$$

5. **Grassmannian:**
$$G_0(Gr(2,4)) \cong \mathbb{Z}^{\oplus 6}.$$
This comes from the decomposition $G_0(Gr(2,4)) \cong \mathbb{A}_0 \oplus \mathbb{A}_1 \oplus (\mathbb{A}_2 \oplus \mathbb{A}_2) \oplus \mathbb{A}_3 \oplus \mathbb{A}_4$ determined by a choice of *flag*.[3]

6. **Elliptic curve:** For an elliptic curve E, we have $G_0(E) \cong \mathbb{Z} \oplus \operatorname{Pic}(E)$ where $\mathbb{Z} \cong \{n[\mathcal{O}]\}$ and $\operatorname{Pic}(E) \cong \{[\mathcal{O}(D)] - [\mathcal{O}]\}$. There is an explicit bijection
$$\begin{aligned} \mathbb{Z} \oplus E &\xrightarrow{\sim} \operatorname{Pic}(E) \\ (n, x) &\mapsto \mathcal{O}(x + (n-1)x_0) \end{aligned}$$
for some fixed point x_0.

7. **Smooth curves:** More generally, for a smooth projective curve C we have $G_0(C) \cong \mathbb{Z} \oplus \operatorname{Pic}(C)$. The subgroup $\operatorname{Pic}^0(C) = \{\mathcal{O}(D) \mid \deg D = 0\}$ has a canonical structure of smooth projective variety of dimension $g =$ the genus of C.

2.4 Pushforward

Definition 2.4.1. Suppose that $f : X \to Y$ is in $\mathcal{Q}\mathrm{Proj}$, $F \in \mathcal{Q}\mathrm{Coh}(X)$. We define f_*F via
$$(f_*F)(V) = \varprojlim_{f(U) \subseteq V} F(U)$$

[2] If X is an affine variety then $Z \subseteq X$ is globally a regular embedding if there exists $f_1, \dots, f_c \in \mathcal{O}_X(X)$ such that $Z = V(f_1, \dots, f_c)$ and each f_{i+1} is a nonzero divisor in $\mathcal{O}_X(X)/\langle f_1, \dots, f_i\rangle$. In general, $Z \subseteq X$ is a regular embedding if $Z \cap V \to V$ is globally a regular embedding for every basic open $V \subseteq X$.

[3] A *flag* is a sequence of subspaces $\{0\} = V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 = V$ with $\dim V_i = i$. In the case of $Gr(2,4)$ we have $d = 4$ and:

(a) $\mathbb{A}_0$ is $\{V_2\}$,

(b) $\mathbb{A}_0 \cup \mathbb{A}_1 =$ is the set of planes W with $V_1 \subset W \subset V_3$,

(c) One $\mathbb{A}_0 \cup \mathbb{A}_1 \cup \mathbb{A}_2$ is $\{W \mid V_1 \subset W\}$,

(d) The other $\mathbb{A}_0 \cup \mathbb{A}_1 \cup \mathbb{A}_2$ is $\{W \mid W \subset V_3\}$,

(e) $\mathbb{A}_0 \cup \mathbb{A}_1 \cup (\mathbb{A}_2 \cup \mathbb{A}_2) \cup \mathbb{A}_3 = \{W \mid W \cap V_2 \neq \{0\}\}$,

(f) $\mathbb{A}_0 \cup \mathbb{A}_1 \cup (\mathbb{A}_2 \cup \mathbb{A}_2) \cup \mathbb{A}_3 \cup \mathbb{A}_4 = Gr(2,4)$.

where the limit is over basic opens U contained in $f^{-1}V$. That is, an element of $(f_*F)(V)$ is a sequence $(s_U)_{f(U)\subseteq V}$ of $s_U \in F(U)$, such that for each $U' \subseteq U$, the transition function sends s_U to $s_{U'}$.

Example 2.4.2.

1. Let X be a smooth curve, D a divisor, and $p : X \to \mathbb{A}^0$ the canonical projection to the base. Then $\mathcal{Q}\mathrm{Coh}(\mathbb{A}^0) \cong \mathcal{V}ec_k$ and
$$p_*\mathcal{O}(D) \cong H^0(X, \mathcal{O}(D)).$$

2. Let $\iota : Z \subseteq X$ be a closed subvariety. Then
$$\iota_*\mathcal{O}_Z \cong \mathcal{O}_X/\mathcal{I}_Z.$$

Definition 2.4.3 (Projective morphism)**.** A morphism $f : X \to Y$ of quasi-projective varieties is called *projective* if it factors as
$$X \overset{\iota}{\hookrightarrow} \mathbb{P}^n \times Y \overset{\text{proj}}{\to} Y$$
where ι is a closed embedding and proj is the projection to the second factor.

Proposition 2.4.4. *If $f : X \to Y \in \mathcal{Q}\mathrm{Proj}$ is projective, then $f_* : \mathcal{Q}\mathrm{Coh}(X) \to \mathcal{Q}\mathrm{Coh}(Y)$ sends coherent sheaves to coherent sheaves.*

Proposition 2.4.5. *There is a unique collection of morphisms of abelian groups $f_* : G_0(X) \to G_0(Y)$ associated to projective morphisms $f : X \to Y$ satisfying the following properties.*

1. *For closed immersions $\iota : Z \hookrightarrow X$, we have $\iota_*([F]) = [\iota_*F]$.*

2. *For projections $\pi : \mathbb{P}^n \times Y \to Y$ we have $\pi_*([\mathcal{O}(i) \otimes \pi^*F]) = [F]$ for $i = 0, \ldots, n$.*

3. *Functoriality: $(g \circ f)_* = g_* \circ f_*$ for composable projective morphisms.*

2.5 Pullbacks

Proposition 2.5.1. *Suppose that $f : X \to Y$ is in $\mathcal{Q}\mathrm{Proj}$ and $G \in \mathcal{Q}\mathrm{Coh}(Y)$. Then there exists a unique $f^*G \in \mathcal{Q}\mathrm{Coh}(X)$ such that:*

1. *If $U \in \mathcal{B}(X)$ and $f(U) \subseteq V$ for some $V \in \mathcal{B}(Y)$, then*
$$(f^*G)(U) = G(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U)$$
where we use the induced ring morphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$.

2. *If $\{U_\lambda\}_{\lambda\in\Lambda}$ is a family of basic opens, closed under intersection, and $U = \cup_{\lambda\in\Lambda}U_\lambda$ is also a basic open, then*
$$(f^*G)(U) = \varprojlim_{\lambda\in\Lambda}(f^*G)(U_\lambda)$$

Remark 2.5.2. The above proposition is a consequence of the sheaf property mentioned in Remark 2.3.2 and the fact that for any basic open $V \subseteq Y$ the preimage $f^{-1}V$ is a union of basic opens.

Example 2.5.3 (Pullback examples)**.**

1. For any morphism $f : X \to Y$, we have
$$f^*\mathcal{O}_Y = \mathcal{O}_X.$$

2. If $\iota : U \to X$ is an open subvariety and $F \in \mathcal{Q}\mathrm{Coh}(X)$, then
$$\iota^* F \cong F|_U$$
where $F|_U$ is simply the functor F restricted to basic opens contained in U.

3. If $p : X \to \mathbb{A}^0$ is the canonical projection and $V \cong k^{\oplus I} \in \mathcal{V}ec_k$ is a vector space with basis or cardinality I. Then
$$p^* V \cong \mathcal{O}_X^{\oplus I}.$$

Recall that there is a very clean description for finitely generated abelian groups up to isomorphism. Namely, they are of the form $\mathbb{Z}^r \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_k$. Coherent sheaves are slightly more complicated, but still quite accessible.

Remark 2.5.4 (Flat pullback)**.** If $j : U \to X$ is an open subvariety, there is an induced group homomorphism $G_0(X) \to G_0(U)$; $[F] \mapsto [j^* F]$. More generally, if $f : Y \to X$ is *flat* in the sense that $f^* : \mathcal{C}\mathrm{oh}(X) \to \mathcal{C}\mathrm{oh}(Y)$ sends exact sequences to exact sequences, then we get a group homomorphism.
$$\begin{aligned} G_0(X) &\to G_0(Y) \\ [F] &\mapsto [j^* F]. \end{aligned}$$

Remark 2.5.5 (Stratification of coherent sheaves)**.** Suppose X is a quasi-projective variety and $F \in \mathcal{C}\mathrm{oh}(X)$. Then there exists a sequence of closed subvarieties $\varnothing = Z_{-1} \subset Z_0 \subset \cdots \subset Z_s = X$ such that if $\iota_i : W_i = Z_i \setminus Z_{i-1} \to X$ is the inclusion, we have
$$\iota_i^* F \cong \mathcal{O}_{W_i}^{\oplus r_i}$$
for some $r_0 \geq r_1 \geq \cdots \geq r_s \in \mathbb{N}$. Geometrically, $\mathcal{O}_X^{\oplus r}$ is the module of sections s of the projection
$$\begin{array}{c} X \times \mathbb{A}^r \\ {\scriptstyle s}\uparrow\ \downarrow{\scriptstyle p} \\ X \end{array}$$
So we can/should think of the coherent sheaf F as the varieties $W_i \times \mathbb{A}^{r_i}$ glued together in some way.

Next lecture we will be concerned with vector bundles, namely, coherent $\mathcal{O}_X$-modules where the rank is locally constant.

Definition 2.5.6 (Vector bundle)**.** A *vector bundle* on a quasi-projective variety X is a coherent $\mathcal{O}_X$-module E such that for every point $x \in X$, there exists a basic open $U \ni x$ and an isomorphism $E|_U \cong \mathcal{O}_U^{\oplus r}$ for some $r \geq 0$.

2.6 Cotangent sheaf

Definition 2.6.1 (Cotangent sheaf)**.** Let X be a quasi-projective variety. Consider the diagonal morphism $\Delta : X \to X \times X;\ x \mapsto (x, x)$ and let $\mathcal{I}_\Delta \subseteq \mathcal{O}_{X\times X}$ be the ideal sheaf of the diagonal. The *cotangent bundle* of X is defined as

$$\Omega_X := \Delta^*(\mathcal{I}/\mathcal{I}^2)$$

where Δ^* denotes pullback along the diagonal morphism. The *tangent bundle* is the dual

$$\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X).$$

Remark 2.6.2. More explicitly, for any basic open $U \subseteq X$ we can find a basic open $V \subseteq X \times X$ such that $V \cap \Delta(X) = U$. In this case,

$$\Omega_X(U) = I/I^2$$

where $I = \{\phi : V \to k \mid \phi(U) = 0\}$.

Remark 2.6.3 (Geometric interpretation)**.** Intuitively, if $Z \subseteq Y$ is a closed subvariety with sheaf of ideals $\mathcal{I}_Z$, then $\mathcal{I}_Z/\mathcal{I}_Z^2$ captures the linear part of functions vanishing along Z. This controls tangent information about the directions perpendicular to Z in Y. When $Z = X$ and $Y = X \times X$, this turns out to be the same as the cotangent bundle.

Example 2.6.4 (Examples of cotangent sheaves)**.**

1. **Affine space:** For $X = \mathbb{A}^n$ we have $\Omega_{\mathbb{A}^n} \cong \mathcal{O}_{\mathbb{A}^n}^{\oplus n}$.

2. **Projective line:** For $X = \mathbb{P}^1$, we have $\Omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$. This can be computed using the Euler sequence:

$$0 \to \Omega_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1} \to 0$$

   The degree -2 reflects the fact that $\mathbb{P}^1$ has "negative curvature" in the sense that it has no global vector fields.

3. **Node curve:** Consider the curve $X = V(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2$ from Lecture 1. At smooth points x, $\dim(\Omega_X)_x = 1$. However, at the singular point 0, the fiber $(\Omega_X)_{(0,0)}$ has dimension 2.

Definition 2.6.5 (Smooth variety)**.** A quasi-projective variety X is called *smooth* of dimension d at a point x if there is a basic open $x \in U$ such that $\Omega_X(U) \cong \mathcal{O}_X(U)^{\oplus d}$. It is called *smooth* if it is smooth at every point.

2.7 Chow groups

Definition 2.7.1 (Dimension and cycles)**.** An irreducible variety Z has *dimension* d if, generically, there are d-linearly independent differential forms. That is, for any non-empty basic open U we have

$$\dim_{K_Z} K_Z \otimes_{\mathcal{O}_Z(U)} \Omega_Z(U) = d.$$

For a quasi-projective variety X, let $X_{(d)}$ denote the set of irreducible subvarieties of X of dimension d. The free abelian group generated by $X_{(d)}$ is denoted

$$\mathcal{Z}_d(X) = \{\Sigma_{i=1}^N n_i[W_i] \mid N, n_i \in \mathbb{N}, W_i \in X_{(d)}\} \cong \bigoplus_{W \in X_{(d)}} \mathbb{Z}$$

An element of $\mathcal{Z}_d(X)$ is called a *d-cycle*.

Example 2.7.2.

1. If X is a smooth curve we have $\mathcal{Z}_0(X) = \mathrm{Div}(X)$.
2. If $Z \to X$ is a closed subvariety, we have a canonical morphism
$$\mathcal{Z}_d(Z) \to \mathcal{Z}_d(X).$$
For a general projective morphism $f : X \to Y$, there is a pushforward $f_* : \mathcal{Z}_d(X) \to \mathcal{Z}_d(Y)$ determined by
$$f_*([Z]) = \begin{cases} [K_Z : K_{f(Z)}] \cdot [f(Z)] & \text{if } \dim Z = \dim f(Z) \\ 0 & \text{otherwise} \end{cases}$$
Here $[K_Z : K_{f(Z)}]$ is the degree of the finite extension of fields $K_{f(Z)} \subseteq K_Z$.
3. **Flat pullback:** If $f : Y \to X$ is a flat morphism between irreducible varieties (see Remark 2.5.4), then there is a pullback map $f^* : \mathcal{Z}_d(X) \to \mathcal{Z}_{d+\dim Y-\dim X}(Y)$. For an irreducible subvariety $Z \subseteq X$ of dimension d, the preimage $f^{-1}(Z)$ may have multiple irreducible components W_i. We define $f^*([Z]) = \sum_i m_i[W_i]$ where m_i are appropriate multiplicities to account for *ramification*. See [Sta25, Tag 0AZE] for more details.
4. **Divisors from functions:** If W is an irreducible variety of dimension $d{+}1$ and $f \in K_W^*$, then f defines a d-cycle $\mathrm{div}(f) \in \mathcal{Z}_d(W)$ given by
$$\mathrm{div}(f) = \sum_{Z \in W_{(d)}} \mathrm{ord}_Z(f) \cdot [Z]$$
where $\mathrm{ord}_Z(f)$ is the order of vanishing of f along Z. See [Sta25, Tag 02AR] for the algebraic definition of $\mathrm{ord}_Z(f)$.
5. Let $D = \sum_i n_i[Z_i] \in \mathcal{Z}_{d-1}(X)$ where X is smooth of dimension d. As for smooth curves, we define the line bundle $\mathcal{O}_X(D)$ by
$$\mathcal{O}_X(D)(U) = \{f \in K_X : \mathrm{div}(f)|_U + D|_U \geq 0\}$$

Definition 2.7.3 (Rational equivalence and Chow groups)**.** The *Chow group* $A_d(X)$ is defined by the exact sequence
$$\bigoplus_{W \in X_{(d+1)}} K_W^* \xrightarrow{\mathrm{div}} \overbrace{\bigoplus_{Z \in X_{(d)}} \mathbb{Z}}^{=\mathcal{Z}_d(X)} \to A_d(X) \to 0.$$

Remark 2.7.4 (Intersection product)**.** Suppose X is irreducible of dimension d. The graded abelian group $\oplus_{i \in \mathbb{N}} A_{d-i}(X)$ admits a structure of graded ring. (Note that we have placed A_i is in degree $d - i$. That is, we are grading by *codimension* codim $= d -$ dim not dimension). We would like to define a structure of graded ring on this graded abelian group using intersection $[V] \cdot [W]$" = "$[V \cap W]$. There are a number of obstacles to this definition.

Firstly, $V \cap W$ may be a union of more than one irreducible subvariety $V \cap W = \cup_r T_r$. Worse, the T_r may not be of codimension $\operatorname{codim} V + \operatorname{codim} W$.

It is a quite technical classical theorem in intersection that for any classes $\alpha \in A_{d-i}(X)$, $\beta \in A_{d-j}(X)$ we *can* find representatives $\alpha = \sum n_k[V_k]$ and $\beta = \sum m_\ell[W_\ell]$ such that the irreducible components $T_{k\ell r}$ of the intersections $V_k \cap W_\ell$ have codimension $i+j$. Even then, we need to account for the fact that the intersections might have some multiplicity. For such cycles in *good position*, the defintion of the intersection product is

$$\alpha \cdot \beta = \sum_{k,\ell,m} n_k m_\ell \cdot i(V_k, W_\ell; T_{k\ell m})[T_{k\ell m}]$$

where the multiplicities come from *Serre's Tor formula.* See [Sta25, Tag 0B08] for more details.

Example 2.7.5 (Examples of Chow groups)**.**

1. For an irreducible variety X of dimension d, we have $A_d(X) \cong \mathbb{Z}$.
2. For a smooth variety X of dimension d the assignment $D \mapsto \mathcal{O}(D)$ induces an isomorphism
$$A_{d-1}(X) \xrightarrow{\sim} \mathrm{Pic}(X)$$
where $\mathrm{Pic}(X) = \{\mathcal{O}(D)\}/\cong$ is the set of isomorphism classes of $\mathcal{O}(D)$ equipped with $\otimes$. For any $L \cong \mathcal{O}(D)$ in $\mathrm{Pic}(X)$, the class $D \in A_{d-1}(X)$ is called the *first Chern class* of L and denoted
$$c_1(L).$$

Now we are going to extend the isomorphism $A_{d-1}(X) \cong \mathrm{Pic}(X)$ to the isomorphism in the GRR theorem. For an abelian group A we write

$$A_{\mathbb{Q}} := A \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Theorem 2.7.6 (Universal property of Chern character)**.** *There exists a unique natural transformation*

$$\begin{aligned} G_0(X) &\to A_*(X)_{\mathbb{Q}} \\ \alpha &\mapsto ch(\alpha) \end{aligned}$$

on smooth quasi-projective varieties X such that:

1. *For line bundles L, we have $ch([L]) = e^{c_1(L)} := \sum_{n\in\mathbb{N}} \frac{1}{n!} c_1(L)^n$.*
2. *For $\alpha, \beta \in G_0(X)$, we have $ch(\alpha+\beta) = ch(\alpha) + ch(\beta)$.*
3. *For flat morphisms $f : Y \to X$ (see Remark 2.5.4) and vector bundles E (see Definition 2.5.6), we have*
$$ch(f^*[E]) = f^*(ch([E])).$$

These morphisms induce isomorphisms

$$\mathrm{ch} : G_0(X)_{\mathbb{Q}} \cong A_*(X)_{\mathbb{Q}}$$

Remark 2.7.7. The groups $G_0(X)$ and $A_*(X)$ are contravariantly functorial for flat morphisms and ch is actually a natural transformation for this functoriality. That is, $\mathrm{ch}(f^*\alpha) = f^*\mathrm{ch}(\alpha)$ when f is flat. In this lecture we are interested in projective pushfowards. In order to make ch natural in projective pushfowards we need to use Todd classes.

Theorem 2.7.8 (Universal property of Todd classes). *There exists a unique natural transformation*

$$\begin{aligned} G_0(X) &\to A_*(X)_{\mathbb{Q}} \\ \alpha &\mapsto \mathrm{td}(\alpha) \end{aligned}$$

on smooth quasi-projective varieties X such that:

1. *For line bundles L, we have* $\mathrm{td}([L]) = \frac{c_1(L)}{1-e^{-c_1(L)}}$.[4]
2. *For $\alpha, \beta \in G_0(X)$, we have* $\mathrm{td}(\alpha+\beta) = \mathrm{td}(\alpha)\cdot\mathrm{td}(\beta)$.
3. *For flat morphisms $f : Y \to X$ (see Remark 2.5.4) and vector bundles E (see Definition 2.5.6), we have*

$$\mathrm{td}(f^*[E]) = f^*(\mathrm{td}([E])).$$

Remark 2.7.9 (Splitting principle). To prove existence and uniqueness of Chern and Todd classes, one uses the *splitting principle*: any vector bundle E of rank r on X can be pulled back to a sum of line bundles $L_1 \oplus \cdots \oplus L_r$ via some (flat projective surjective) $f : Y \to X$ that induces an injection $f^* : A_*(X) \to A_*(Y)$. This reduces the problem to line bundles, where the classes are explicitly defined.

2.8 Restatement

We can now restate the Grothendieck-Riemann-Roch theorem with all the machinery we've developed:

Theorem 2.8.1 (Grothendieck–Riemann–Roch, Restated). *Suppose $f : X \to Y$ is a projective morphism of smooth quasi-projective varieties. Then the following square commutes, and the horizontal morphisms are isomorphisms.*

$$\begin{array}{ccccc} G_0(X)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}} & A_*(X)_{\mathbb{Q}} & \xrightarrow{\mathrm{td}(\mathcal{T}_X)\cdot -} & A_*(X)_{\mathbb{Q}} \\ \downarrow f_* & & & & \downarrow f_* \\ G_0(Y)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}} & A_*(Y)_{\mathbb{Q}} & \xrightarrow{\mathrm{td}(\mathcal{T}_Y)\cdot -} & A_*(Y)_{\mathbb{Q}} \end{array}$$

Remark 2.8.2. When X is a smooth projective curve and $Y = \mathbb{A}^0$, this recovers the classical Riemann–Roch theorem from Lecture 1. In this case we have:

- $f_* : \underbrace{G_0(X)}_{\cong \mathbb{Z}\oplus \mathrm{Pic}(X)} \to \underbrace{G_0(\mathbb{A}^0)}_{\cong\mathbb{Z}}$ sends $L \in \mathrm{Pic}(X)$ to

$$\dim H^0(X, L) - \dim H^0(X, \mathcal{H}om(L, \Omega_X)).$$

  This comes from *Serre duality.*

[4]The power series $\frac{x}{1-e^{-x}} \in \mathbb{Q}[[x]]$ is defined to be the inverse of the power series $\frac{1-e^{-x}}{x} = 1 - \frac{x}{2} + \frac{x^2}{6} - \ldots$.

- For $D \in \mathrm{Div}(X)$ we have $f_*(D) = \deg D$. This follows from the definition.
- $\mathrm{td}(\mathcal{T}_X) = 1 + \frac{1}{2}c_1(T_X) = 1 - \frac{1}{2}K$ where $K = \mathrm{div}(\Omega_X)$. This follows from the definitions.
- $\mathrm{td}(\mathcal{T}_Y) = 1$.
- We have $\deg K = 2g - 2$. This can be obtained in various ways, but all of them involve some kind of theorem.

So for $L \cong \mathcal{O}(D)$, the square in the statement becomes

$$L \longmapsto 1 + D$$

$$\begin{array}{ccccccc}
L & & \mathbb{Z} \oplus \mathrm{Pic}(X) & \xrightarrow{1+c_1} & \mathbb{Z} \oplus A_0(X) & \xrightarrow{(1-\frac{1}{2}K)\cdot -} & \mathbb{Z} \oplus A_0(X) \\
\downarrow & & \downarrow f_* & & & & \downarrow (0,\deg) \\
\begin{array}{c}\dim H^0(X,L) \\ - \dim H^0(X,\mathcal{H}om(L,\Omega_X))\end{array} & & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z}
\end{array}$$

and the GRR formula becomes:

$$\begin{aligned}
\dim H^0(X, L) - \dim H^0(X, \mathcal{H}om(L, \Omega_X)) \\
&= \mathrm{ch}(f_*[L]) \cdot \mathrm{td}(\mathcal{T}_Y) \\
&\overset{[GRR]}{=} f_*(\mathrm{ch}([L]) \cdot \mathrm{td}(\mathcal{T}_X)) \\
&= f_*((1 + D) \cdot (1 - \tfrac{1}{2}K)) \\
&= f_*(1 + D - \tfrac{1}{2}K) \\
&= \deg D - \tfrac{1}{2}\deg K \\
&= \deg(D) + 1 - g
\end{aligned}$$

Remark 2.8.3 (Sketch of proof)**.** The proof proceeds by:

1. Reducing to the case where f is a closed embedding or a projection using the factorization of projective morphisms
2. For closed embeddings, use *deformation to the normal cone* to reduce to the case of a regular closed immersion. That is, a closed immersion which locally looks like a zero section $Z \to Z \times \mathbb{A}^c$. In this case, one does a concrete calculation.
3. For projections $\mathbb{P}^n \times Y \to Y$, one uses the explicit description of $G_0(\mathbb{P}^n \times Y)$ and the fact that $\mathrm{td}(\Omega_{\mathbb{P}^n}) = (1 + H + H^2 + \ldots + H^n)$ where H is the class of a hyperplane.

3 $K_{\leq 1}$

A reference for this lecture is [Wei13].

Recall that if $X \subseteq \mathbb{A}^n$ is an affine variety, then all information (except the embedding into $\mathbb{A}^n$) is contained in the ring $\mathcal{O}_X(X)$. That is, up to isomorphism, we can reconstruct the variety X from the ring $\mathcal{O}_X(X)$. More precisely, we have an equivalence of categories:

$$\{\text{affine } k\text{-varieties}\} \simeq \{\text{finitely generated } k\text{-algebras}\}^{op}$$

In this lecture I want to work with the larger category of *affine schemes*. This is equivalent to, and sometimes defined as, the opposite of the category $\mathcal{R}\text{ing}$ of commutative rings with unit. That is, in this lecture we will work with rings. If I want to think of a ring as a geometric object I will write $\operatorname{Spec}(R)$, but in this lecture you should just think of this as notation. I don't want to talk about locally ringed topological spaces.

$$\{\text{affine schemes}\} \cong \mathcal{R}\text{ing}^{op}$$
$$\operatorname{Spec}(R) \leftrightarrow R$$

3.1 K_0

Last time we considered $G_0(X)$. For a ring R, this is defined as:

$$G_0(R) = \frac{\mathbb{Z}\left[\begin{array}{c}\text{finitely generated}\\ R\text{-modules}\end{array}\right]}{\left\langle [M]-[L]-[N] \;\middle|\; \begin{array}{c} 0 \to L \to M \to N \to 0 \\ \text{is exact}\end{array}\right\rangle}$$

The group G_0 is good for many things, but not everything.

Example 3.1.1 (Limitations of G_0)**.**

1. G_0 **doesn't detect nilpotent elements**: Consider $R = k[x]/(x^2)$ where k is a field. Every $M \in \mathcal{C}\text{oh}(R)$ is a finite direct sum of copies of

   $$M = k[x]/(x^2) = R \quad \text{or} \quad M' = R/(x) \cong k.$$

   We have an exact sequence $0 \to k \xrightarrow{x} R \to k \to 0$, so in $G_0(R)$ we get $[R] = [k] + [k] = 2[k]$. Thus $G_0(R) \cong \mathbb{Z} \cong G_0(k)$. More generally, for a Noetherian ring R with nilradical $\operatorname{Nil}(R)$, we have

   $$G_0(R) \cong G_0(R/\operatorname{Nil}(R)).$$

   We will see below that for local rings R we have $K_1(R) = R^*$. In particular, K_1 can see nilpotents.

2. G_0 **cannot see certain singularities.** For example, consider the cusp $X = V(y^2 - x^3) \subseteq \mathbb{A}^2$ and the affine line $Y = \mathbb{A}^1$. Both have $G_0(X) \cong G_0(Y) \cong \mathbb{Z}$, even though X has a cusp singularity while Y is smooth. However, for integral Noetherian rings of (Krull) dimension one, we have $K_0(R) \cong \mathbb{Z} \oplus \operatorname{Pic}(R)$, [Weibel II.2.6.3]. Since $\operatorname{Pic}(X) \neq 0$ while $\operatorname{Pic}(\mathbb{A}^1) = 0$, we get $K_0(X) \neq K_0(\mathbb{A}^1)$, so K_0 (defined below) can distinguish these cases where G_0 cannot.

3. **Functoriality**: The functor G_0 has functoriality of a cohomology *with compact support* rather than a "cohomology theory". More precisely, G_0 is covariant $G_0(X) \to G_0(Y)$ for projective morphisms $X \to Y$ of varieties, and contravariant $G_0(X) \to G_0(Y)$ for flat morphisms $Y \to X$, but it is not contravariant for all morphisms.

4. **No ring structure in general**: The semiring structure on $\mathcal{C}\mathrm{oh}(X)_{/\cong}$ coming from $\otimes$ does not descend in general to a ring structure on $G_0(X)$.

In this lecture instead of all coherent sheaves we will focus on vector bundles.

Definition 3.1.2. A *vector bundle* on a variety X is a coherent sheaf $\mathcal{E}$ that is locally of constant rank, meaning that for every point $x \in X$, there exists a basic open $U \ni x$ such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$ for some integer $r \geq 0$.

Example 3.1.3 (Examples of vector bundles)**.**

1. The structure sheaf $\mathcal{O}_X$ is a vector bundle of rank 1.

2. The $\mathcal{O}(D)$ (for $D \in \mathcal{Z}_{d-1}(X)$ on a smooth irreducible X of dimension d) are vector bundles of rank 1.

3. The $\mathcal{O}(d)$ in $\mathcal{C}\mathrm{oh}(\mathbb{P}^n)$ are vector bundles of rank 1.

4. A variety X is smooth of dimension d if and only if Ω_X is a vector bundle of rank d.

5. If E and F are vector bundles, then $E \oplus F$ is a vector bundle.

6. If $E \oplus F \cong G$ where G is a vector bundle, then both E and F are vector bundles.

Recall that for affine varieties (and more generally, for affine schemes) the category of $\mathcal{C}\mathrm{oh}(X)$ is equivalent to the category of finitely generated $\mathcal{O}_X(X)$-modules. We can also identify the subcategory of vector bundles.

Proposition 3.1.4. *Let X be an affine variety and $R = \mathcal{O}_X(X)$. Then we have an equivalence of categories:*

$$\left\{ \begin{array}{c} \textit{vector bundles} \\ \textit{on } X \end{array} \right\} \simeq \left\{ \begin{array}{c} \textit{finitely generated} \\ \textit{projective } R\textit{-modules} \end{array} \right\}$$

Note that a module P is finitely generated and projective *if and only if there exists some Q and an isomorphism $P \oplus Q \cong R^{\oplus n}$.*

Algebraic K-theory of a ring R is then defined as follows.

Definition 3.1.5 (K_0)**.** For a ring R, we define $K_0(R)$ as:

$$K_0(R) = \frac{\mathbb{Z}\left[\begin{array}{c} \text{finitely generated} \\ \text{projective } R\text{-modules} \end{array} \right]}{\left\langle [P] - [N] - [Q] : \begin{array}{c} 0 \to N \to P \to Q \to 0 \text{ exact} \\ \text{with } N, P, Q \text{ projective} \end{array} \right\rangle}$$

Remark 3.1.6. Since surjections $P \twoheadrightarrow Q$ towards projective modules Q have sections $P \hookleftarrow Q$, for sequences as above we have $P \cong N \oplus Q$ and so $K_0(R)$ can also be defined as:

$$K_0(R) = \frac{\mathbb{Z}\begin{bmatrix} \text{finitely generated} \\ \text{projective } R\text{-modules} \end{bmatrix}}{\langle [P \oplus Q] - [P] - [Q] \rangle}$$

This description shows that $K_0(R)$ is the group completion of the abelian monoid $(\mathrm{Proj}(R)_{/\cong}, \oplus)$ of isomorphism classes of projective R-modules. That is, the map $(\mathrm{Proj}(R)_{/\cong}, \oplus) \to K_0(R)$ is the unique homomorphism of abelian monoids such that for every abelian group A,

$$\hom_{\mathcal{A}\mathrm{b}}(K_0(R), A) \xrightarrow{\sim} \hom_{\mathrm{CommMon}}(\mathrm{Proj}(R)_{/\cong}, A)$$

Definition 3.1.7 (Regular ring)**.** A Noetherian ring R is called *regular* if every finitely generated R-module admits a finite resolution by finitely generated projective R-modules. That is, for every $M \in \mathcal{C}\mathrm{oh}(R)$ there exists an exact sequence

$$0 \to P_0 \to P_1 \to \cdots \to P_n \to M \to 0$$

with each $P_i \in \mathrm{Proj}(R)$.

Remark 3.1.8. Usually regularity is defined in terms of regular sequences. The equivalence to the above definition is an actual theorem requiring substantial commutative algebra, [Sta25, Tag 00O7].

Corollary 3.1.9. *For regular Noetherian rings R, e.g., $R = \mathcal{O}_X(X)$ when X is a smooth affine variety, we have*

$$G_0(R) = K_0(R).$$

Remark 3.1.10 (Sketch of proof)**.** By induction, we see that for resolutions as in the above definition, we have $[M] = \sum_{i=0}^{n} (-1)^i [P_i]$ in $G_0(R)$. In particular, $K_0(R) \to G_0(R)$ is surjective. Similarly, any relation in $G_0(R)$ can be replaced by a relation in $K_0(R)$ only involving projective modules.

3.2 K_1

Definition 3.2.1 (Milnor square, [Wei13, Exam.I.2.6])**.** A *Milnor square* is a pullback square of surjections

$$\begin{array}{ccc} R & \overset{p}{\twoheadrightarrow} & R' \\ \downarrow & & \downarrow \\ S & \underset{q}{\twoheadrightarrow} & S' \end{array}$$

Remark 3.2.2.

1. Explicitly, we are asking that $R = \ker(S \oplus R' \to S')$.

2. Often in the definition of Milnor squares there is the condition that $\ker(p) \cong \ker(q)$, but this is automatic from the above formulation.

Remark 3.2.3. Surjections of rings correspond to closed immersions of schemes, and pullbacks of rings correspond to pushouts of schemes. That is $R = S \times_{S'} R'$ means $\mathrm{Spec}(R) = \mathrm{Spec}(S) \sqcup_{\mathrm{Spec}(S')} \mathrm{Spec}(R')$.

Theorem 3.2.4 ([Wei13, Thm.II.2.9]). *Suppose we have a Milnor square as above. Then there is a long exact sequence*

$$GL_\infty(S') \to K_0(R) \to K_0(S) \oplus K_0(R') \to K_0(S')$$

where $GL_\infty(S') = \varinjlim(GL_1(S') \to GL_2(S') \to GL_3(S') \to \dots)$.

Remark 3.2.5. In fact, one might expect that such a sequence exists because the category $\mathrm{Proj}(R)$ is equivalent to a category whose objects are triples (P, Q, ϕ) consisting of an S-module P, an R'-module Q and an isomorphism $\phi : P \otimes_S S' \cong Q \otimes_{R'} S'$, [Wei13, Theorem I.2.7].

The failure of injectivity suggests that we need to keep track of more information than just isomorphism classes. Automorphisms seem to be important.

Observation 3.2.6.

1. $K_0(R)$ is the group completion of the monoid $(\mathrm{Proj}(R)_{/\cong}, \oplus)$.
2. Automorphisms seem to be important (Theorem 3.2.4).

Instead of working with isomorphism classes $\mathrm{Proj}(R)_{/\cong}$, let's consider the *groupoid*

$$\mathrm{Proj}(R)^{\cong}.$$

This is the category whose objects are finitely generated projective modules and whose morphisms are isomorphisms.

The groupoid $\mathrm{Proj}(R)^{\cong}$ has a symmetric monoidal structure

$$\mathrm{Proj}(R)^{\cong} \times \mathrm{Proj}(R)^{\cong} \to \mathrm{Proj}(R)^{\cong}$$

given by direct sum $\oplus$ and the isomorphisms $P \oplus Q \cong Q \oplus P$. We want to form its group completion. That is, a universal functor

$$\mathrm{Proj}(R)^{\cong} \to \mathcal{G}$$

towards a *grouplike* symmetric monoidal groupoid. That is, a symmetric monoidal groupoid such that for every object X the functor $X \oplus -$ is an equivalence. *Universal* means that for any grouplike symmetric monoidal groupoid $\mathcal{G}'$ it should induce an equivalence of groupoids

$$\mathrm{Fun}(\mathcal{G}, \mathcal{G}') \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Proj}(R)^{\cong}, \mathcal{G}')$$

where Fun is the groupoid of monoidal functors.

Observation 3.2.7. Suppose that $\Phi : \mathrm{Proj}(R)^{\cong} \to \mathcal{G}$ is a functor towards a grouplike symmetric monoidal groupoid. Use $\oplus$ for operations on both groupoids and $\mathbb{O}$ for the unit object. So $- \oplus \mathbb{O}$ is isomorphic to the identity functor.

1. Since $X \oplus - : \mathcal{G} \to \mathcal{G}$ is an equivalence for any X, we have

   $$\mathrm{Aut}_{\mathcal{G}}(\mathbb{O}) \cong \mathrm{Aut}_{\mathcal{G}}(X)$$

   for all objects X.

2. For each n, we have a group homomorphism $\mathrm{GL}_n(R) = \mathrm{Aut}(R^{\oplus n}) \to \mathrm{Aut}_{\mathcal{G}}(\Phi(R^{\oplus n})) \cong \mathrm{Aut}_{\mathcal{G}}(\mathbb{O})$ compatible with inclusions $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$, in the sense that we have commutative triangles

$$\begin{array}{ccccccc} GL_1(R) & \longrightarrow & GL_2(R) & \longrightarrow & GL_3(R) & \longrightarrow & \dots \\ & \searrow & & \searrow & \downarrow & & \\ & & & & \mathrm{Aut}_{\mathcal{G}}(\mathbb{O}) & & \end{array}$$

This gives a map

$$\mathrm{GL}_\infty(R) \to \mathrm{Aut}_{\mathcal{G}}(\mathbb{O}).$$

3. By the Eckmann-Hilton argument, since composition $\circ$ and direct sum $\oplus$ both give operations on the set $\mathrm{Aut}(\mathbb{O})$, and they distribute over each other:

$$(\alpha \oplus \beta) \circ (\gamma \oplus \delta) = (\alpha \circ \gamma) \oplus (\beta \circ \delta)$$

the group $\mathrm{Aut}(\mathbb{O})$ is abelian. Therefore we get a group homomorphism

$$\frac{\mathrm{GL}_\infty(R)}{[\mathrm{GL}_\infty(R), \mathrm{GL}_\infty(R)]} \to \mathrm{Aut}_{\mathcal{G}}(\mathbb{O})$$

Here we write $[G, G] = \langle ghg^{-1}h^{-1} \rangle$ for the commutator subgroup of a group G. So $G \to G/[G, G]$ is the largest abelien quotient of G.

Definition 3.2.8 (K_1)**.** For a ring R, we define:

$$K_1(R) = \frac{\mathrm{GL}_\infty(R)}{[\mathrm{GL}_\infty(R), \mathrm{GL}_\infty(R)]}.$$

Here is some evidence that this is a good definition.

Theorem 3.2.9 ([Wei13, Thm.III.2.6])**.** *Suppose we have a Milnor square as above. Then there is a long exact sequence*

$$K_1(R) \longrightarrow K_1(S) \oplus K_1(R') \longrightarrow K_1(S') \longrightarrow K_0(R) \longrightarrow K_0(S) \oplus K_0(R') \longrightarrow K_0(S')$$

3.3 $K_{<0}$

Now we move on to negative K-theory. One motivation for negative K-theory comes from trying to extend exact sequence to the right.

Here is our (admittedly weak) hint as to what a good definition might be.

Theorem 3.3.1 (Fundamental Theorem for K_1, [Wei13, III.3.6])**.** *For every ring R, there is an exact sequence*

$$0 \to K_1(R) \to K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t, t^{-1}]) \to K_0(R) \to 0.$$

Definition 3.3.2 (Negative K-theory)**.** For a ring R and $n > 0$, we inductively define $K_{-n}(R)$ to be the cokernel

$$K_{-n}(R) := \operatorname{coker}\Big(K_{-n+1}(R[t]) \oplus K_{-n+1}(R[t^{-1}]) \to K_{-n+1}(R[t, t^{-1}]) \Big).$$

Theorem 3.3.3 ([Wei13, III.4.3])**.** *Suppose we are given a Milnor square as above. Then the sequence of Theorem 3.2.9 continues as :*

$$\begin{aligned}
\dots \longrightarrow K_0(R) \longrightarrow K_0(S) \oplus K_0(R') \longrightarrow K_0(S') \longrightarrow{}& \\
\longrightarrow K_{-1}(R) \longrightarrow K_{-1}(S) \oplus K_{-1}(R') \longrightarrow K_{-1}(S') \longrightarrow{}& \\
\cdots \qquad & \\
\dashrightarrow K_{-n}(R) \longrightarrow \dots &
\end{aligned}$$

4 K-theory as the universal localising invariant

In the last lecture we generalised K_0 to the pair K_0, K_1 by moving from the set $\mathrm{Proj}(R)_{/\cong}$ to the groupoid $\mathrm{Proj}(R)^{\cong}$. In this lecture we generalise this process to get higher $K_0, K_1, K_2, \ldots$ by considering ∞-groupoids. These form an ∞-category.

Remark 4.0.1 (Historical development)**.** Jumping directly to localising invariants is a bit misleading. The development of algebraic K-theory has several key phases: Grothendieck introduced K_0 for vector bundles in the 1950s; Bass [Bas68] extended this to K_0 of rings and introduced K_1 in 1968; Milnor defined K_2 using the Steinberg group in the early 1970s; Bass also introduced negative K-theory in the 1970s; Quillen revolutionized the field with his "plus construction" giving higher K-groups in the mid-1970s; Waldhausen developed K-theory for categories with cofibrations and weak equivalences in the 1980s; and Thomason–Trobaugh [TT90] established K-theory for schemes using perfect complexes.

4.1 Simplicial sets

References for simplicial sets and ∞-groupoids (which used to be called Kan complexes):

1. May, Simplicial objects in algebraic topology, [May67].
2. Bousfield–Kan, Homotopy limits, completions and localizations, [BK72].
3. Goerss–Jardine, Simplicial homotopy theory, [GJ99].

Groups (groupoids with one object)	$\subseteq$	Groupoids (categories for which every morphism is invertible)	$\subseteq$	Categories	$\to$	Directed graphs
∞-groups	$\subseteq$	∞-groupoids	$\subseteq$	∞-categories	$\subseteq$	Simplicial sets

Recall that a directed graph consists of a set G_0 of vertices, a set G_1 of edges and two morphisms

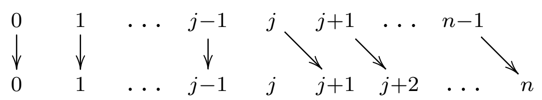
$$d_0, d_1 : G_1 \rightrightarrows G_0$$

which associate to each edge $e \in G_1$ a source $d_1 e \in G_0$ and a target $d_0 e \in G_0$.

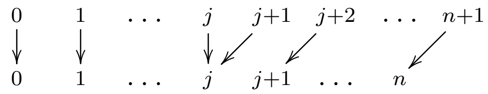
We can generalise this in higher dimensions by allowing "n-dimensional edges" for all $n \in \mathbb{N}$. The information of all these higher edges and how they are related to each other is organised in the concept of a *simplicial set*.

Definition 4.1.1. We write $\Delta \subseteq \mathrm{LinOrdSet}$ for the full subcategory of the category LinOrdSet of linearly ordered sets whose objects are finite and non-empty. In other words, those linearly ordered sets which are isomorphic to the linearly ordered set $[n] = \{0 < 1 < \cdots < n\}$ for some $n \geq 0$. Morphisms are those morphisms of sets $p : [n] \to [m]$ such that $i \leq j \implies p(i) \leq p(j)$.

Example 4.1.2. For each $0 \leq j \leq n$ with $n \neq 0$, the *face* morphism $\delta_j : [n-1] \to [n]$ are defined as the unique injection which does not have j in its image.



Example 4.1.3. For each $0 \leq j \leq n$ the *degeneracy* morphism $\sigma_j : [n{+}1] \to [n]$ is defined as the unique surjection which sends both j and $j{+}1$ to j.



Exercise. Show that every morphism $[n] \to [m]$ can be written as a composition of face and degeneracy morphisms.

Definition 4.1.4. The category of simplicial sets $\mathcal{S}\mathrm{et}_\Delta$ is the category of functors $\Delta^{op} \to \mathcal{S}\mathrm{et}$, so

$$\mathcal{S}\mathrm{et}_\Delta := \mathrm{PSh}(\Delta)$$

Given such a functor $X : \Delta^{op} \to \mathcal{S}\mathrm{et}$ we write $X_n := X([n])$. Elements of X_n are called *n-simplices* of X..

Example 4.1.5. For any simplicial set $X : \Delta^{op} \to \mathcal{S}\mathrm{et}$ the morphisms

$$d_j : X_n \to X_{n-1}.$$

corresponding to the δ_j are called *face* morphisms. For $x \in X_n$ we call $d_j x$ the *jth face of x*. The morphisms

$$s_j : X_n \to X_{n+1}.$$

corresponding to the σ_j are called *degeneracy* morphisms.

Example 4.1.6 (Δ^n)**.** For each n, the functor $\Delta^n := \hom_\Delta(-, [n]) : \Delta^{op} \to \mathcal{S}\mathrm{et}$ defines a simplicial set. By Yoneda's Lemma, for any $X \in \mathcal{S}\mathrm{et}_\Delta$,

$$\hom_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n, X) \cong X_n.$$

Example 4.1.7 ($\partial\Delta^n$)**.** Consider the morphisms of simplicial sets $\delta_j : \Delta^{n-1} \to \Delta^n$. We define

$$\partial\Delta^n = \bigcup_{j=0}^{n} \delta_j(\Delta^{n-1})$$

as the union of these faces. Explicitly, $(\partial\Delta^n)_j \subseteq (\Delta^n)_j = \hom_\Delta([j], [n])$ is the set of morphisms $[j] \to [n]$ of linearly ordered sets which are not surjective. This can also be described as the colimit

$$\partial\Delta^n = \varinjlim_{[i] \subsetneq [n]} \Delta^i$$

In particular, for any other simplicial set X we have

$$\hom(\partial\Delta^n, X) = \varprojlim_{\substack{[i] \subsetneq [n] \\ n-2 \leq i \leq n-1}} X_i.$$

That is, a morphism $\partial\Delta^n \to X$ is the same thing as a set of $(n{-}1)$-simplices $x_0, \ldots, x_n \in X_{n-1}$ satisfying $d_i x_j = d_j x_i$.

Definition 4.1.8 (Λ^n_j)**.** For $0 \leq j \leq n$ we define the *jth horn* as the union

$$\Lambda^n_j = \bigcup_{i \neq j} \delta_i(\Delta^{n-1}).$$

Equivalently, $(\Lambda^n_j)_i \subseteq (\Delta^n)_i = \hom_\Delta([i], [n])$ is the set of those $[i] \to [n]$ whose image does *not contain* the subset $\{0, 1, \ldots, j{-}1, j{+}1, \ldots, n\}$.

Example 4.1.9 (Sing X)**.** Define

$$\Delta^n_{\text{top}} := \left\{ (x_0, \dots, x_n) \mid 0 \le x_i \le 1; \sum_{i=0}^{n} x_i = 1 \right\} \subseteq \mathbb{R}^{n+1}$$

to be the convex hull of the standard basis vectors $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. So Δ^0_{top} is a point, Δ^1_{top} is a line segment, Δ^2_{top} is a triangle, Δ^3_{top} is a tetrahedron, ...

Any morphism $p : [n] \to [m]$ in Δ defines an $\mathbb{R}$-linear morphism $\mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$; $e_i \mapsto e_{p(i)}$, which restricts to a continuous morphism $\Delta^n_{\text{top}} \to \Delta^m_{\text{top}}$. In this way we get a functor

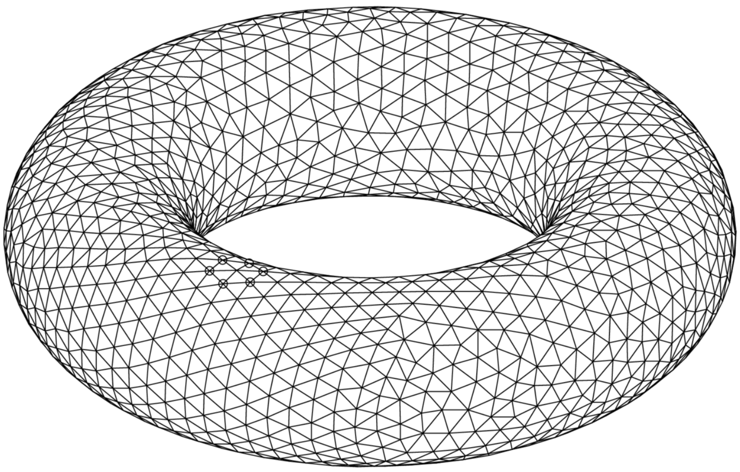
$$\Delta \to \mathcal{T}\text{op}; \qquad [n] \mapsto \Delta^n_{\text{top}}$$

from Δ to the category of topological spaces. For any other topological space X, the assignment

$$\operatorname{Sing} X : [n] \mapsto \hom_{\mathcal{T}\text{op}}(\Delta^n_{\text{top}}, X)$$

defines a simplicial set. Explicitly,

1. $\operatorname{Sing}_0 X$ is the set of points of X,
2. $\operatorname{Sing}_1 X$ is the set of paths in X,
3. $\operatorname{Sing}_2 X$ is the set of triangles in X,
4. ...



Source: Wikimedia Commons, CC BY-SA license

Example 4.1.10 (Product of simplicial sets)**.** For simplicial sets X and Y, their product $X \times Y$ is defined by

$$(X \times Y)_n = X_n \times Y_n$$

with structure morphisms acting componentwise.

Example 4.1.11 (Mapping simplicial sets)**.** For simplicial sets X and Y, the *mapping simplicial set* $\operatorname{Map}(X, Y)$ has n-simplices given by

$$\operatorname{Map}_{\mathcal{S}\text{et}_\Delta}(X, Y)_n = \hom_{\mathcal{S}\text{et}_\Delta}(X \times \Delta^n, Y)$$

The maps associated to $[n] \to [m]$ are induced by the corresponding $\Delta^n \to \Delta^m$.

4.2 Infinity groupoids

Definition 4.2.1 (Kan fibration)**.** A morphism $f : X \to Y$ of simplicial sets is a *Kan fibration* if for every $0 \leq j \leq n$ with $0 \neq n$ and commutative square

$$\begin{array}{ccc} \Lambda^n_j & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & Y \end{array}$$

a dashed morphism exists making two triangles commutative. A simplicial set X is an ∞*-groupoid* if the canonical morphism $X \to \Delta^0$ is a Kan fibration.

Example 4.2.2. If X is a topological space, then $\operatorname{Sing} X$ is an ∞-groupoid. In fact, by the homotopy hypothesis, there is an equivalence of ∞-categories between the ∞-category of topological spaces and the ∞-category of ∞-groupoids.

Definition 4.2.3 (Homotopy groups)**.** For an ∞-groupoid X and a basepoint $x \in X_0$, the nth homotopy group $\pi_n(X, x)$ is defined as follows. Consider

$$Z_{n,x} := \left\{ \Delta^n \xrightarrow{f} X \;\middle|\; \begin{array}{ccc} \partial\Delta^n & \longrightarrow & x \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow{f} & X \end{array} \text{ commutes} \right\}$$

We define an equivalence relation on $Z_{n,x}$: two morphisms $f, g \in Z_{n,x}$ are equivalent if there exists a morphism $H : \Delta^{n+1} \to X$ such that

$$\begin{aligned} H|_{\delta_0(\Delta^n)} &= f \\ H|_{\delta_i(\Delta^n)} &= x \qquad i = 1, \ldots, n-1 \\ H|_{\delta_n(\Delta^n)} &= g \end{aligned}$$

Then

$$\pi_n(X, x) = Z_{n,x}/\sim .$$

Remark 4.2.4. The lifting property defining ∞-groupoids ensures this actually is an equivalence relation. It is not an equivalence relation in a general simplicial set.

Example 4.2.5. If $n = 0$ then $\pi_0(X, x)$ is the set of connected components of the ∞-groupoid X.

Example 4.2.6. Suppose $p : E \to B$ is a Kan fibration between ∞-groupoids, $e \in E$, $b = p(e)$, and $F = \{b\} \times_B E$. Then there is a long exact sequence of groups

$$\ldots \pi_{n+1}(B, b) \to \pi_n(F, e) \to \pi_n(E, e) \to \pi_n(B, b) \to \ldots$$

for $n > 0$ ending with an exact sequence of pointed sets

$$\cdots \to \pi_1(B, b) \to \pi_0(F, e) \to \pi_0(E, e) \to \pi_0(B, b).$$

Here, a sequence $(A, a) \xrightarrow{f} (B, b) \xrightarrow{g} (C, c)$ of pointed sets is exact if $f(A) = g^{-1}(c)$.

Definition 4.2.7 (Weak equivalence)**.** A morphism $f : X \to Y$ of ∞-groupoids is a *weak equivalence* if it induces isomorphisms on all homotopy groups:

$$\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is a bijection for all $n \geq 0$ and all basepoints $x \in X_0$, and $\pi_0 X \to \pi_0 Y$ is surjective.

Example 4.2.8. In the notation of Example 4.2.6, if all homotopy groups of F are trivial and $\pi_0(E) \to \pi_0(B)$ is surjective, then $E \to B$ is a weak equivalence.

Example 4.2.9. The inclusion

$$X = \{z \in \mathbb{C} \mid |z| = 1\} \to \mathbb{C} \setminus \{0\} = Y$$

induces a weak equivalence $\operatorname{Sing} X \to \operatorname{Sing} Y$ of ∞-groupoids. Indeed, the existence of a deformation retract $Y \to X$; $z \mapsto z/|z|$ implies all homotopy groups are isomorphic.

Example 4.2.10 (Homotopy equivalence)**.** Two ∞-groupoids X and Y are *homotopy equivalent* if there exist morphisms $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the respective identity morphisms. Here, two morphisms $a, b : A \to B$ are *homotopic* if there exists a morphism $H : A \times \Delta^1 \to B$ such that $H|_{A \times \{0\}} = a$ and $H|_{A \times \{1\}} = b$. By Whitehead's theorem for ∞-groupoids, a morphism is a homotopy equivalence if and only if it is a weak equivalence.

4.3 Infinity categories

The standard reference for ∞-categories is Lurie, Higher Topos Theory, [Lur09]. There are also a number of other texts, e.g., Haugseng, Yet another introduction to ∞-categories, [Hau25].

Definition 4.3.1 (Boardman, Vogt, 1973)**.** An *∞-category* is a simplicial set C such that for every $0 < i < n$ and each diagram

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & C \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

there exists a (not necessarily unique) dashed arrow making a commutative triangle.

Example 4.3.2. If $I \in \mathcal{S}\mathrm{et}_\Delta$ and $C \in \mathcal{C}\mathrm{at}_\infty$, then $\operatorname{Map}(I, C) \in \mathcal{C}\mathrm{at}_\infty$. A morphism $D \to C$ between two ∞-categories is called a *functor*.

(Exercise: Show that for any $X \in \mathcal{S}\mathrm{et}_\Delta$, if Y is an ∞-groupoid, resp. ∞-category, then so is $\operatorname{Map}_{\mathcal{S}\mathrm{et}_\Delta}(X, Y)$)

Example 4.3.3. Let C be a small category. Considering the ordered sets $[n]$ as categories[5] $\{0 \to 1 \to \cdots \to n\}$ the assignment

$$N : [n] \mapsto \operatorname{Fun}([n], C)$$

sending $[n]$ to the set of functors $[n] \to C$ defines a simplicial set. This is called the *nerve* of C. Explicitly,

[5] So, for $0 \leq i, j \leq n$ there is exactly one morphism $i \to j$ if $i \leq j$, and no morphisms otherwise.

1. $N(C)_0$ is the set of objects of C,
2. $N(C)_1$ is the set of (all) morphisms in C,
3. The two morphisms $N(C)_1 \rightrightarrows N(C)_0$ induced by the two functors $[0] \rightrightarrows [1]$ send morphisms in $N(C)_1$ to their source and target.

$$(X \xrightarrow{f} Y) \qquad \mapsto \qquad X, Y$$

4. The morphism $N(C)_0 \to N(C)_1$ induced by $[1] \to [0]$ sends each object to its identity morphism.

$$X \qquad \mapsto \qquad (X \xrightarrow{\mathrm{id}_X} X)$$

5. $N(C)_2$ is the set of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$.
6. The three maps $d_0, d_1, d_2 : N(C)_2 \rightrightarrows N(C)_1$ induced by the three faithful functors $[1] \rightrightarrows [2]$ send $\xrightarrow{f}\xrightarrow{g}$ to g, $g \circ f$, and f respectively.

$$\begin{array}{ccc} & Y & \\ {}^{f}\nearrow & & \searrow^{g} \\ X & \xrightarrow[g\circ f]{} & Z \end{array} \qquad \mapsto \qquad (Y \xrightarrow{g} Z), \quad (X \xrightarrow{g\circ f} Y), \quad (X \xrightarrow{f} Y)$$

7. More generally, $N(C)_n$ is the set of sequences of n composable morphisms $\xrightarrow{f_1} \dots \xrightarrow{f_n}$ and the various maps $N(C)_n \to N(C)_m$ come from various combinations of composition and inserting identities.

Definition 4.3.4. Let $C \in \mathcal{C}\mathrm{at}_\infty$. Elements of C_0 are called *objects* and elements of C_1 are called 1-morphisms, or often just *morphisms*. Given two morphisms $f, g \in C_1$ such that $d_0 f = d_1 g$ (equivalently, a morphism of simplicial sets $\Lambda^2_1 \to C$), for any factorisation $\Lambda^2_1 \to \Delta^2 \overset{\sigma}{\dashrightarrow} C$, the morphism $d_1\sigma \in C_1$ will be called a *composition* of g and f. For any object $X \in C_0$, the morphism $s_0 X \in C_1$ is called the *identity morphism* of X, and written id_X.

Example 4.3.5. A morphism $f : X \to Y$ in an ∞-category is called an *equivalence* if there exists a morphism $g : Y \to X$ and 2-cells σ and τ of the form

$$\begin{array}{ccc} & Y & \\ {}^{f}\nearrow & \sigma & \searrow^{g} \\ X & \xrightarrow[\mathrm{id}_X]{} & X \end{array} \qquad\qquad \begin{array}{ccc} Y & \xrightarrow{\mathrm{id}_Y} & Y \\ {}_{g}\searrow & \tau & \nearrow_{f} \\ & X & \end{array}$$

Definition 4.3.6 (Mapping space)**.** For an ∞-category C and objects $x, y \in C_0$, the *mapping space* $\mathrm{Map}_C(x, y)$ is defined as the pullback

$$\mathrm{Map}_C(x, y) := \{x\} \times_C \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^1, C) \times_C \{y\}$$

in the 1-category $\mathcal{S}\mathrm{et}_\Delta$ where the fiber products are taken with respect to the source and target maps $d_1, d_0 : C_1 \to C_0$. The *morphism set* is

$$\hom_C(x, y) = \pi_0 \mathrm{Map}_C(x, y).$$

Example 4.3.7. Any ∞-groupoid is an ∞-category. In particular, for any topological space X, the simplicial set $\operatorname{Sing} X$ is an ∞-category.

Example 4.3.8. There exists an ∞-category $\mathcal{G}\mathrm{pd}_\infty$ whose objects are small ∞-groupoids and whose mapping spaces are equivalent to the mapping simplicial set defined above.

$$\operatorname{Map}_{\mathcal{G}\mathrm{pd}_\infty}(X,Y) \simeq \operatorname{Map}_{\mathcal{S}\mathrm{et}_\Delta}(X,Y).$$

Example 4.3.9. For any ∞-category C, there is a maximal sub-∞-groupoid

$$C^{\cong} \subseteq C$$

called the *core* of C. It has the same objects as C, but only the invertible morphisms. More precisely, $(C^{\cong})_n$ consists of those n-simplices $x \in C_n$ such that all images of x in C_1 are invertible morphisms in C.

Example 4.3.10. There exists an ∞-category $\mathcal{C}\mathrm{at}_\infty$ whose objects are small ∞-categories and whose mapping spaces are equivalent to

$$\operatorname{Map}_{\mathcal{C}\mathrm{at}_\infty}(C,D) \simeq \operatorname{Map}_{\mathcal{S}\mathrm{et}_\Delta}(C,D)^{\cong}.$$

Example 4.3.11. Let $R \in \mathcal{R}\mathrm{ing}$. A *bounded chain complex* of projectives is a sequence of morphisms

$$[\dots \overset{d_{n+1}}{\to} P_n \overset{d_n}{\to} P_{n-1} \overset{d_{n-1}}{\to} \dots\dots]$$

in $\operatorname{Proj}(R)$ such that $d_{n-1}d_n = 0$ and only finitely many P_n are non-zero. A *morphism* of chain complexes $f_\bullet : P_\bullet \to Q_\bullet$ is a sequence a morphisms $f_n : P_n \to Q_n$ making commutative squares. A *homotopy* of morphisms $h : f_\bullet \sim g_\bullet$ is a sequence of morphisms $h_n : P_n \to Q_{n+1}$ such that $f_n - g_n = d_{n+1}h_n + h_{n-1}d_n$.

There exists an ∞-category $D^b(R)$ whose objects are bounded chain complexes of projectives and morphisms are

$$\hom_{D^b(R)}(P_\bullet, Q_\bullet) \cong \frac{\{f_\bullet : P_\bullet \to Q_\bullet\}}{\text{homotopy}}$$

For more details on this example, see Lurie, Higher Algebra, [Lur17, §1.3.1 and §1.3.2].

Example 4.3.12. The simplicial set Δ^1 is an ∞-category but not an ∞-groupoid (the unique non-identity morphism is not invertible). The pushout $\Delta^0 \sqcup_{\partial\Delta^1} \Delta^1$ of simplicial sets is not an ∞-category. (The unique non-degenerate 1-simplex cannot be composed with itself).

4.4 Limits

Definition 4.4.1 (Limits in ∞-categories)**.** Let $C \in \mathcal{C}\mathrm{at}_\infty$ and $I \in \mathcal{S}\mathrm{et}_\Delta$. Given an object $X \in C$, write $\gamma(X) \in \operatorname{Fun}(I,C)$ for the constant functor $I \to \Delta^0 \overset{X}{\to} C$.

For a morphism $F : I \to C$, a *limit* of F is an object $\varprojlim F \in C$ together with a morphism $\gamma(\varprojlim F) \to F$ in $\operatorname{Fun}(I,C)$ such that for any object $X \in C$, the natural map

$$\operatorname{Map}_C(X, \varprojlim F) \to \operatorname{Map}_{\operatorname{Fun}(I,C)}(\gamma(X), F)$$

is an equivalence of ∞-groupoids. Dually, a *colimit* of F is an object $\varinjlim F \in C$ together with a natural transformation $F \to \gamma(\varinjlim F)$ such that for any object $X \in C$, the natural map

$$\operatorname{Map}_C(\varinjlim F, X) \to \operatorname{Map}_{\operatorname{Fun}(I,C)}(F, \gamma(X))$$

is an equivalence of ∞-groupoids.

Example 4.4.2 (Initial and terminal objects)**.** An *initial object* $\varnothing$, resp. *terminal object* $*$, in an ∞-category C is a limit, resp. colimit, of the unique functor $\varnothing \to C$ from the empty ∞-category. Equivalently, it is an object such that for any $X \in C$, the mapping space $\mathrm{Map}_C(\varnothing, X)$, resp. $\mathrm{Map}_C(X, *)$, is *contractible*, i.e., all homotopy groups are trivial, or equivalently, $\mathrm{Map} \cong \Delta^0$.

1. In the ∞-category of ∞-groupoids $\mathcal{G}\mathrm{pd}_\infty$:
   - Initial object $\varnothing$, resp. terminal object $*$: the empty ∞-groupoid, resp. the point Δ^0. Note that just as we can have very large categories which are equivalent to the punctual category, we can have quite large ∞-groupoids which are terminal objects. For example,
     (a) for any n, the ∞-groupoid $\mathrm{Sing}\,\mathbb{R}^n$ is a terminal object of $\mathcal{G}\mathrm{pd}_\infty$.
     (b) For any $X \in \mathcal{G}\mathrm{pd}_\infty$ and $x \in X$, the ∞-groupoid $\mathrm{Map}(\Delta^1, X) \times_X \{x\}$ of paths towards x is a terminal object of $\mathcal{G}\mathrm{pd}_\infty$.
2. In the ∞-category of pointed ∞-groupoids:
   - Initial and terminal object: Δ^0 (the point, which is both initial and terminal, making this a pointed category)
3. In the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories:
   - Initial object $\varnothing$, resp. terminal object $*$: the empty ∞-category $\varnothing$, resp. the terminal ∞-category Δ^0 with one object and only identity morphisms
   - Terminal object: $*$ (the terminal ∞-category with one object and only identity morphisms)
4. In the derived category $D^b(R)$:
   - Initial object: 0 (the zero chain complex)
   - Terminal object: 0 (the zero chain complex)

   As in $\mathcal{G}\mathrm{pd}_\infty$ we can have "large" objects which are also initial / terminal. For example $[\cdots \to 0 \to P \overset{=}{\to} P \to 0 \to \dots]$ is equivalent to 0 for any P. So it is also an intial / terminal object.

Example 4.4.3 (Products and disjoint unions)**.** Products and Coproducts are limits and colimits over $\Delta^1 = \Delta^0 \sqcup \Delta^0$.

1. In $\mathcal{G}\mathrm{pd}_\infty$ and $\mathcal{C}\mathrm{at}_\infty$, products and coproducts are as in the 1-category $\mathcal{S}\mathrm{et}_\Delta$.
2. In $D^b(R)$ coproducts and products are isomorphic:

$$(P_\bullet \times Q_\bullet)_n = P_n \oplus Q_n = (P_\bullet \sqcup Q_\bullet)_n.$$

Example 4.4.4 (Pullbacks)**.**

1. In $\mathcal{G}\mathrm{pd}_\infty$ the limit of a diagram $X \to Z \leftarrow Y$ is modelled by the simplicial set

$$X \times_Z \mathrm{Map}(\Delta^1, Z) \times_Z Y.$$

   If $X = \{z\} = Y$ is a vertex of Z, then the limit $\{z\} \times_Z \{z\}$ in $\mathcal{G}\mathrm{pd}_\infty$ is the space of loops from z to z.

2. In $\mathcal{C}\mathrm{at}_\infty$: The limit of a diagram $C \to D \leftarrow E$ is modelled by the simplicial set

$$C \times_D \mathrm{Map}(\mathrm{Iso}(\Delta^1), D) \times_D E.$$

where $\mathrm{Iso}(\Delta^1)$ is the free ∞-groupoid generated by Δ^1. Explicitly, we have $\mathrm{Iso}(\Delta^1)_m = \prod_{i \in [m]}\{0, 1\}$

3. In $D^b(R)$: For chain complexes $P_\bullet \xrightarrow{f} R_\bullet \xleftarrow{g} Q_\bullet$, the limit is modelled by the complex

$$\mathrm{Cone}(P_\bullet \oplus Q_\bullet \xrightarrow{(f,-g)} R_\bullet)[-1].$$

Here, for a general morphism of complexes $A_\bullet \xrightarrow{f} B_\bullet$ one defines

$$\mathrm{Cone}(A_\bullet \xrightarrow{f} B_\bullet)_n = B_n \oplus A_{n-1}$$

with differential $\left[\begin{smallmatrix} d_B & f \\ 0 & d_A \end{smallmatrix}\right]$ and

$$A_\bullet[m]_n = A_{n-m}$$

Example 4.4.5. Suppose that a quasi-projective variety X is the union of two basic opens $X = U \cup V$. Then the derived category $D^b(X)$ is equivalent to the pullback

$$D^b(X) = D^b(U) \times_{D^b(U \cap V)} D^b(V)$$

in $\mathcal{C}\mathrm{at}_\infty$ where we define $D^b(Y) := D^b(\mathcal{O}_Y(Y))$ for affine varieties. More generally, if X is a union of finitely many basic opens $X = \cup_{i=1}^n U_i$ then the derived category is the limit

$$D^b(X) = \varprojlim_{i_0 < \cdots < i_m} D^b(U_{i_0} \cap \cdots \cap U_{i_n})$$

Pushouts in $\mathcal{C}\mathrm{at}_\infty$ and $\mathcal{G}\mathrm{pd}_\infty$ are not so explicit in general.

Example 4.4.6. The pushout of a diagram $P_\bullet \leftarrow R_\bullet \to Q_\bullet$ in $D^b(R)$ is modelled by the chain complex

$$\mathrm{Cone}(R_\bullet \to P_\bullet \oplus Q_\bullet).$$

4.5 Stable infinity categories

Definition 4.5.1. An ∞-category C is *stable* if it satisfies the following conditions:

(Sta0) It is pointed. That is, it admits both an initial object $\varnothing$ and a final object $*$ and an equivalence $\varnothing \cong *$. We write 0 for such an object.

(Sta1) It admits *fibres* and *cofibres*. That is, for every $f : X \to Y$, both

$$\mathrm{fib}(f) := X \times_Y 0 \quad \text{and} \quad \mathrm{cof}(f) = 0 \sqcup_X Y$$

exist.

(Sta2) A commutative square of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array} \qquad (9)$$

is cartesian if and only if it is cocartesian.

Example 4.5.2. For $R \in \mathcal{R}\text{ing}$ the category $D^b(R)$ is stable.

Example 4.5.3. Suppose that C is a pointed ∞-category. The stabilisation $\mathrm{Sp}(C)$ is defined as the limit

$$\mathrm{Sp}(C) = \varprojlim(\dots \xrightarrow{\Omega} C \xrightarrow{\Omega} C \xrightarrow{\Omega} C)$$

in $\mathcal{C}\mathrm{at}_\infty$ where $\Omega X := * \times_X *$. Every object in $\mathrm{Sp}(C)$ gives rise to a sequence of objects $E_0, E_1, \dots$ in C equipped with equivalences $E_{n-1} \xrightarrow{\sim} \Omega E_n$. We also have a canonical functor

$$\Omega^\infty : \mathrm{Sp}(C) \to C$$

namely, projection to the last component.

Proposition 4.5.4 ([HA, Cor.1.4.2.23])**.** *Let C be an ∞-category which admits finite limits, and T a stable quasi-category. Then composition with the functor Ω^∞ induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{Lex}(T, \mathrm{Sp}(C)) \to \mathrm{Fun}^{Lex}(T, C)$$

where Fun^{Lex} *means the full subcategory of functors sending finite limits to finite limits, i.e.,* left exact *functors.*

Definition 4.5.5. The stabilisation of the category of pointed ∞-groupoids is called the category of *spectra* and is denoted

$$\mathcal{S}\mathrm{pt} = \mathrm{Sp}(\mathcal{G}\mathrm{pd}_{\infty,*}).$$

Remark 4.5.6. The equivalences $E_n \xrightarrow{\sim} \Omega E_{n+1}$ induces isomorphisms

$$\pi_i E_n \xrightarrow{\sim} \pi_{i+1} E_{n+1}.$$

Given a spectrum E we define

$$\pi_i E := \pi_{i+j} E_j; \qquad i \in \mathbb{Z}$$

for any j such that the right hand side is defined.

Example 4.5.7. Suppose that $C \subseteq D$ in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$ is a full sub-∞-category closed under finite limits and finite colimits. The *Verdier localisation* D/C is a stable ∞-category with the same objects as D and mapping spaces

$$\mathrm{Map}_{D/C}(X, Y) = \varinjlim_{X' \to X} \mathrm{Map}_D(X', Y)$$

where the colimit is over the ∞-category of morphisms $s : X' \to X$ such that $\mathrm{fib}(s) \in C$.

Example 4.5.8. The canonical functor $D^b(\mathbb{Z}) \to D^b(\mathbb{Q})$ identifies $D^b(\mathbb{Q})$ with the Verdier quotient of $D^b(\mathbb{Z})$ by the full sub-∞-category $D^b(\mathbb{Z})_{\mathrm{tor}} \subseteq D^b(\mathbb{Z})$ of those complexes $P_\bullet$ whose homology groups $H_n(P) = \frac{\ker(P_n \to P_{n-1})}{\mathrm{im}(P_{n+1} \to P_n)}$ are torsion.

$$D^b(\mathbb{Z})/D^b(\mathbb{Z})_{\mathrm{tor}} \cong D^b(\mathbb{Q}).$$

Example 4.5.9. Let $X \in \mathcal{Q}\mathrm{Proj}$. The coherent $\mathcal{O}_{\mathbb{P}^n}$-modules $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$ induce a sequence of stable sub-∞-categories

$$0 = C_{-1} \subset C_0 \subset \cdots \subset C_n = D^b(\mathbb{P}^n \times X).$$

Namely, C_i is generated by $\mathcal{O}, \ldots, \mathcal{O}(i)$. We have equivalences

$$C_i/C_{i-1} \cong D^b(X).$$

via the functors

$$\begin{aligned} D^b(X) &\to D^b(\mathbb{P}^n \times X) \\ F &\mapsto \mathcal{O}(i) \otimes p^* F \end{aligned}$$

4.6 Localising invariants

The original reference for the characterisation of algebraic K-theory as the universal localising invariant is [BGT13]. For a graduate course about this see [HW21]

Write

$$\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$$

for the ∞-category whose objects are stably ∞-categories and morphisms are exact functors. That is, functors which preserve finite limits and finite colimits.

Definition 4.6.1 (Idempotent completion)**.** A stable ∞-category C is *idempotent complete* if every idempotent endomorphism $p : X \to X$ (i.e., $p \circ p \simeq p$) admits a splitting. That is, there exists an isomorphism

$$X \cong Y \oplus Z$$

such that p is identified with the composition $X \to Y \to X$. We write $\mathcal{C}\mathrm{at}_\infty^{\mathrm{perf}} \subseteq \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$ for the full subcategory of idempotent complete stable ∞-categories. The inclusion admits a left adjoint

$$(-)^\natural : \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{perf}}$$

called idempotent completion.

Proposition 4.6.2. *A square*

$$\begin{array}{ccc} C & \xrightarrow{i} & D \\ \downarrow & & \downarrow{\scriptstyle p} \\ 0 & \longrightarrow & E \end{array}$$

of small stable ∞-categories is a bifibre square in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$ if it is a cartesian square in $\mathcal{C}\mathrm{at}_\infty$ and E is (equivalent to) the Verdier localisation D/C of D along C. Such a square is a fibre square in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{perf}}$ if $E \cong (D/C)^\natural$.

Definition 4.6.3. A functor $F : \mathcal{C}\mathrm{at}_\infty^{\mathrm{perf}} \to \mathcal{S}\mathrm{pt}$ is called a *localising invariant* if it sends bifibre squares to bifibre squares.

4.7 K-theory

Theorem 4.7.1. *The ∞-category of localising functors $F : \mathcal{C}\mathrm{at}_\infty^{\mathrm{perf}} \to \mathcal{S}\mathrm{pt}$ equipped with a natural transformation*

$$(-)^{\cong} \to \Omega^\infty F$$

admits an initial object K.

Remark 4.7.2. That is, K is the localising invariant "as close as possible" to the core functor which sends $C \in \mathcal{C}\mathrm{at}_\infty^{\mathrm{perf}}$ to its associated ∞-groupoid $C^{\cong}$. Note that in a precise mathematical sense, $D^b(R)$ is the closest stable ∞-category to $\mathrm{Proj}(R)$, [Lur17, Thm.1.3.3.2, Thm.1.3.3.8].

Theorem 4.7.3. *For $n \leq 1$ and $R \in \mathcal{R}\mathrm{ing}$, the homotopy groups*

$$\pi_n K(D^b(R))$$

are the groups $K_n(R)$ we saw last time.

Example 4.7.4. Using the decomposition associated to $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$, we have

$$K(D^b(\mathbb{P}^n_R)) \cong \bigoplus_{i=0}^{n} K(R).$$

Example 4.7.5. For a finite field $\mathbb{F}_q$ with q elements, we have:

$$K_n(\mathbb{F}_q) = 0 \quad \text{for } n \leq -1 \tag{10}$$
$$K_0(\mathbb{F}_q) \cong \mathbb{Z} \tag{11}$$
$$K_1(\mathbb{F}_q) \cong \mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)\mathbb{Z} \tag{12}$$
$$K_{2i}(\mathbb{F}_q) = 0 \quad \text{for } i \geq 1 \tag{13}$$
$$K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1)\mathbb{Z} \quad \text{for } i \geq 1 \tag{14}$$

The pattern in positive degrees follows from Quillen's computation, while negative K-groups vanish since $\mathbb{F}_q$ has finite global dimension.

Example 4.7.6. For the field of rational numbers $\mathbb{Q}$:

$$K_n(\mathbb{Q}) = 0 \quad \text{for } n \leq -1 \tag{15}$$
$$K_0(\mathbb{Q}) \cong \mathbb{Z} \tag{16}$$
$$K_1(\mathbb{Q}) \cong \mathbb{Q}^\times \tag{17}$$
$$K_2(\mathbb{Q}) \cong (\mathbb{Z}/4)^* \times \prod_{p \text{ odd prime}} (\mathbb{Z}/p)^* \tag{18}$$

The computation of $K_2(\mathbb{Q})$ is due to Tate and follows from Gauss's first proof of quadratic reciprocity. For higher K-groups, Borel proved that (modulo torsion):

$$K_{4k+1}(\mathbb{Z})/\mathrm{tors} = \mathbb{Z} \quad \text{for } k > 0$$
$$K_i(\mathbb{Z})/\mathrm{tors} = 0 \quad \text{for } i > 2; i \neq 4k+1$$

Example 4.7.7 (Matsumoto's theorem)**.** For any field k, Matsumoto's theorem states that the second K-group is given by

$$K_2(k) = \frac{k^\times \otimes_{\mathbb{Z}} k^\times}{\langle a \otimes (1-a) \mid a \neq 0, 1\rangle}.$$

The relations $a \otimes (1-a) = 0$ are called the *Steinberg relations.*

5 Recent advances; open problems

5.1 Weibel's Conjecture

Theorem 5.1.1 (Weibel's Conjecture [Wei80], Kerz–Strunk–Tamme [KST18])**.** *For a Noetherian scheme X of finite Krull dimension d, we have $K_i(X) = 0$ for all $i < -d$.*

There were many previous partial cases of Theorem 5.1.1. A non-exhaustive list is: [Bas68], [Wei89], [Hae04], [Kri10a], [Cis13], [Kel14], [KS17], [KS22]. Many of them relied on versions of Theorem 5.1.2.

Theorem 5.1.2 (Descent for blow-ups in regular centres [SGA71, Exposé VII], [TT90])**.** *Suppose $Z \to X$ is a regular immersion of classical schemes. Then there is a cartesian square*

$$\begin{array}{ccc} K(X) & \longrightarrow & K(Z) \\ \downarrow & & \downarrow \\ K(Bl_X Z) & \longrightarrow & K(E) \end{array}$$

where $Bl_X Z$ is the blow-up of X along Z, and E is the exceptional divisor.

If $Z \to X$ is not regular, then the above theorem fails! This can be corrected by using formal completions. The idea to use formal completions can be considered as a version of Grothendieck's *theorem on formal functions.* Such formal completions in K-theory were considered by many authors such as [KS02], [Cor06], [Kri10b], [GH06], [GH11], [Mor16], [Mor18].

Theorem 5.1.3 ([KST18, Thm. A])**.** *Suppose X is a Noetherian scheme, $Y \to X$ is proper, $Z \to X$ is a closed immersion, and the induced morphism $Y \setminus E \to X \setminus Z$ is an isomorphism, then*

$$\begin{array}{ccc} K(X) & \longrightarrow & \varprojlim_n K(Z_n) \\ \downarrow & & \downarrow \\ K(Y) & \longrightarrow & \varprojlim_n K(E_n) \end{array}$$

is a cartesian square of spectra, where Y_n and E_n denote the nth infinitesimal thickenings of Z in X and E in Y respectively.

Theorem 5.1.3 was proven using *derived schemes.* These are schemes where instead of rings, we use *derived rings* such as simplicial rings or animated rings.

5.2 Prismatic cohomology and K-theory of finite rings

Theorem 5.2.1 (Antieau–Krause–Nikolaus, [AKN22])**.** *We can now calculate the K-theory of rings of the form $\mathbb{Z}/p^k$ to quite high degrees using computers.*

As $K(\mathbb{Z}/p^k; \mathbb{Z}_p) \simeq \tau_{\geq 0} TC(\mathbb{Z}/p^k; \mathbb{Z}_p)$ by [DGM13, HM03], it is enough to determine TC of these rings. To do so, we use the filtration on TC constructed by Bhatt–Morrow–Scholze in [BMS19]. If R is a quasisyntomic ring, there is a complete decreasing filtration $F^{\geq \star}_{\text{syn}} TC(R; \mathbb{Z}_p)$ with associated graded pieces

$$F^{=i}_{\text{syn}} TC(R; \mathbb{Z}_p) \simeq \mathbb{Z}_p(i)(R)[2i],$$

where $\mathbb{Z}_p(i)(R)$ is the weight i syntomic cohomology of R introduced in [BMS19]. The syntomic complexes provide a p-adic analogue of the motivic filtration on K-theory.

There is a spectral sequence which larger vanishes and we end up with a three term complex.

Theorem 5.2.2 (Antieau-Krause-Nikolaus [AKN22]). *For $i > 1$, there is an explicit cochain complex*

$$\mathbb{Z}_p^{in-1} \xrightarrow{syn_0} \mathbb{Z}_p^{2(in-1)} \xrightarrow{syn_1} \mathbb{Z}_p^{in-1}$$

quasi-isomorphic to $\mathbb{Z}_p(i)(\mathbb{Z}/p^n)$. The terms are free $\mathbb{Z}_p$-modules of the given ranks in cohomological degrees 0, 1, *and* 2.

This can be put into a computer. A new theorem coming out of this is:

Theorem 5.2.3 (Even vanishing theorem [AKN22]). *If $i > \frac{p}{2(p-1)^2}(p^n-1)$, then $H^2(\mathbb{Z}_p(i)(\mathbb{Z}/p^n)) = 0$ and hence $K_{2i-2}(\mathbb{Z}/p^n) = 0$ if additionally $i > 2$.*

This gives a quantitative bound on the vanishing of even K-groups, extending earlier results of Angeltveit [Ang11].

Corollary 5.2.4 (Order formula [AKN22]). *For any $\mathbb{Z}/p^n$,*

$$\frac{\#K_{2i-1}(\mathbb{Z}/p^n; \mathbb{Z}_p)}{\#K_{2i-2}(\mathbb{Z}/p^n; \mathbb{Z}_p)} = p^{i(n-1)}.$$

5.3 Failure of the telescope conjecture

The telescope conjecture was one of the seven Ravenel conjectures from 1984 concerning the stable homotopy groups of spheres and chromatic homotopy theory. While six of the seven conjectures were eventually proven, the telescope conjecture remained open until 2023.

One way to understand the objects appearing in the conjecture is in terms of *tensor triangulated geometry*. The category of spectra $\mathcal{S}$pt as a monoidal structure, and we can pretend it is $D^b(X)$ for some (hypothetical) topological space X. The points of the topological space Spec($\mathcal{S}$pt) correspond to certain monoid objects in $\mathcal{S}$pt. These are indexed by primes p and a natural number n. In this setting one implicitly fixes a prime p, and talks about the *n-th Morava K-theory* $K(n)$. Just as with $D^b(R)$ in scheme theory, we can perform certain “localisations” in $\mathcal{S}$pt.

Conjecture 5.3.1 (Ravenel's telescope conjecture, [Rav84]). For each prime p and height $n \geq 0$, the telescopic localization $L_{T(n)}$ agrees with the $K(n)$-localization $L_{K(n)}$ on the category of spectra.

Here, $T(n)$ denotes the nth telescope, a spectrum built from the Bousfield-Kuhn functor, and $K(n)$ is the nth Morava K-theory spectrum.

Theorem 5.3.2 (Burklund-Hahn-Levy-Schlank, [BHLS23]). *The telescope conjecture is false. For each prime p and height $n+1 \geq 2$, there exist spectra X such that $L_{T(n+1)}X \not\simeq L_{K(n+1)}X$.*

Proof sketch. The authors construct explicit counterexamples using algebraic K-theory. They show that the $T(n+1)$-localized algebraic K-theory of $BP\langle n\rangle^{(h\mathbb{Z})}$ is not $K(n+1)$-local, where $BP\langle n\rangle^{(h\mathbb{Z})}$ denotes a certain truncated Brown-Peterson spectrum with additional structure. □

5.4 Stable homotopy groups of spheres

There should be a section here about recent work of Isaksen, Wang, and Xu [IWX20, IWX23] calculating stable homotopy groups of spheres using motivic homotopy theory. However, I can't possibly improve on Piotr Pstragowski's beautiful talk: `https://www.ms.u-tokyo.ac.jp/video/conference/2023Motives_in_Tokyo/cf2022-111.html` so you should just go watch that.

5.5 Atiyah-Hirzebruch spectral sequence

There is a generalisation of the Grothendieck–Riemann-Roch isomorphism that was conjectured by Lichtenbaum, Quillen, and Beilinson in various forms. It was constructed by multiple authors. See [Weibel, VI.4.4] for a historical account.

Theorem 5.5.1. *For X smooth over a field, there are functors $\mathbb{Z}(n) : Sm_k^{op} \to D(\mathbb{Z})$ and a spectral sequence*

$$H^{p-q}(X, \mathbb{Z}(-q)) \implies K_{-p-q}(X)$$

which degenerates rationally to give isomorphisms

$$K_n(X)_{\mathbb{Q}} \cong \oplus_{i=0}^{d} H^{2i-n}(X, \mathbb{Q}(i)) \cong \oplus_{i=0}^{d} CH_{d-i}(X, n)_{\mathbb{Q}}.$$

Here, Bloch's higher Chow groups $CH_*(X, n)$ are more general versions of the Chow groups $A_*(X)$ from the second lecture. The group $CH_j(X, n)$ is generated by cycles in $\mathcal{Z}_j(X \times \Delta_k^n)$ which intersect the boundary properly. Here $\Delta_k^n = V(\sum x_m - 1) \subseteq \mathbb{A}^{n+1}$ and the boundary is the intersection with the axes $V(x_0 x_1 \ldots x_n)$ (Similar to Δ_{top}^n from the last lecture).

Example 5.5.2.

1. $\mathbb{Z}(n) = 0$; $n < 0$
2. $\mathbb{Z}(0) = \mathbb{Z}$;
3. $\mathbb{Z}(1) \cong \mathbb{G}_m[1]$.

There are a number of models for the complexes $\mathbb{Z}(n)$. As mentioned above, one of them is via Bloch's higher Chow groups, [Blo86]. Another is via the *slice filtration*, [Voe02]. This is a sequence of presheaves of spectra

$$\cdots \to f_{n+1}K \to f_n K \to \cdots \to f_0 K = K \in \mathrm{PSh}(Sm_k, \mathcal{S}\mathrm{pt})$$

such that

$$\mathbb{Z}(n)[2n] = \mathrm{cofib}(f_{n+1}K \to f_n K),$$

[Lev08]. The presheaf $f_n K$ is essentially, the colimit of all maps the form $(\mathbb{P}^1, \infty)^{\wedge n} \wedge (\Sigma^\infty X_+)[i] \to K$ in the Morel-Voevodsky stable homotopy category.

We would like to extend this spectral sequence to non-smooth schemes. In fact, there is a version for quasi-projective varieties converging to G-theory, but we would like one that captures K-theory.

Definition 5.5.3. A presheaf of spectra F is a *procdh sheaf* if it sends formal abstract blowup squares to cartesian squares of spectra.[6]

[6] We also require that it be a Nisnevich sheaf, but I don't want to get distracted with that here.

The canonical forgetful functor admits a left adjoint

$$\mathrm{Shv}_{procdh}(\{\text{qcqs schemes}\}) \leftrightarrows \mathrm{PSh}(Sm) : L_{procdh}$$

The following theorem is a combination of an observation of Bhatt–Lurie, and a result of K.-Saito.

Theorem 5.5.4 ([KS24, Thm.1.8])**.** *The left adjoint sends K-theory to K-theory.*

If we push the slice filtration through this left adjoint, then we obtain a spectral sequence with graded pieces $L_{procdh}\mathbb{Z}(n)[2n]$. Consequently, we obtain a spectral sequence.

Corollary 5.5.5 ([KS24])**.** *For any Noetherian scheme of finite dimension there is a convergent spectral sequence*

$$H^{2i-j}_{procdh}(X, \mathbb{Z}(i)_{procdh}) \implies K_j(X)$$

Remark 5.5.6. [EM23] (over a field), [Bou24] (mixed characteristic), show that $\mathbb{Z}(i)_{procdh}$ can be obtained from $\mathbb{Z}(i)_{cdh}$ and $\mathbb{Z}(i)^{TC}$, and also prove many properties about this (e.g., projective bundle formula).

5.6 Open problems

Here is a somewhat random selection of open conjectures.

Conjecture 5.6.1 (Parshin's conjecture)**.** For any smooth projective variety X defined over a finite field, the higher algebraic K-groups vanish up to torsion:

$$K_i(X) \otimes \mathbb{Q} = 0, \quad i > 0$$

Conjecture 5.6.2 (Finite generation conjecture for K-theory)**.** For any ring R that is finitely generated over $\mathbb{Z}$, the groups $K_n(R)$ should be finitely generated.

Conjecture 5.6.3 (Beilinson–Soulé vanishing)**.** For X a smooth variety, for all $i < 0$ one has

$$H^i(X, \mathbb{Z}(n)) = 0.$$

Conjecture 5.6.4 (Vandiver's conjecture, see [Wei13, Conj.VI.10.8])**.** If ℓ is an irregular prime, then the group $\mathrm{Pic}(\mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}])$ has no ℓ-torsion.

Vandiver's conjecture has been verified for all primes up to 163 million.

Theorem 5.6.5 (Connection to K-theory, [Wei13, Thm.VI.10.10])**.** *If Vandiver's conjecture holds for ℓ, then the ℓ-primary torsion subgroup of $K_{4k-2}(\mathbb{Z})$ is cyclic for all k.*

If Vandiver's conjecture holds for all ℓ, then the groups $K_{4k-2}(\mathbb{Z})$ are cyclic for all k.

Conjecture 5.6.6 ((One of) Beilinson's conjecture(s), [Nek94, Conj.6.5(2)])**.** Let X be a smooth projective variety over $\mathbb{Q}$. Then

$$\mathrm{ord}_{s=n} L(h^{2n-1}(X), s) = \dim_{\mathbb{Q}} CH^n(X)_0 \otimes \mathbb{Q}$$

where $L(h^{2n-1}(X), s)$ is the L-function associated to the motive $h^{2n-1}(X)$, and $CH^n(X)_0$ denotes the Chow group of codimension n cycles on X that are homologically equivalent to zero.

References

[AKN22] Benjamin Antieau, Achim Krause, and Thomas Nikolaus. On the K-theory of $\mathbb{Z}/p^n$ – announcement. *arXiv preprint arXiv:2204.03420*, 2022.

[Ang11] Vigleik Angeltveit. On the algebraic K-theory of Witt vectors of finite length. *arXiv preprint arXiv:1101.1866*, 2011.

[Bas68] Hyman Bass. *Algebraic K-theory*. Benjamin, New York, 1968.

[BGT13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic K-theory. *Geometry & Topology*, 17(2):733–838, 2013.

[BHLS23] Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M. Schlank. K-theoretic counterexamples to Ravenel's telescope conjecture. *arXiv preprint arXiv:2310.17459*, 2023.

[BK72] A. K. Bousfield and D. M. Kan. *Homotopy Limits, Completions and Localizations*, volume 304 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1972.

[Blo86] Spencer Bloch. Algebraic cycles and higher K-theory. *Advances in Mathematics*, 61(3):267–304, 1986.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral p-adic Hodge theory. *Publications mathématiques de l'IHÉS*, 129:199–310, 2019.

[Bou24] Tess Bouis. Motivic cohomology of mixed characteristic schemes. *arXiv preprint arXiv:2412.06635*, 2024.

[Cis13] Denis-Charles Cisinski. Descente par éclatements en K-théorie invariante par homotopie. *Annals of Mathematics*, 177(2):425–448, 2013.

[Cor06] Guillermo Cortiñas. The obstruction to excision in K-theory and in cyclic homology. *Inventiones mathematicae*, 164(1):143–173, 2006.

[DGM13] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*, volume 18 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2013.

[EM23] Elden Elmanto and Matthew Morrow. Motivic cohomology of equicharacteristic schemes. *arXiv preprint arXiv:2309.08463*, 2023.

[Ful69] William Fulton. *Algebraic Curves: An Introduction to Algebraic Geometry*. W. A. Benjamin, New York, 1969.

[Ful84] William Fulton. *Intersection Theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 1984.

[GH06] Thomas Geisser and Lars Hesselholt. Bi-relative algebraic K-theory and topological cyclic homology. *Inventiones mathematicae*, 166(2):359–395, 2006.

[GH11] Thomas Geisser and Lars Hesselholt. On relative and bi-relative algebraic K-theory of rings of finite characteristic. *Journal of the American Mathematical Society*, 24(1):29–49, 2011.

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser, Basel, 1999.

[Hae04] Christian Haesemeyer. Descent properties of homotopy K-theory. *Duke Mathematical Journal*, 125(3):589–620, 2004.

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1977.

[Hau25] Rune Haugseng. Yet another introduction to ∞-categories, 2025. Available at `https://runegha.folk.ntnu.no/naivecat_web.pdf`.

[HM03] Lars Hesselholt and Ib Madsen. On the K-theory of local fields. *Annals of Mathematics*, 158(1):1–113, 2003.

[HW21] Fabian Hebestreit and Matthias Wagner. Algebraic and Hermitian K-Theory, 2020/21. Winter Term 2020/21, University of Bonn.

[IWX20] Daniel C. Isaksen, Guozhen Wang, and Zhouli Xu. Stable homotopy groups of spheres. *Proceedings of the National Academy of Sciences*, 117(40):24757–24763, 2020.

[IWX23] Daniel C. Isaksen, Guozhen Wang, and Zhouli Xu. Stable homotopy groups of spheres: from dimension 0 to 90. *Publications mathématiques de l'IHÉS*, 137:107–243, 2023.

[Kel14] Shane Kelly. Vanishing of negative K-theory in positive characteristic. *Compositio Mathematica*, 150(8):1425–1434, 2014.

[Kri10a] Amalendu Krishna. An Artin-Rees theorem in K-theory and applications to zero cycles. *Journal of Algebraic Geometry*, 19(3):555–598, 2010.

[Kri10b] Amalendu Krishna. An Artin-Rees theorem in K-theory and applications to zero cycles. *Journal of Algebraic Geometry*, 19(3):555–598, 2010.

[KS02] Amalendu Krishna and Vasudevan Srinivas. Zero-cycles and K-theory on normal surfaces. *Annals of Mathematics*, 156(1):155–195, 2002.

[KS17] Moritz Kerz and Florian Strunk. On the vanishing of negative homotopy K-theory. *Journal of Pure and Applied Algebra*, 221(7):1641–1644, 2017.

[KS22] Moritz Kerz and Florian Strunk. On negative K-groups in positive characteristic. *Mathematische Zeitschrift*, 301:2071–2085, 2022.

[KS24] Shane Kelly and Shuji Saito. A procdh topology. *arXiv preprint arXiv:2401.02699*, 2024.

[KST18] Moritz Kerz, Florian Strunk, and Georg Tamme. Algebraic K-theory and descent for blow-ups. *Inventiones mathematicae*, 211(2):523–577, 2018.

[Lev08] Marc Levine. The homotopy coniveau tower. *Journal of Topology*, 1(1):217–267, 2008.

[Lur09] Jacob Lurie. *Higher Topos Theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, 2009.

[Lur17] Jacob Lurie. *Higher Algebra*. Princeton University Press, Princeton, 2017.

[May67] J. Peter May. *Simplicial Objects in Algebraic Topology*, volume 11 of *Van Nostrand Mathematical Studies*. D. Van Nostrand Company, Princeton, New Jersey, 1967.

[Mor16] Matthew Morrow. Pro *cdh*-descent for cyclic homology and K-theory. *Journal of the Institute of Mathematics of Jussieu*, 15(3):539–567, 2016.

[Mor18] Matthew Morrow. Pro unitality and pro excision in algebraic K-theory and cyclic homology. *Journal für die reine und angewandte Mathematik*, 736:95–139, 2018.

[Nek94] Jan Nekováø. Beilinson's conjectures. In *Motives*, volume 55 of *Proceedings of Symposia in Pure Mathematics*, pages 537–570. American Mathematical Society, Providence, RI, 1994.

[Rav84] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. *American Journal of Mathematics*, 106(2):351–414, 1984.

[SGA71] *Théorie des intersections et théorème de Riemann-Roch*, volume 225 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1971. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre.

[Sta25] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2025.

[TT90] R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. *The Grothendieck Festschrift*, III:247–435, 1990.

[Voe02] Vladimir Voevodsky. Open problems in the motivic stable homotopy theory. I. *Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998)*, 3:3–34, 2002.

[Wei80] Charles A. Weibel. K-theory and analytic isomorphisms. *Inventiones mathematicae*, 61(2):177–197, 1980.

[Wei89] Charles A. Weibel. Homotopy algebraic K-theory. *Contemporary Mathematics*, 83:461–488, 1989.

[Wei13] Charles A. Weibel. *The K-book: An introduction to algebraic K-theory*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, 2013.